GENERIC EXAMPLE PROVING CRITERIA FOR ALL

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Our goal is to provide criteria for determining if an example in an argument is being used as a generic example. We write in response to Leron and Zaslavsky's (2013) discussion of generic examples, which we agree with in some ways and disagree with in others. We agree with Leron and Zaslavsky's pedagogical advice about choosing examples that can serve as generic examples to develop proofs, but we contend that this advice is insufficient for judging whether an example used in an argument is generic or not. We also agree with Leron and Zaslavsky's purpose of discussing how generic proof relates to canonical notions of proof, but we believe that this discussion reveals some ambiguities that pervade our research community's discourse about generic example arguments.

In our interpretation, Leron and Zaslavsky use two criteria for a generic example, both of which help a teacher make a good choice when developing an example-based argument. One of these criteria is based on judging whether an example is a good representative of a class of objects, which we will call the "representativeness criterion." The second is based on judging whether an argument that relies on an example is a "generic proof," which we will call the "generic proof" criterion. We detail these criteria in the next section. We contend that these criteria are valuable for the purpose of pedagogical advice, but do not reliably extend to a framework for judging whether a generic example argument is valid or acceptable as proof, particularly when the framework is to be used by an arguer or observer who is not a content expert. In other words, we believe that these criteria are irreconcilable when the context is changed from choosing a pedagogically useful example to judging whether an argument is viable.

In order to extend the notion to this latter context, we develop different criteria for generic example argumentation. Fundamental to these criteria is a philosophy that we derived from our work on generic example proving. This philosophy holds that in an argument for a general claim, the representation of the claim's conditions is *not* necessarily where the generality of the argument lies. In many arguments, the generality lies in the warrants, that is, in how the representation is appealed to in supporting the claim. To us, the use of a quantified variable x in some arguments is merely an agreed-upon symbol that signals to the reader that the arguer is being general in her or his expression of the claim's conditions. We propose that in some arguments, features of the example can be viewed as placeholders that function analogously to variables, provided that the arguer is clear about this.

For example, Euclid's proofs invariably use a specific example as a referent, in the form of a lettered diagram. Special features are necessarily instantiated in the diagram, yet the argument is still considered a proof since none of these special features, such as specific lengths or angle measures, are appealed to. The diagram is not incidental, since after the claim is stated generally (*protasis*) it is always restated in reference to the specific diagram (*ekthesis*), and the rest of the proof is done in reference to that diagram (Netz, 1999). Remove the diagram, and we have what Balacheff (1988) might call a *thought experiment,* which Balacheff considers a type of proof. Has the argument been improved because the diagram has been removed? We say not.

In the next section, we detail these issues and discuss how they reflect ambiguities in our community's discourse about generic example arguments. We then propose different criteria for determining when an example is used generically in an argument and we tackle the question of when such a generic example argument can be acceptable as a "proof." We then offer a Lakatosian dialogue, of the kind Leron and Zaslavsky use, to illustrate viable generic example argumentation in the context of Leron and Zaslavsky's episode about Lagrange's theorem for finite groups.

Leron and Zaslavsky's criteria for generic example proving

Leron and Zaslavsky (2013) present three mathematical case study examples illustrating various facets of generic proving, which they describe as "any mathematical or educational activity surrounding a generic proof" (p. 24). The case studies are episodes from hypothetical undergraduate mathematics classrooms. In the third episode, the authors summarize a classroom-developed argument for Lagrange's Theorem that relies on an exploration of the cyclic group Z_{12}. In their discussion of this episode, Leron and Zaslavsky claim that Z_{12} is not a generic example:

> Indeed, the group Z_{12}, being a *cyclic* group (generated by a single element) is the simplest kind of group imaginable and thus definitely *not* a generic example of a finite group. If someone asked you for a generic example of a "typical" finite group, you would definitely not think of giving them this example. Still it does a fair job of exemplifying the main ideas of the proof of Lagrange's theorem. (p. 12, italics in original)

Here, the example Z_{12} is judged not to be a generic example of a finite group because the fact that it is cyclic, a special feature, is not "typical" of finite groups. This is what we

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have called the Leron and Zaslavsky *representativeness criterion:* that an example must be typical of the broader class of cases it exemplifies.

One problem with extending this criterion to judging an argument is that no object is unambiguously a representative of a larger class. By being a specific example, it will always have instantiated features not shared by all cases. No finite group is inherently representative of all finite groups. But there is a broader issue raised by the representativeness criterion. Judging whether an example is "typical" is fundamentally based on a person's concept image of the class of objects. Suppose a student has seen several cyclic groups and only one non-cyclic group. Based on this limited awareness, this student may reasonably consider Z_{12} "typical" of "group". Thus, this criterion is not just ambiguous; it is also problematic. Using the "representativeness" criterion to judge whether an example is used generically *precludes* students who do not share the teacher's concept image of "group" from being able to judge whether an example is used generically (Yopp & Ely, in press). Such a student has no clear basis for considering genericity, and it diverts their attention away from the crucial issue of the generality of the argument, by focusing it on the example itself.

Shortly after they use the representativeness criterion, Leron and Zaslavsky appeal to the *generic proof* criterion:

> Finally, this mathematical case study also highlights the fact that the test of genericity should be applied not to the example itself (Z_{12} is not a good generic example of a group), but rather to the proving process that this example generates: the process of partitioning G by its cosets, and the properties of this partition, are quite general, though the group to which we are applying this process is not. (p. 27)

Here, an example is considered generic if it generates a general proving process, or generic proof. Later the authors clarify what this is: a generic proof does not constitute a "complete proof" by itself, but it serves as a "recipe for the learner on how to construct the complete proof" (p. 28). This is a useful consideration for the teacher to keep in mind. Yet, it is fundamentally problematic for a student to understand and apply when judging example-based arguments in their own right. Using this criterion, the only way to really judge if an example is generic is to wait to see if a complete formal proof can be made from it.

We see similar problems in both of Leron and Zaslavsky's criteria if the objective is to develop criteria for generic examples that both students and teachers can use to create and evaluate arguments. The *representativeness criterion* presupposes that the arguer or evaluator has a rich example-base of the objects at hand (*e.g.*, group). The *generic proof criterion* presupposes that a "complete proof" (which we take to mean canonical proof, perhaps using variables or other placeholder symbols) is in place before the arguer can go back and judge whether the example used is generic. This makes the evaluation of the generic example argument inaccessible to anyone who does have the "complete proof" developed, and it classifies all example-based arguments as non-proofs.

Our criteria for assessing generic examples

Leron and Zaslavsky's treatment of generic examples draws substantially upon prior usage (*e.g.,* Balacheff, 1988; Mason & Pimm, 1984; Rowland, 1998). We believe that addressing the ambiguities we have identified will add to our entire field's understanding of the topic.

For our criteria, we first distinguish between (a) judging whether the example is used generically in the argument and (b) judging whether the argument can be accepted as proof. In this section, we discuss the first of these judgments. Our purpose is to get away from basing judgments on an observer's ability to see how the argument for the specific case applies (*mutatis mutandis*) to all cases, and to get away from basing judgments on the observer's opinions about "typical" examples. These judgments rely too much on observers' individual concept images. The key to our method is to consider only the manner in which the argument appeals to the example.

Our method relies on reconstructing the generic example argument using Toulmin's (2003) argument analysis scheme. The observer reconstructs the argument by noting plausible data and a plausible chain of warrants that link the data to the claim. We use the term plausible because argument analysis requires some inferences by the observer (Aberdein, 2006).

For an argument to use an example generically, none of the warrants can appeal to features of the example not shared by all cases in the domain of the claim. This means that warrants must appeal only to properties of the objects in the claim's hypothesis (*e.g.*, defining features), or properties that follow logically from them. A violation of this criterion occurs when the arguer appeals to some specific feature of the example.

For example, consider a person who creates a particular triangle, cuts off the angles, and arranges them adjacently so that the two outer angle sides form what appears to be a straight line and then uses this construction as an argument for the claim that *all triangles have interior angle sums of 180 degrees.* This person violates our criterion because the warrant for their argument appeals to a feature specific to the example: the particular measures of its angles.

Now suppose a different person starts with the same triangle, but instead chooses a base and draws a line parallel to it through the opposite vertex. They then argue that in two places the alternate interior angles formed between these parallel lines are equal. As a result the three vertex angles of the triangle are equal to three other angles that together constitute a straight angle. This arguer is making a general appeal to the example and the construction. The reason it is generic is not because the argument is more "sophisticated" or because it involves more advanced content knowledge than the other. The reason is that the warrants appeal only to properties of the objects in the claim's hypothesis or follow logically from them. Whether the argument can be accepted as a proof requires a bit more work, which at a minimum requires checking that all the logical necessities are in place to show how the conditions imply the conclusion. This proof criterion of logical necessity applies to any direct argument, not just an example-based argument. In the next section, we will contend that generic-example arguments can be accepted as proofs, in opposition to Leron and Zaslavsky's claim.

Can an example-based argument be accepted as proof?

Leron and Zaslavsky (2013) do not view generic-example arguments (generic proof) as proofs:

> The main weakness of a generic proof is, obviously, that it does not really prove the theorem. The "fussiness" of the full, formal, deductive proof is necessary to ensure that the theorem's conclusion infallibly follows from its premises. In fact, some of the more subtle points of a proof are prone to be glossed over in the context of the generic proof: some steps which "just happen" in the example, may require a special argument in the complete proof to explain *why* they happen, and to ensure that they will *always* happen. (p. 27, italics in original)

Using our criteria above for generic example, we contest this assertion, and propose that students can make and judge the viability of generic example arguments and that in certain situations these arguments can be accepted as proof. For example, when the conditions of the argument have a relatively simple mathematical structure that can be expressed easily in an example, the example-based argument demonstrates how the conclusion of the claim follows as a necessity from the conditions, and the features of the example can be viewed as placeholders where any of their "kind" can go without altering the argument. According to Leron and Zaslavsky, an important trait of a full deductive proof, that a generic proof lacks, is fussiness: a generic proof necessarily leaves out important steps. We contend that this happens no more necessarily for generic proofs than for any other kind of proof done in a class. In either kind of argument, a prover may leave out a reason for a step because they do not believe or do not realize that the step needs backing.

We acknowledge that there is extensive debate in our community about what constitutes "proof" (see Weber, 2014, for a summary), and we are not going to take on that debate here. We do however contend that the criteria for "proof" or "complete proof" laid out by Leron and Zaslavsky can be met with example-based arguments. We worry that Leron and Zaslavsky may be, perhaps inadvertently, prioritizing general representations (such as ones that use quantified variables or symbolic placeholders) when evaluating the validity and soundness of arguments, as opposed to evaluating whether the representations are being appealed to in a general manner. If "fussiness" is what matters, it should not be assumed that canonical representations afford fussiness and that others do not.

We agree that sufficient fussiness is important for an argument to be acceptable as a proof, whether the argument uses a generic example or not. The criterion of fussiness is inherently ambiguous, since some steps can be skipped if the arguer justifiably assumes the reader does not need them spelled out. The point is that when a proof skips steps, it should be because the arguer knows that a logical argument for these steps could be provided, if needed, not because the arguer is unaware that steps are being skipped. Arguers must still check whether steps are being skipped and demonstrate that conclusions follow from conditions when constructing a generic proof. In both example-based arguments and formal proofs, the arguer must pause and ask, "Did I just assume something about the objects that has not already been established by the community or readers, or in an earlier step?" In the case of a generic example argument, they must also ask two other questions: "Have I appealed to a feature of the example that is special to that example *i.e.*, is not in, or implied by, the definition(s) the example exemplifies?" By attending to this question, we contend that a generic proof can avoid gaps in reasoning just as well as a "formal" proof.

Leron and Zaslavsky demonstrate that generic proofs can emerge organically from student exploration in ways that formal proofs sometimes do not. One might ask whether our criteria for generic example use sacrifices this naturalness by which a generic proof can emerge in a classroom? In order to show that it need not, we provide an example of how generic proving might develop in a classroom and ultimately result in a proof. For the sake of length, our version is "fast-forwarded" a bit, and we omit some dead ends the discussion might go into in a real classroom. We provide a hypothetical dialogue from a classroom resulting in an argument for Lagrange's Theorem that uses Z_{12} as a generic example and that could reasonably be accepted as a proof. To be clear, we entirely agree with Leron and Zaslavsky that Z_{12} is *not* the best pedagogical choice for a generic example for this purpose. We use Z_{12} to demonstrate that it *can* be used as a generic example, contrary to Leron and Zaslavsky's claim. This example illustrates our main point: that it is not an inherent property of the example that makes an example-based argument "proof," but rather how the example is used in the argument.

Mathematical case study: developing a generic example argument for Lagrange's Theorem

The students in an undergraduate group theory class have been asked to prove Lagrange's Theorem: If H is a subgroup of a finite group G, then the order of H divides the order of G.

Alpha: Let's check an example. For instance, Z_{12} has a subgroup $\{0, 3, 6, 9\}$, and its size, 4, divides 12.

Beta: What good is checking one example?

Alpha: To get a feel for how the argument goes. How about a different example, like…

Beta: Stop! Empiricist! How many examples will you generate? What good will any of that do us? So it works for Z_{12} and your chosen H. Suppose you show me it works for a thousand examples. So what? You will never convince me it works for all.

Gamma: Maybe an example can help us see *why* the theorem works in general.

Teacher: This is a good point. Proofs can emerge by starting with general representations like "G" and "H," and proofs can also emerge from exploring examples. Let's try Alpha's example, but let me point out a strategy that isn't totally obvious. Take a look at all of the "shifts" of your subgroup $H = \{0, 3, 6, 9\}$ inside the larger group Z_{12}. Take your subgroup and "shift" everything in it by adding a given element of your original group. You'll get a shifted version of your subgroup. Try it for Z_{12}. By the way, we call these shifts *cosets* of H in Z_{12}.

[*The class writes out all of the shifts of* H *in* Z_{12} *(Figure 1).*]

Alpha: I see something! You only need three "shifts" to get the entire original group. There are 4 elements in H. Shift H once; there are 4 more. Shift it again; there are 4 more. You fill up the whole group with 3 chunks of size 4. The order of the subgroup (4) divides the whole group (12).

Beta: Well good for you. You have shown that everything goes smoothly for this group and subgroup. You are about 100 kilometers from an argument and light years from a proof!

Gamma: Maybe we should write down what makes this shifting idea work. Perhaps we can make a case for all other groups as well.

Delta: The shifts had to be the same size or else you can't just multiply by something to fill up the order of the whole group.

Alpha: But we know this. When you shift by something like 2, all of the things in H just go up by 2 and you get $\{2, 5, 8, 11\}$—still 4 things.

Beta: Stop that!

Alpha: Stop what?

Beta: What you just said relies on the fact that the group operation on Z_{12} is addition. Everything shifts by the same amount, cycling around if needed. That's what cyclic does! Cyclic groups cycle around. Other groups don't necessarily do that.

Gamma: [*Has been writing out the group table for* Z_{12} *in the corner of the board—see figure 2*] I think we actually don't have to use the idea of "goes up by" here. Just look at the group table. In each row (and each column) each element of the group must appear. That happens for any group. Because all elements must appear in each row, they can only appear once.

Take a subgroup like H here and hit it with a 2. You'll get four different things in this row of the group table. [*Gamma highlights the four elements in Figure 1.*] Since they're in the same row, they have to be different from each other. The shift has to be the same size as the original subgroup.

Beta: Fine. But in Alpha's example we're using more than just the fact that the cosets all have the same number of elements. The cosets are either disjoint or equal, and there are three of them. That's what makes the formula work. Twelve elements divided by 3 cosets equals 4 elements in the group. This part of your example relies on "cyclic". I've got you on this!

Gamma: Ah…so we have to explain why any two shifts of a subgroup are either disjoint or equal?

Teacher: Here's a hint: suppose you knew that two shifts had at least one element in common. What could you conclude then?

Alpha: Okay…$4 + H$ and $7 + H$ both contain 10. Oh I get it! You can verify that they each contain *all* the same things as each other.

$0+H = \{0, 3, 6, 9\}$
$1+H = \{1, 4, 7, 10\}$
$2+H = \{2, 5, 8, 11\}$
$3+H = \{3, 6, 9, 0\}$
$4+H = \{4, 7, 10, 1\}$
$5+H = \{5, 8, 11, 2\}$
$6+H = \{6, 9, 0, 3\}$
$7+H = \{7, 10, 1, 4\}$
$8+H = \{8, 11, 2, 5\}$
$9+H = \{9, 0, 3, 6\}$
$10+H = \{10, 1, 4, 7\}$
$11+H = \{11, 2, 5, 8\}$

Figure 1. Cosets of Z_{12}.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

Figure 2. Group table for Z_{12} *with Gamma's highlighting.*

Beta:	Goodness. Now you're back to just checking Z_{12}!
Gamma:	Look, 4 + 6 = 10 and 7 + 3 = 10. So 7 + 3 = 4 + 6 and thus 7 = 4 + 6 – 3. We know that 4 + *H* and 7 + *H* are equal because 7 is just 4 + 3.
	[*Gamma writes:* 7 + *H* = (4 + 3) + *H* = 4 + (3 + *H*) = 4 + *H*.]
Delta:	Why can you just say that 3 + *H* is the same as *H*?
Gamma:	Because *H* is a subgroup. It is closed under the group operation. If you hit it with anything in *H*, like 3, you only get stuff already in *H*. By the previous argument it'll be the same size as *H*, so everything in *H* will be in there.
Beta:	Using your table [*smirks*].
Gamma:	No, what the table represents. Every element of a group (or subgroup) occurs exactly once in each row. This is something we know to be true of every group. That doesn't use any special feature of our example. It uses the fact that *H* is a group.
Delta:	Okay, I see why that part will work for any group. But, you had to be able to write 7 as 4 + (something in *H*). How do you know you can do that for any finite group and subgroup?
Beta:	Maybe it has something to do with knowing that both 4 + *H* and 7 + *H* share at least one element.
Gamma:	Yes, that's it. Alpha pointed out they both share 10. That means that we can write 10 as 7 + <u>3</u> (because it is in 7 + *H*) and as 4 + <u>6</u> (because it is in 4 + *H*). So 7 + <u>3</u> = 4 + <u>6</u>. So 7 = 4 + <u>6</u> – <u>3</u>. And <u>6</u> – <u>3</u> is 3, which is something in *H*.
Delta:	What do you mean by the minus sign? We can only use the group operation +.
Gamma:	Good point. I should have written 7 + <u>3</u> = 4 + <u>6</u>. So 7 = 4 + <u>6</u> + (<u>3</u>–). So 7 = 4 + (<u>6</u> + (<u>3</u>–)). And (<u>6</u> + (3–)) = 3, which is something in H. It works because every group element has an inverse.
Beta:	You used special properties of Z_{12}—that 6 + 3– = 6 + 9 = 3 and that *H* is a subgroup—to conclude that 3 and 3– are in *H*.
Delta:	No, Gamma used the fact that any element's inverse is in the group and so is the combination of any two group elements.
Alpha:	Why is this generic?
Delta:	Because our two shifts 4 + *H* and 7 + *H* share an element, we can use that element to write the 7 in terms of a 4 + something in *H*. That shows why 7 is in 4 + *H*. We didn't need to use anything special about the 7 or 4, or *H*. Look:
	[*Delta writes:* 7 + *H* = 4 + (something in *H*) + *H*, and (something in *H*) + *H* is just *H*.]
Beta:	That makes sense.
Alpha:	Why is Beta not grouchy any more?
Beta:	Well let me return to my usual demeanor, but contribute. You also used the associative property of groups.
Gamma:	Now we know why all of the shifts are either disjoint or equal, and why they all have to be the same size. The original group is filled up by non-overlapping shifts of the subgroup that are all the same size. The order of the group is just the order of the subgroup times some number of shifts. We're done!
Delta:	Sorry to spoil the fun, but how do you know that everything in your group is in one of the shifts?
Gamma:	Don't worry. We know each element of the group, for instance 8, is going to be in at least one of the shifts. *H* is a subgroup, it contains the identity 0, so 8 + *H* will contain 8 + 0, which is 8.
Teacher:	Can you extract the relevant pieces of your argument and make them into a proof?

Extracting the generic proof from the classroom episode

Claim: For any finite group *G* and subgroup *H* of *G*, the order of *G* is equal to the order of *H* times the number of distinct cosets of *H* in *G*.

Support: Consider Z_{12} and its subgroup *H* = {0, 3, 6, 9}. The group table in Figure 3 shows all possible cosets (indicated by shaded columns).

1. These cosets all have four elements, the number of elements in *H*.

2. These cosets are either disjoint or equal.

3. Every element of *G* is contained in at least one coset.

The Z_{12} table is a referent that allows the relational structure between the conditions and conclusions of the claim to be expressed.

	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

Figure 3. Group table of Z_{12} with cosets of the subgroup $H = \{0, 3, 6, 9\}$.

Observation 1 is true because every row of the group table contains every element of the group exactly once. We appeal to this as a previously established result. (This is based on the fact that the group operation on Z_{12} combines two elements to form an element in Z_{12}. Each row has one of these "combined" elements fixed. When one element is fixed, the group operation on Z_{12} is one-to-one.)

Observation 2 is true because if $7 + H$ and $4 + H$ have an element in common, say 10. We can write 10 as 7 + 3 and as 4 + 6. But this means that 7 = 4 + 6 + 3– = 4 + 3, which shows that 7 can be written as 4 + (an element of H). This means that $7 + H$ is 4 + (something in H) + H, which is $4 + H$. This uses inverse, associative, and closure properties, which are true for every group, not just Z_{12}. So if two cosets overlap, they are actually equal. All cosets of all finite groups have the properties used to show 7 is an element of $4 + H$.

Observation 3 is true because 0 (the identity for this group) is in H, and every group has an identity element.

Can we accept the Z_{12}-based argument for Lagrange's Theorem as proof?

We believe that the generic example argument that emerged from our hypothetical class's exploration of Z_{12} can be taken as proof of Lagrange's Theorem for finite groups. First, we check to see if the example is being used generically. To do this we need to establish the definition of finite group that the group is using. Suppose it is:

> A set of elements and an operation that combines two elements of the set to form an element of the set is a group if and only if the operation and the set have the properties of closure, associativity, identity, and inverses. A group is a finite group if and only if it has a finite number of elements.

Observation 1 appeals to a feature in the table shared by all finite groups, which had apparently been established in an earlier class session. Observation 2 uses only features of the table shared by all finite groups: closure, associativity, and inverses. Observation 3 appeals only to the identity element, which all groups have by definition.

At this point, we can stamp the example use as generic. This is a necessary but not sufficient condition for accepting the argument as proof. We must also check "fussiness". In other words, we need to make sure no important steps have been skipped. Again, we note this determination is ambiguous, but all types of arguments can have such ambiguities.

The extracted argument logically derives its conclusions only from non-specific features of a finite group—either ones established in class or directly from the definition of group—but it does not entirely acknowledge how Observation 1 requires the group to be *finite*. The "appeal" in that step is true only for finite groups. We contend that if this step is added, the argument may be accepted as proof. The argument demonstrates, with appropriate fussiness, that the claim's conclusion follows necessarily from its hypotheses, even though the reasoning is illustrated with an example.

Conclusion

We acknowledge the usefulness of Leron and Zaslavsky's (2013) work and its value as guidance for a teacher making choices of examples for proving when teaching. We have presented alternative criteria for generic examples in order to judge when an example is *used* generically in an argument. With these criteria in place, we argue that a generic example argument can be accepted as proof provided that additional criteria are in place. This is in contrast to the stance expressed by Leron and Zaslavsky.

Our goal is more than just to improve the accuracy of such judgments of researchers, teachers, and students. By removing criteria that rely heavily on robust observer concept images, we can empower students and others to assess their own reasoning rather than appealing to the teacher to sanction their arguments.

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