Communications

Linguistic and Mathematical Competence

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> *The Study of Algebra may be pursued in three very different schools, the Practical, the Philological, or the Theoretical, according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; according as ease of operation, or symmetry of expression, or clearness of thought, (the **agere**, the **fari**, or the **sapere**,) is eminently prized and sought for.*
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> *William Rowan Hamilton, 1837*

In 1932, A.F. Bentley published a erudite treatise in mathematical logic entitled *Linguistic Analysis of Mathematics*. This book was an attempt to achieve consistency in the foundations of mathematics by formalizing the linguistic "environment" within which mathematics is embedded. Unfortunately, Bentley's project was proved impossible by Gödel's incompleteness theorem which had reached press in the preceding year.

Not all of the attempts to meld linguistic and mathematical methods have fared so badly. In the field of computational linguistics, for example, mathematical theory is routinely applied to the theory of grammatical construction. (See for example, Eilenberg [1974]).

Within mathematics education, some investigators have toyed with the notion that arithmetic or algebraic "sentences" share an implicit linguistic structure with natural language sentences. (See Aiken [1972] p. 25-28 for a list of references). The perceived lack of utility of such approaches has lead Austin and Howson [1979] to comment:

> It has frequently been pointed out that mathematics itself is a formalized language and it has been suggested that it should be taught as such... Such statements possess a degree of validity, but would appear to be somewhat dangerous and potentially confusing. Mathematics is not a language — a means of communication — but an activity and a treasure house of knowledge acquired over many centuries. (p. 176)

More succinctly David Wheeler [1983] p. 86) states: "I shall keep well away from the region signposted *Mathematics is a Language*. I believe it to be uninhabited."

Recently, investigators have considered directly applying the methods of linguistics to the subject matter of mathematics. Maralyn Matz [1980] provided an outline of a computational (information processing) theory of algebraic performance using the linguist's tools:

> This article employs two classic methods for probing the content and mechanisms underlying a competence: analyzing the errors people make in its use and studying the acquisition of that competence. With evidence from these two sources we can begin to piece together a theory of algebraic competence in much the same manner as linguistic researchers who have formulated a theory of linguistic competence using grammaticality judgments and considerations of children's acquisition of language. (p. 95)

One reason for adopting such an approach is articulated by Nesher [1981] p. 27.

> My claim is that articulating the general problem of the acquisition of mathematics as an acquisition of a language system will enable us to compare it with the acquisition of a natural language and other human skills which seem to be examples of a successful learning, and we might gain deep insights by drawing these fruitful analogies.

The rigorous application of generative linguistic methods to the analysis of algebraic skill has been taken up by the author in his dissertation (unpublished). The technical basis for the application of generative linguistic methods to mathematical systems is found in the linguist's characterization of language.

> From now on I will consider a language to be a set (finite or infinite) of sentences, each finite in length and constructed out of a finite set of elements. All natural languages in their spoken or written form are languages in this sense, since each natural language has a finite number of phonemes.... Similarly, the set of "sentences" of some formalized system of mathematics can be considered a language. (Chomsky [1957] p. 2)

A major advantage of a formal generative linguistic paradigm is that preconceptions of mathematical competence must be rigorously operationalized in order to claim a place in the theory. The formalism of these methods may suggest new perspectives to previously unrecognized problems. Within a "grammar" of elementary algebra symbol manipulation, the author has called into question several traditional assumptions about the role of the curriculum in skill development, by arguing that the rules which the fluent algebraist manifests are unlike those which he/she has been taught. This raises the obvious question as to the source of the rules which are so manifest. This article explores the possibility that the rules manifest in algebraic competence are derived from, or related to, natural language competence. Two examples from the author's research are analysed below.

Unexplained behavior

The first example comes from the "syntax" of elementary algebra. Consider the expression $1 + 3x^2$. The correct syntactic structure is $1 + [3(x^2)]$ with the "x" squared, that result tripled, and the product added to 1. This interpretation must be made in preference to other possibilities such as $(1 + 3)(x^2)$, $[1 + (3x)]^2$, etc.

In a recent study, subjects were asked to evaluate such expression presented in one of two capital letter nonce notations. Half of the participants (randomly selected) were presented with 1A3MxE2 and the others were given 1 A 3 M xE2. Each group was told that the value of x was 2, and instructed that the capital letters were abbreviations for the standard operations of algebra ("A" for addition, "M" for multiplication, "E" for exponentiation, etc.).

It proved to be significantly easier for subjects to transfer their ordinary notation competence to the spaced notation than to the notation without spaces. But the spaced notation is more similar to standard notation than is the unspaced notation. This provides some evidence that the knowledge of syntactic rules, for some students at least, is tied up in the configuration of symbols rather than in knowledge of operation hierarchies. Since instruction in the schools is exclusively oriented towards propositional statements about operations, expressions, etc., we can rightly wonder how students, on their own, have come to represent syntactic rules in terms of the spacing of symbols.

The second example concerns the properties of the real numbers. Schwartzman [1977] introduced the notion of a "generalized distributive law." He noted that several real number properties seem to incorporate an aspect of "distribution." (Exponent over multiplication, radical over division, etc.).

$a(b + c) = ab + ac$ $\qquad$ $(ab)^n = a^n + b^n$

$a(b - c) = ab - ac$ $\qquad$ $(a/b)^n = a^n/b^n$

$(b + c)/a = b/a + c/a$ $\qquad$ $\sqrt[n]{(ab)} = \sqrt[n]{a}\cdot\sqrt[n]{b}$

$(b - c)/a = b/a - c/a$ $\qquad$ $\sqrt[n]{(a/b)} = \sqrt[n]{a}/\sqrt[n]{b}$

A survey of the literature shows that each of the following error types is common for beginning algebraists:

$(x + y)^n = x^n + y^n$ $\qquad$ $\sqrt[n]{(x + y)} = \sqrt[n]{x} + \sqrt[n]{y}$

$(x - y)^n = x^n - y^n$ $\qquad$ $\sqrt[n]{(x - y)} = \sqrt[n]{x} - \sqrt[n]{y}$

$a(bc) = ab\cdot ac$ $\qquad$ $a^{m+n} = a^m + a^n$ $\qquad$ $a^{mn} = a^m a^n$

(See Budden [1972] p. 8; Davis & McKnight [1979] p. 37 and p. 98; Laursen [1978] p. 194; Matz [1980] pp. 98-99; Schwartzman [1977] p. 595; and Smith [1981] p. 310) These all seem to be examples of overgeneralizing distributivity. We might therefore conclude that successful students tend to go through a phase of overgeneralizing before achieving fluency in manipulative skill. This suggests that the correct instances of distributivity (above) are associated to one another in the mind of the competent algebraist. Since this association is not reinforced in typical instruction, we may ask what is the source of students' apparently spontaneous tendency to perceive relationships of distributivity.

Linguistic explanations

For both of these problems, it is possible to find somewhat analogous structures in natural language processing. In the first example, the spacing of the symbols proved to be a significant determinant of syntactic interpretation. But this is similar to the syntax of natural language. Consider, for example, the phonetic string /light house keeper/. The ambiguity of this construction can be resolved by temporal spacing of the lexical entries: /light house keeper/ versus /light house keeper/. Thus we have an example in which the rules of spacing which are manifest in the way in which algebraic rules are encoded have a parallel construction in natural language.

The predisposition to "distribute" discussed in the second example, may also have its roots within natural language competence. The ambiguity of the sentence "The old man and woman walked down the stairs." ("[old man] and [old woman]" or "[old man] and woman") shows that English speakers must constantly make decision about "distributivity" in normal discourse. Clark [1974] has noted the tendency to resolve this ambiguity in favour of distributing:

> Another very clear finding [in linguistic development] is that people have a preference for consistency. If an operation should be performed on one element of a statement, but not on the other, the person, be he child or adult, is likely to perform it on both. For example "Not A and B" will not be interpreted as the absence of A together with the presence of B, but will be interpreted as "Neither A nor B". Similarly, children expect larger objects to be heavier and stronger. In addition, the conjunctions "but", "although" and "unless" which draw attention to exceptions to a general rule are particularly difficult to comprehend. (p. 78)

These examples are only suggestive. They do not prove that mathematical competence and natural language competence are related by common or parallel structures. This theory does, however, offer a possible explanation for a fact which is becoming increasingly clear; that the mathematical competence which is developed through exposure to algebraic symbols is not as closely related to instruction in the schools as mathematics educators might like to imagine.

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Four Operations? or Ten?

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This contribution aims to relate two recurring themes in the literature. One concerns the way so many children start school quite experienced in mathematical thinking (including problem solving), and with quite a valuable outfit of mathematical ideas. This theme laments the way school fails to build on these foundations, and substitutes instead a "subject" which many children find meaningless, so that they can no longer use their thinking powers and instead take refuge in rote-learning.

The other concerns the extraordinary difficulties children seem to encounter in coming to terms with the ideas of ratio and proportion. The second of these themes is the subject of high-powered research, but the first is an area ripe for development. Current research [e.g. Carpenter et al, 1982] seems to be on quite a different track.

The operations of everyday life with everyday things

Beth Blackall gave a very stimulating talk at the 1975 annual conference of the Mathematical Association of Victoria. She described how in her junior primary classroom she dealt with nine operations: three forms of subtraction, two of division, one each of addition and multiplication, and also two others, namely doubling and halving. This struck a chord with me as I had recently contributed an article about seven of them to *Mathematics in School* 4, 5.

Clearly doubling and halving are valuable, even essential, additions to my seven. They make use of the very special number *two* and provide a way forward for mastering numbers of all kinds, both large and small. As an example now well known to most adults the "rule of 72" gives you a quick guide to how long it takes for prices to *double* (or for the value of money to *halve*) due to inflation. At the beginners' level, facility with doubling and halving, together with multiplication by ten, reduces the number of "table facts" that need attention to a mere ten: and if the square numbers are known as friends this number is further reduced to five, namely those with products 21, 27, 42, 54 and 63.

The surprising thing is the way so many people, including teachers, are quite unaware that subtraction comes in several forms in real life. Over the years I have introduced teachers and future teachers to some of these forms in various ways. The most successful way seems to be to treat the various manifestations as *different* parts of the furniture of the world and them, only then, after plenty of discussion and practical work, to point out their connection with subtraction (which everyone knows is exactly the same as "take away"!)

Similarly with the two forms of division. For some reason adults find it hard to distinguish between them, or to recognise them in real life, even before their connection with division is pointed out. And as in the case of subtraction it does seem necessary to point it out. Accommodation to this new way of thinking about the operations does not come easily. When I, or others, have used a straightforward analytical approach it seems to lack the impact needed to change well-established understandings about the nature of these operations.

Another everyday operation with everyday things

Since those days it has become apparent that the number of everyday operations is ten, not nine. The extra one (see figure) is multiplicative comparison. On the figure this has been shown with the form of words *(times) as much as*. The "times" is in brackets because it is only used when the number concerned is greater than one. Thus "petrol costs of the nature of mathematics on the other, the professional as an individual as opposed to the professional as a member of an institution, must prevail. One way the teacher may develop sufficient professional autonomy to bring this about is by engaging in curriculum development with a curriculum developer on equal terms as suggested earlier. It may be that by coming together with other professionals and mutually sharing reflections on their work in an in-service situation, they may also gain clearer sight of the action to be taken.

With these thoughts in mind, it would seem that what is needed in mathematics education to add to the quality of the professional life of the mathematics teacher perhaps is not a new rhetoric of reform and change, but rather a clearly identified theoretical rationale for reform and change (which echoes Bauersfeld's [1980] plea for the need for a more theoretical orientation in mathematics education). It was suggested earlier that in attempting to bring about reform and innovation in school, a rhetoric of reform and change is adopted and, in the process, the rules and meanings that underlie institutional life are somehow filtered out leaving the impression that a consensus prevails. With a clearly articulated and well thought out rationale (relating to their perceptions of their subject, amongst other considerations), teachers may become more aware of what happens at this level and of how their intentions in bringing about change may be obstructed. The social and institutional context of schools does create powerful images that may "dull our senses" but the identification of such a rationale may, in turn, lead to the identification of strategies for the increased professionalism of teachers so that senses could become heightened rather than dulled.

What has emerged from our considerations here is a common basis that has to do with the notion of the social context in which mathematics education takes place, at three different levels. Firstly, examination of the part of the researcher studying classroom interaction has identified the importance of the different contexts produced by the adoption of different metaphors in how we approach our task as mathematics educators. Secondly, we have been concerned with the teacher and curriculum developer working side by side in the classroom to produce one kind of context. Finally, our attention has been drawn to the context of the institutional framework. What must not be neglected is the recognition that all three levels are contained in a particular cultural context which is the reality of the everyday life of our pupils. There has been some mention made not only of a degree of alienation of teachers from their profession but also of pupils from school and mathematics. The former might not occur to the degree that it does if we were to pay more attention to the latter. Mellin-Olsen [1984], in writing of pupils' perceptions of mathematics schooling, suggests that "In the case of cultures foreign to standard school cultures, it is vital to learn about not only the imagery and spatial systems of its members, but also of the kinds of contexts within which they are developed" (p. 157). This suggests reaching out to understand the reality of the pupil and, as teachers, creating contexts for the teaching and learning of mathematics that relate to that reality. It is possible, then, that the more humanistic approach advocated earlier with respect to research could pervade mathematics education more generally and lead to a concern for the social contexts created at a variety of levels. This could help to alleviate alienation by making our subject more accessible and meaningful to our pupils as well as to ourselves. What could better enhance our professional lives as teachers of mathematics?

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