A Japanese Approach to Arithmetic

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Reconceptualizing arithmetic

In 1981, my wife Elizabeth, who speaks Japanese, and I spent four months in a Tokyo elementary school observing and recording the teaching of mathematics Because we had become increasingly concerned over the "trap" into which many third and fourth graders in American schools have fallen who have come to depend on counting, we wanted to see in detail how arithmetic was introduced to children in the first two grades without counting [Hatano, 1980, a and b] Counting means different things, but what worries us, for example, is the repetition of number names in correspondence with markers or with fingers to solve two-digit addition and subtraction problems. We chose Kitamaeno School because there seemed to be a relatively low level of parental pressure on children, e.g , few students attend after-school academic tutoring classes (*juku*) and few of its graduates went on to post-secondary schools What surprised us most was their habit of giving young children challenging problems and organizing them into groups to evaluate and analyze their own work [Easley and Easley, in press, a and b] It appears there is a powerful advantage in a reflective, problem-solving approach to a reconceptualized arithmetic which seems to have developed throughout Japan in the past 20 years. However, the most profound difference between the Kitamaeno School and typical American elementary schools lies in the social atmosphere and high level of pupil responsibilities in this school which highlights these other differences in the math curriculum.

Teachers at Kitamaeno School, a regular public school serving about 600 children, belong to Zenkoku Seikatsu Shido Kenkyukai, a voluntary teachers association promoting child study which holds annual meetings in the summer to study ways of improving teaching Through our continuing contacts with the academic chairman of the faculty, Mr Seiichi Watanabe, and the math resource teacher, Mrs Yumiko Hashimoto, we have learned that the focus on child development in this association has spread to as many as 20% of the schools in their district and to many schools in other districts as well.

Kitamaeno School is located in a relatively low-income section of Tokyo (70% of the pupils receive lunch at government expense). It includes grades 1-6, and serves a community of mainly wage earners, temporary employees, and keepers of small shops of all kinds, representing a rather independent-minded segment of Japanese society Few parents, if any, were tenured employees of big companies, where upward mobility and company loyalties are reported to be high.

In terms of mathematical content the Kitamaeno School teachers have been influenced by ideas promoted by another voluntary organization, The Association of Mathematical Instruction (AMI), which Professor Toyama founded and is now headed by Professor Ginbayashi. (*Sugaku-Kyoiku-Kyogikai* (SKK) is the Japanese name of this organization of more than 2,000 members which has been working for twenty years [Miyamoto, 1980, Hatano 1980 a and b].) AMI developed an approach, using manipulative tiles, that is based on partitioning quantities and recombining them as polynomials, and on a modern idea of continuous quantity which mathematicians like Toyama and Ginbayashi have advanced. Hatano [1980a and b] points out the relationship which partitioning focussed on 5 has with the Japanese abacus (soroban), a cultural artifact which certainly goes far back in history and undoubtedly originated as a modification of the Chinese abacus However, the abacus is not actually used in introducing quantity in Japanese schools but is itself introduced late in third grade as an application of the polynomial structure of multidigit calculations that the children understand and use well, by that time, in calculations with whole numbers Hatano also reports that the leading Japanese teachers union has endorsed these changes.

Although the AMI tiles bear a superficial resemblance to the Cuisenaire rods introduced by Gattegno, and the base-ten blocks of Dienes, the partitioning of numbers and the recombination of their parts begins with five, and eliminates any need for counting algorithms, which makes it quite distinctive The Japanese national curriculum and the authorized textbooks we examined do not encourage any use of counting as a basis for understanding addition and subtraction.

A contrast of partitioning with the counting model for adding and subtracting can be readily grasped by visualizing domino patterns For example, the familiar pattern for five can be seen directly as composed of a four pattern with a one in the center, or as overlapping 3 + 2 patterns The

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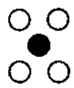
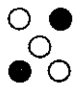
results of such calculations as 1 + 4, 5 – 1, or 5 – 4, 3 + 2, 5 – 2, or 5 – 3, are evident without counting.

Thus 5 is seen as 4 and 1: 

or 5 is seen as 3 and 2: 

and 3 becomes 2 and 1: 

Similarly, the partitioning of 4 and 2 can be seen with little difficulty. For flexibility in visualizing these partitions, rows or other arrangements of tiles, plastic flowers, fingers and other materials were used in first grade at Kitamaeno School. Also, each of the numbers from 0 – 5 was studied to discover what examples could be found in the classroom.

How partitioning is interpreted by first grade children

A very interesting classroom debate we observed concerned whether it was right to partition 3 into 2 and 1 or to partition 5 into 3 and 2. The arguments seemed to reflect a concern for visual symmetry and suggested an intense, but brief, struggle in learning to perceive patterns and partitions corresponding to the first group of numbers, 0 – 5

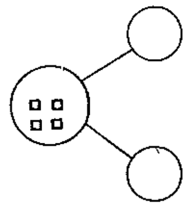
> On Thursday of the third week of school, Watanabe sensei's first grade class (with all 39 pupils present) continued work on partitioning numbers up to 5. The children each had 5 thin cardboard tiles on their desks, instead of the plastic flowers they had used on Wednesday before they were interrupted by a fire drill. On the board, Watanabe sensei had larger magnetic tiles of plastic, and following some of the suggestions of the children, he gradually built up a "staircase" from 1 to 5. When reviewing the partitions of tiles into two parts, however, he encountered the same problem he had the day before. One child came up to the board and tried to make 3 groups with 3 tiles.
>
> Watanabe sensei asked, "Isn't it all right to have one left over?" (Presumably, 2 in one group and 1 in the other.)
>
> A boy volunteered, came up front, and tried adding a fourth tile to the three before dividing them into two equal groups.
>
> Sensei said to him, "You are going ahead to what we'll do next."
>
> When this boy went back to his seat, the boy who sat behind him complained that he couldn't hear what he had said, evidently picking up on the emphasis that sensei gave to speaking clearly and with confidence. In reply, he got a whispered, "*Baka*!" (stupid)

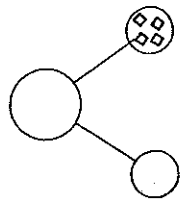
Going on to 4, Watanabe sensei asked them to set up 4 of their own tiles like his and "divide them into two groups."

"Divided," chorused back the class with one voice.

(One boy, however, made a domino pattern of 5, evidently bored with such simple tasks.)

A girl was called up to demonstrate the division of the 4 tiles sensei had placed in a large circle on the board into two smaller circles. She put all 4 into the top circle.





"Is it all right?" Watanabe sensei asked the class.

"Wrong!" chorused the class loudly.

As the girl was embarrassed, Watanabe sensei asked another girl from the same group to come up and "rescue her." The second girl put two tiles in the top circle and two in the bottom one.

"Is it all right?" sensei asked.

"It's right!" the class chorused.

A child who could write numerals was then asked to come up and write the 4, 2, and 2 by the appropriate circles.

Putting 5 magnetic tiles in the large circle, Watanabe sensei then asked the children to divide their 5 tiles into two groups. One made the domino pattern with corners touching, another a cross, and others made more open patterns of various kinds, some with two distinct groups and others with three. A 2-1-2 pattern caught Watanabe sensei's eye and he asked the boy who made it to put it on the board with the magnetic tiles. When he put 5 tiles in a 2-1-2 pattern in the lower circle, sensei asked if this was dividing the five tiles into two groups.

The boy replied that he had just made it up in his head and wasn't listening.

Sensei then called on a girl who didn't have her hand up, but had a 3-2 partition on her desk.

She came up with lowered head, but sensei said, "Don't worry, just do it the way you did at your desk. She put 3 in the lower circle and 2 in the upper one.

There were some shouts of "Wrong!" but some applause also.

Another child put up a 4-1 arrangement, which was approved by the class, then another repeated the 3-2 arrangement, to which the class shouted, "Same!" and it was changed to 2-3, which pleased some but not others.

The boy who had earlier wanted to add 1 to 3 and so go on to 4, now asked for another tile so he could do 6. Watanabe sensei declared the lesson was over, and called on two students who shouted, "Attention!" and when satisfied that everyone was sitting straight, "Bow!" The unison bow of the class, returned by sensei, signaled the end of the class, and everyone noisily left to take their twenty-minute recess, Watanabe sensei included.

The domino patterns and other patterns, which many of the Kitamaeno children loved to make with their tiles and plastic flowers, suggest a concept of discrete quantity, although the staircase sensei made with his larger collection and some of the compact or corners-touching patterns the children made involve the idea of continuous quantity that AMI emphasizes. What is typical of Kitamaeno school math teaching in the above lesson is that the children are expected to make their own judgements about what is right and what is wrong and the teacher's efforts to lead them to better ideas are deliberately limited and often ignored by pupils in their enthusiasm for their own ideas. When Watanabe sensei noticed several days later that many children still remained unhappy about partitioning 3 into 2 and 1 or partitioning 5 into 3 and 2, he called a meeting with the other first grade teachers to discuss it. They all reported similar experiences and, after considering various ways of helping children accept such "unfair" partitions, it was decided to drop this matter and simply go on the next topic in the curriculum, subtraction. As it turned out, the worrisome conceptual difficulty in partitioning caused no trouble in subtraction – taking away 1 or 2 from 3 or 2 or 3 from 5 left the proper remainders. Children used their tiles or looked at their fingers – only very rarely making any rhythmic movements or sounds that would suggest counting.

The emphasis all teachers placed on speaking clearly and with confidence seemed to be developed in first grade by the practice of choral responses, which were not memorized but clued by the words used in the questions, and shy individuals were supported by encouragement from their groups. Groups were used both to support timid or confused children and to teach listening skills to the brash and domineering ones. While interpersonal or social skills were often a bit roughcut in new first graders, they were quite effective in second graders and very polished in children in the higher grades. It was amazing how well mathematical debates were conducted by children once they began to catch on to the expectation that they had the responsibility to decide on what they would believe and that they shared in determining how things were to be explained.

There were very few errors made in finding sums and differences from 0 – 5, and the few that were made seemed to come from the residual counting procedures that a very few of the first graders initially relied on whenever they were on the spot. Whether the quick recall of addition and subtraction facts involving small numbers is just quick silent counting [Suppes, 1967; Gelman and Galistelli, 1978] or "subitizing", as staring at patterns of tiles or fingers suggests, is difficult to decide. The advantage which such visual patterns have over counting methods for the practical teaching of arithmetic lies in the way in which alternative partitionings, such as 4 and 1 or 3 and 2, model both the process of regrouping in algorithms involving two or more digits and provide quick intermediate results in more complex calculations, as we shall explain.

The objection raised by some advocates of counting methods, that children learning visual patterns are just memorizing these basic facts without comprehending them, did not seem justified in the context of the classrooms at Kitamaeno School, where lots of physical representations of these partitions were used.

Developing alternative algorithms for multi-digit computation provides a greater opportunity for mathematical reasoning than mechanical counting does and presumably peoples who later do very well in mathematics have, largely on their own, cut their mathematical "teeth" on it. Story problems and computational algorithms are the source of much pain for American children. They'd be better off, like the children at Kitamaeno School, trying a new approach to understanding algorithms than sticking with the counting basis of our present curriculum just because children take to counting so naturally. That counting is natural cannot be denied. The problem is that it doesn't regularly lead naturally to algorithms.

"The Great Mystery"

Terry Denny reported [1978] that teachers in a Texas school were very familiar with what they called "The Great Mystery". They told him that many children in second, third, or even fourth grade often would learn to perform an algorithm perfectly, but then a week later appeared never to have even heard of it, or else made terrible mistakes in doing it. This could be interpreted, again, as lack of sufficient drill and practice in the algorithm, or it could tell us that something may be conceptually wrong with the curriculum. The first clue I had as to what might be wrong came from the Steuben Primary Center in Kankakee.

A very perceptive second grade teacher in that school, Prudy Kimery, several years ago, pointed out to me that her children had a great deal of difficulty using counting and place value at the same time. I had not been aware of this conflict before she pointed it out. She thought perhaps she might be able to teach better if she didn't have to keep switching back and forth between exercises that can be done by counting and exercises that require recognizing the place values of parts of numbers. She thought of rearranging the sequence of topics to minimize the switching back and forth between these two concepts – which form the rational, but easily reconciled, bases of our arithmetic. After trying several ways to connect these two into an integral whole, we can see how much more smoothly calculations develop by discarding counting.

We know, as adults, that there is no real conflict between counting and place value, for the Arabic numerals we use in counting are generated by the place value system in a standard order, just as the odometers on our cars or the counters on our tape recorders generate the sequence of Arabic numerals as the units wheel is turned, each wheel advancing the next wheel to the left by one number as it completes one revolution. So anyone who knows the place value system for generating the sequence of numbers can count up and down as much as they like. The conflict is therefore not a logical one, but must be a psychological incompatibility for children who have learned to count numbers past 9 without feeling the "bump" as the counting in the ones column starts over each decade [Hendrix, 1962; Whittaker, 1978]

Braunfeld [1979], referring to the base-10 structure, summarizes: "Depending on the arithmetic context, children are required either to exploit this structure to their advantage or to suppress it in their minds are irrelevant. This, I believe, creates a conflict that accounts for some of the difficulties we see in children's efforts to count and do elementary arithmetic."

Mieko Kamii [1981, 1982], who studied children's idea of place value from the point of view of developmental psychology, suggests that first grade is too early for most children to understand place value. However, if you learn to use counting to add and subtract before you understand the process of generating number names, place value must be more difficult to learn, because you have to go back and reinterpret what you were doing. In counting, ten, eleven, twelve, are just seen and heard as the next three numbers after nine, not as the start of a whole new cycle of constructing numbers. The way they are written, 10, 11, 12, should signal that the cycle is starting over, but it can be mistaken as simply an arbitrary way of writing numbers – something like spelling. These alternative conceptions may be what makes the development of place value so difficult for so many children, and even adults, to grasp.

Adding zero

Adding zero required more discussion at Kitamaeno School than did the other number facts from 0 – 5.

> On Tuesday of the seventh week of school, with 38 pupils present, Watanabe sensei galvanized the class to attention by announcing that he had a very strange thing to be added. Putting three large red magnetic counters together on the board, and saying that they represented red apples, he drew a basket underneath. Then writing a plus sign and drawing another basket followed by an equal sign, he asked, "How many?" Several children said, "Three."
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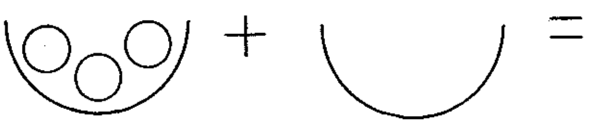
> 
>
> When asked for an explanation, Komatsu san said, "There are three apples in one and there are none in the other, so three apples in all."
>
> Sato san, another girl, then explained, "There are three here, but since there are zero there, you can't add, therefore there are only three."
>
> Kaneko kun, a boy, said, "There are three in one basket, none in the other, so you can move one apple into the empty basket and have 2 + 1, or move two and have 1 + 2, which still makes three."
>
> Katakura kun said, "You can't move apples to other baskets because it changes the meaning (of the problem)." Sensei said, "Let's think about this and discuss it again tomorrow."

Comment: If a teacher, or textbook author, had never presented children with the problem of finding all the combinations that make 3, but only with pairs of numbers to be added, then one could assume that every problem must have fixed addends. However, that was not the case in Watanabe sensei's class, so this debate was understandable.

> On the next day, with all 39 pupils present, the discussion was continued. When Watanabe sensei brought three red magnetic markers down from the top to the center of the board and drew a basket underneath, the children all said they remembered. Before asking for the answer, sensei drew the empty basket, the plus and equal sign as before. Then, to reconstruct the last few minutes of the previous lesson, sensei called on Kaneko to show what he had done yesterday. Kaneko went to the board and moved the counters successively into two other arrangements, explaining that either way the answer was still three.
>
> Next Katakura was called up to explain his argument. Sensei reminded the class, "Yesterday, Katakura said, "It's funny," and "The meaning is different, isn't it?" "
>
> Katakura now added, "It doesn't mean exactly three."
>
> Sensei asked him, "Katakura, is your three this one in the basket of this one over here in the answer box?"
>
> Katakura pointed to the latter.
>
> "The three on the right is the answer to the addition," sensei continued. "This is the result of adding this to this. He says this doesn't have the meaning of three. Please discuss it. I wonder whether Katakura's or Kaneko's ideas are the best, or whether there are more ideas."
>
> The children talked seriously in their groups for about a minute. Sensei took a vote: Those who thought Kaneko kun's thinking was good: 10. Katakura's: 28. Sensei then asked, "Who didn't put up their hand?"
>
> Ichimura kun: "You can't add. You can't add 3 + 0. Therefore, there's no answer."
>
> Sensei asked, "Do you mean the answer is zero?"
>
> "Yes."
>
> "Who agrees?" (eight) "Oh, some of you have changed your minds."
>
> Sato san went up front. "You can't add 3 and 0, therefore your answer is three."
>
> "You can't add," sensei asked "or there's nothing to add – which is it?"
>
> "There is nothing to add," she replied.

Sekine kun agreed with Ichimura that the answer was zero

Akutsu went up and said, "Since there are three here, even if there are none here, there are three."

Watanabe sensei evidently understood from the previous lesson, a point we had missed, that zero was the problem bothering Katakura kun, for he came prepared with collecting jars for a thought experiment, in which he dramatically poured no water into a partially filled container.

It is certainly impressive that a number of first grade children participated so effectively in this debate. Although two or three of the 39 children needed individual tutoring by the teacher, if they were ever to catch up with the others, the results with at least 35 of the children were very impressive, for they were beginning to show that they understand what they were doing. This is evidenced in self correction. Simply pointing to an error on their page usually resulted in their correcting it confidently right away.

Using five to build facts

At this point, nearly all the first grade children at Katamaeno School could add the following array of sums:

0+0	0+1	0+2	0+3	0+4	0+5
1+0	1+1	1+2	1+3	1+4	
2+0	2+1	2+2	2+3		
3+0	3+1	3+2			
4+0	4+1				
5+0					

And using the same partitions they could find the following differences:

0–0					
1–1	1–0				
2–2	2–1	2–0			
3–3	3–2	3–1	3–0		
4–4	4–3	4–2	4–1	4–0	
5–5	5–4	5–3	5–2	5–1	5–0

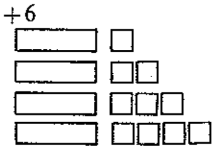
The cardboard five-tiles and ten-tiles used at Kitamaeno School had small divisions marked on one side and no divisions on the other. It was interesting just watching a few children turning over the five-tiles, when they were first introduced, sometimes for reassurance, counting the one-tile divisions on the back, and turning them back. Clearly, some of the children also used some other schemas besides tiles (perhaps even counting) in thinking about sums and differences.

The next step in the AMI approach was to define the numbers from 5 to 9 as combinations of 5 with numbers already studied, thus adding these sums to the bottom row of the array of sums:

5+1 5+2 5+3 5+4

Tile patterns for the four new numbers are shown at the right, below. Without reaching ten yet, these definitions opened up the combinations shown to the right of the diagonal broken line. Making a "5" or substituting a "5 tile" for five ones is the first step in the new combinations, and recognizing the answer visually as one of the newly defined numbers is the second step.

0+0	0+1	0+2	0+3	0+4	0+5	0+6	0+7	0+8	0+9
1+0	1+1	1+2	1+3	1+4	1+5	1+6	1+7	1+8	
2+0	2+1	2+2	2+3	2+4	2+5	2+6	2+7		
3+0	3+1	3+2	3+3	3+4	3+5	3+6			
4+0	4+1	4+2	4+3	4+4	4+5				
5+0	5+1	5+2	5+3	5+4					



Learning the new sums above involves learning instantly to see a five whenever it can be formed and to recognize this five with the ones left over as one of the four new numbers 6, 7, 8, or 9.

The 28 new differences immediately available from this approach are shown below the horizontal line:

0–0									
1–1	1–0								
2–2	2–1	2–0							
3–3	3–2	3–1	3–0						
4–4	4–3	4–2	4–1	4–0					
5–5	5–4	5–3	5–2	5–1	5–0				
6–6	6–5					6–1	6–0		
7–7	7–6	7–5				7–2	7–1	7–0	
8–8	8–7	8–6	8–5			8–3	8–2	8–1	8–0
9–9	9–8	9–7	9–6	9–5	9–4	9–3	9–2	9–1	9–0

None of the new differences above requires breaking up a five but just repartitioning the ones. These differences are easy to see with tiles.

The missing differences needed to fill in the empty triangle in the above array require a two-step process. In these cases, the five used to make the 6, 7, or 8 has to be broken up. One can subtract from the 5 and then add the part of it that is left to the remaining ones. This two-step process is similar to a neat way of borrowing with two-digit numbers and paves the way beautifully for it. Here the steps are these:

6–4 becomes (5–4)+1; 6–3 becomes (5–3)+1; 6–2 becomes (5–2)+1
7–4 becomes (5–4)+2; 7–3 becomes (5–3)+2
8–4 becomes (5–4)+3

Of course, many other recombinations are possible as children begin to remember more and more of the new facts. Working these patterns out with tiles was slow work for new first graders, but after fourteen weeks altogether, beginning with comparison by matching, partitioning 0 – 5, and writing number sentences for story problems involving addition and both take away and comparison subtraction with these facts, they were ready to go on to the study of ten.

From the "reservoir" to the "water consumer".

The processes just outlined form the principal theme of AMI, which they call the "water supply" or "springs," from which the "reservoir" of useful computational ideas is to be filled. (A second theme AMI emphasizes is "enjoyable lessons.") The partitioning and recombination "reservoir" focussed on 5 seems to be sufficient to generate a variety of algorithms for adding and subtracting multi-digit

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0+0 0+1 0+2 0+3 0+4 0+5 0+6 0+7 0+8 0+9 0+10 0+11 0+12 0+13 0+14 0+15 0+16
1+0 1+1 1+2 1+3 1+4 1+5 1+6 1+7 1+8 1+9 1+10 1+11 1+12 1+13 1+14 1+15 1+16
2+0 2+1 2+2 2+3 2+4 2+5 2+6 2+7 2+8 2+9 2+10 2+11 2+12 2+13 2+14 2+15 2+16
3+0 3+1 3+2 3+3 3+4 3+5 3+6 3+7 3+8 3+9 3+10 3+11 3+12 3+13 3+14 3+15 3+16
4+0 4+1 4+2 4+3 4+4 4+5 4+6 4+7 4+8 4+9 4+10 4+11 4+12 4+13 4+14 4+15
5+0 5+1 5+2 5+3 5+4 5+5 5+6 5+7 5+8 5+9 5+10 5+11 5+12 5+13 5+14
6+0 6+1 6+2 6+3 6+4 6+5 6+6 6+7 6+8 6+9 6+10 6+11 6+12 6+13
7+0 7+1 7+2 7+3 7+4 7+5 7+6 7+7 7+8 7+9 7+10 7+11 7+12
8+0 8+1 8+2 8+3 8+4 8+5 8+6 8+7 8+8 8+9 8+10 8+11
9+0 9+1 9+2 9+3 9+4 9+5 9+6 9+7 9+8 9+9 9+10
10 + (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)
11 + (0, 1, 2, 3, 4, 5, 6, 7, 8)
12 + (0, 1, 2, 3, 4, 5, 6, 7)
13 + (0, 1, 2, 3, 4, 5, 6)
14 + (0, 1, 2, 3, 4, 5)
15 + (0, 1, 2, 3, 4)
16 + (0, 1, 2, 3)
17 + (0, 1, 2)
18 + (0, 1)
19 + 0

0-0
1-1 1-0
2-2 2-1 2-0
3-3 3-2 3-1 3-0
4-4 4-3 4-2 4-1 4-0
5-5 5-4 5-3 5-2 5-1 5-0
6-6 6-5 6-4 6-3 6-2 6-1 6-0
7-7 7-6 7-5 7-4 7-3 7-2 7-1 7-0
8-8 8-7 8-6 8-5 8-4 8-3 8-2 8-1 8-0
9-9 9-8 9-7 9-6 9-5 9-4 9-3 9-2 9-1 9-0
10 - (10,  9, 8, 7, 6, 5, 4, 3, 2, 1, 0)
11 - (11, 10,                        1, 0)
12 - (12, 11, 10,                     2, 1, 0)
13 - (13, 12, 11, 10,                  3, 2, 1, 0)
14 - (14, 13, 12, 11, 10,               4, 3, 2, 1, 0)
15 - (15, 14, 13, 12, 11, 10,            5, 4, 3, 2, 1, 0)
16 - (16, 15, 14, 13, 12, 11, 10          6, 5, 4, 3, 2, 1, 0)
17 - (17, 16, 15, 14, 13, 12, 11, 10       7, 6, 5, 4, 3, 2, 1, 0)
18 - (18, 17, 16, 15, 14, 13, 12, 11, 10     8, 7, 6, 5, 4, 3, 2, 1, 0)
19 - (19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0)
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Table 1

One – and Two-step Sums to 19 and Some Corresponding Differences.

numbers, treating calculation polynomially Using the 5-focussed partitioning with multiplication and division doesn't seem necessary after place value has been mastered, but in the Lincoln Consolidated Schools in Michigan, partitioning with 5 for multiplication and division was developed with great success in a second grade classroom with children who had not been through this approach in first grade [Taylor, 1983]

The next step in the Kitamaeno curriculum is to understand ten as a unit. What does "one ten" mean?

When Watanabe sensei introduced ten to his first grade class he made a rare exception to the priority teachers at that school generally gave to letting children decide on the math answers We noticed that some of his students made the same mistake a great many American children make, when asked how many tens they had just made Instead of "one ten," several of them said, "ten tens." Apparently what happens is that "ten" does not immediately suggest a unit. "Ten" is an adjective in its common uses As a noun, it is more likely to suggest a unique position in a sequence or a sound, but such a position does not lend itself easily to being enumerated. The idea that separate parts can combine to make a unit goes fine visually, but when the number one has to be applied to the resulting unit, many children balk. However, something was going on in their heads that made it possible for them within a few months to accomplish what takes a few years for most American children – the comprehension of the place value system. Clearly, the second graders at Kitamaeno School were having very little difficulty understanding borrowing and carrying. A test in the twelfth week on mastery of these operations showed very few errors, and class discussions were rich and thoughtful.

When ten and the remaining numbers up to 19 have been studied with tiles, and story problems, and addition and subtraction number sentences have been written with numbers up to 19, then the doorway is opened to still larger numbers. The steps must be taken carefully, however, and partitioning and regrouping numbers requires full attention. At first, the children were building the teen numbers by adding to ten the numbers 0 – 9. As they went on, adding to 11, the numbers 0 – 8, etc., there was further practice in regrouping into a five plus so-many-ones, e.g., $11 + 4 = 10 + 5$, $12 + 4 = 10 + 5 + 1$, $12 + 6 = 10 + 5 + 3$, etc. Then came subtracting without breaking up the ten. That extended the sums and differences that could be done with five and ones, as partially suggested in Table 1 which shows sums to 19, but using the second addends only up to 16 for lack of space. Some of the two-digit sums and differences are only suggested by parenthetical lists in order to include as many as possible.

The sums and differences blocked off in Table 1 are those involving five or the breaking up of a five. The others can be directly read off from tile patterns in one step. The differences that are still missing in the Table 1 subtractions array require a new three-step process, e.g., subtracting from the ten used in making the first number and recombining the result with the ones that weren't used. For example, $11 - 9 = 10 - 9 + 1$. This anticipates a general three-step process of borrowing when subtracting, e.g., $31 - 9 = ((10 - 9) + 1) + 20 = 22$. This method, subtracting from the borrowed ten, has the advantage that one never has to subtract from a number larger than 10, thus considerably reducing the chances for error.

By the time they are doing these kinds of problems in the beginning of second grade, children are writing explanations in their notebooks. By third grade, they are developing alternative methods of their own based on the variations possible in the order of operations. Moving into adding and subtracting hundreds occurs early in second grade, and fifty and hundred tiles are used by some teachers in developing these algorithms. Others used colored magnetic markers on the chalkboard, with a different color for each column.

Addition, multiplication and division algorithms are developed with similar care, and by fifth grade, children are developing the rules themselves for multiplication of two decimal numbers or division by a decimal number.

Some American teachers we are working with are taking all grade 2, 3, and 4 children back through the "water supply system" with tiles in hopes of straightening out their conceptual difficulties in algorithms. Others are using the focus on 5 without using the tiles themselves. One or two are using groups in much the same way as the Kitamaeno School teachers, and managing whole-class discussions of mathematical topics into a dialogue carried on directly between students rather than having everyone speak to the teacher who then filters and amplifies what is worth repeating to the class. Our hope is that the vast majority of children in primary grades in American schools can, through adaptation of such methods, acquire an enthusiasm for, and confidence in, discussion of their own mathematical ideas in problem solving and concept development. This seems to require primarily a change in attitude rather than increasing knowledge or skill on the part of the teachers, for many teachers are skilled in leading such discussions in social studies and are tolerant of explorations of alternatives by creative, non-targeted children [Grieb and Easley, in press]. Most would not normally think of allowing such discussions in math, perhaps fearing where they might lead and feeling dubious about what they might accomplish. Teachers who have gone through this transition are excited about the responses of children to the challenge of figuring out arithmetic for themselves.

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