grammatical form offers no hint. If one talks about the 'quotient' and the 'remainder' in the situation of division with remainder, the meaning is specified by the context. If only the quotient is mentioned in the discussion, the meaning is ambiguous. However, as the above examples show, students often do not attend to the explicit specifications suggested by the context or by the grammatical form

### Extending the meaning

In a mathematical context, we are used to the idea of 'extending the meaning' for certain words. Multiplication, for example, is originally introduced as 'repeated addition', where the word 'times' (2 times 7) has identical meaning to the words 'multiplied by' (2 multiplied by 7). Later, this meaning of multiplication is extended beyond the whole numbers, to rational, real or complex numbers, to matrices and functions, where we still talk about multiplication but do not mean 'times' any more. This extended meaning can be seen as a metaphorical usage of the word 'multiplication'

Other examples of broadening the meaning or metaphorical usage are in extending exponentiation to negative or fractional exponents or extending 'sine' from the ratio in a right-angle triangle to a function. These metaphors appear naturally as we build our mathematical knowledge and students are often not aware of them

There is an embedded 'luxury' in these extensions: the operations resulting from the extended meaning are consistent with the original meaning. For example, we all think of the 'sum' as the result of the addition operation, without explicitly specifying whether the addends are integers, real numbers or matrices. However, the original definition of the sum (of $a$ and $b$) emerged as the number of elements in the union set of two disjoint sets (where $a$ and $b$ refer to the number of elements in the two disjoint sets respectively). Extending the meaning of 'sum' creates no conflict: 17 is the sum of 12 and 5 regardless of whether the addends are viewed as whole, integer or real

The case of divison is different: extending the meaning of division from whole to rational numbers does not provide the 'luxury' of consistency. I suggest that one solution to this difficulty is through becoming aware ourselves and raising the awareness of our students to this potential discrepancy

### Conclusion

This communication focused on the lexical ambiguity of two words – 'divisor' and 'quotient' – that frequently appear in elementary mathematics classrooms. Excerpts were presented from class discussions and interviews with individual students that outlined the confusion about the meanings of these terms. I suggested that the regular 'tools' to determine meaning, such as context or grammatical form, are not always sufficient.

Durkin and Shire (1992, p. 82) list several ideas for practical implications in the situations of polysemy. They suggest monitoring lexical ambiguity, enriching contextual clues, exploiting ambiguity to advantage and confronting ambiguity. Even though their suggestions refer to the ambiguity across registers, I find them completely applicable in case of divisors and quotients, where the ambiguity is within the mathematics register and even within the elementary classroom sub-register

Most textbooks for pre-service elementary school teachers provide the whole-number-division definition for quotients and remainders and ignore a potential conflict with students' prior knowledge. Such mathematical precision is not in the best interest from a pedagogical perspective. An appropriate didactical activity could awaken the conflict and then resolve it.

### References


Campbell, S. (1998) *Preservice Teachers' Understanding of Elementary Number Theory: Qualitative Constructivist Research Situated within a Kantian Framework for Understanding Educational Inquiry*, Unpublished Ph D Dissertation, Burnaby, BC, Simon Fraser University

Clapham, C. (1996, second edn) *The Concise Oxford Dictionary of Mathematics*, New York, NY, Oxford University Press.

Durkin, K and Shire, B. (1991) 'Lexical ambiguity in mathematical contexts', in Durkin, K and Shire, B (eds), *Language in Mathematical Education: Research and Practice*, Milton Keynes, Bucks, Open University Press, pp. 71-84

Gerber, H (1982) *Mathematics for Elementary School Teachers*, Toronto, ON, Saunders College Publishing

James, G and James, R C. (1968, third edn) *Mathematics Dictionary*, Princeton, NJ, Van Nostrand and Company.

Pimm, D. (1987) *Speaking Mathematically: Communication in Mathematics Classroom*, London, Routledge and Kegan Paul

Random House (1992) *Webster's College Dictionary*, New York, NY, Random House


## Univalence: A Critical or Non-Critical Characteristic of Functions?


RUHAMA EVEN, MAXIM BRUCKHEIMER


There is a long history of attempts to use the function concept as an organising theme for the secondary mathematics curriculum. The appeal of the concept as a central idea is fairly clear. It can serve to unify seemingly different and unrelated topics; it can give meaning to algebraic procedures and notation which, in many other cases, are restricted to acquiring manipulative facility without leading to understanding; it easily lends itself to several different representations (an aspect much appreciated nowadays by the mathematics education community); and today, with the availability of advanced technological tools, its visual aspects are within everyone's reach.

With the current intense focus, both in curriculum research and the development of the function concept, it seems to us that there is one aspect that deserves more careful attention than it receives: the *univalence* requirement, that corresponding to each element in the domain there be only one element (image) in the range. We propose a re-examination of the present place of univalence in school mathematics and its implications, as there may be didactical advantages to postponing its introduction. The following is intended as a contribution to this re-examination

## From multi-valued functions to single-valued functions

Functions are not 'what they used to be'. In the 'good old days', functions were multi-valued But the modern definition explicitly requires univalence. Interestingly, a search of quite a few books and articles on the history of mathematics did not result in an explicit statement of the reason for requiring functions to be univalent. Exceptional in this regard is Freudenthal (1983) who attributes this requirement to the desire by mathematicians to keep things manageable. Keeping track of meanings of multi-valued symbols (such as $\sqrt{\ }$), and taking care that they have the same meaning throughout some context requires considerable care. Therefore, according to Freudenthal, the univalence requirement was added to the definition of function. An approach which conforms to Freudenthal's description can be found in some older textbooks, where functions are allowed to be multi-valued until there is cause to restrict the discussion to single-valued functions (e g the appearance of limits)

## Current approaches to univalence in school mathematics

Today, when students first encounter the concept of function formally, univalence is usually presented to them as an important characteristic, both by textbooks and teachers, presumably many of the latter under the influence of the former. Students then usually spend some time learning to distinguish between functions and non-functions. In spite of this widespread emphasis, we did not find a single textbook that explains why univalence is important Students are not given examples or activities that would help them understand what one can do with functions that one cannot do with relations that are not functions. However, textbooks are only one source from which students form their conceptions of mathematical concepts Teachers are another significant source. How do teachers present univalence to students?

Numerous informal discussions with teachers indicate that many of them think that the univalence characteristic of functions is very important for their students' understanding of the concept. This is further illustrated by a study conducted in the U S (Even, 1993) in which many prospective secondary mathematics teachers claimed that it was very important that their students should be able to distinguish functions from non-functions For instance:

> it's something that's important and you're trying to make a distinction between that and relations. It is almost vital that somebody understand why this is a function and this is not Another thing might be a relation but it's not a function

Moreover, when asked to give a definition of a function and then presented with a situation where a student does not understand that definition and being asked to give an alternative that might help the student understand, many of these pre-service teachers focused on the univalence property only. Thus, they used the 'vertical line test' (a visual representation of univalence on the graph of the function/non-function) as further explanation, ignoring the correspondence aspect of the function concept entirely For example:

> By graphing the function and doing the vertical line test, a line never crosses the graph more than once.

A possible reason why both books and teachers do not (cannot) explain the need for univalence is that many teachers, teachers of teachers, as well as textbook authors, do not explicitly know (or have never thought about) the need for this requirement. We asked several groups of Israeli and American pre-service secondary teachers to explain why, in their opinion, it is necessary to have the univalence requirement in the definition What is its importance, its role? Many of them thought that univalence is important but admitted that they did not know why. One said, for example:

> I don't know why. I don't know why there should be one. It's the way I always learned though

Distinguishing between functions and non-functions is presented as and remains a meaningless ritual

Not knowing why univalence is needed may influence pedagogical content-specific choices, by making it 'reasonable' to present students with easy procedures that over-emphasize procedural knowledge without sufficient concern for meaning This is exactly what many teachers do when they choose to provide students with the 'vertical line test' as a rule to follow and to get the right answers. For example, when asked why she wanted to teach the 'vertical line test' to her students, one prospective teacher said:

> If they're told to figure out whether it's a function or not, using the definition, they probably wouldn't be able to do it. If they know the vertical line test works, even if they don't know why it works, they can see right away why this is a function, because they can go through with a ruler or a straight edge and vertically go across the function, looking for places where there are two points.

We cannot escape the conclusion that univalence, so highly valued at this early stage, lacks rhyme or reason.

## From single-valued functions to multi-valued functions

As we illustrated above, students learn that univalence distinguishes between relations or correspondences that are not functions and those which are. But, students are not told why this distinction is important, and as univalence is seldom mentioned after functions are formally introduced, for many students it does not become part of their function concept image. Therefore, it is not surprising that when presented with four graphs and asked to choose the graph that did *not* represent a function, only half of the Algebra 2 students who participated in the fourth mathematics assessment of the NAEP chose the graph of a circle (Swafford and Brown, 1989). Dominant in students' concept images are the functions they meet during their studies – functions that can be represented by formulae, with 'nice' graphs, and that are 'known to be functions'. From the student point of view, the statement that a 'nice' graph such as a circle is not a function, whereas its upper half (or even a weird-looking graph such as the first in Figure 1) is, may not make much sense. Nor will be being informed that the second represents a function while the third does not, as a parabolic graph may or may not represent a function, depending on 'one's point of view'.

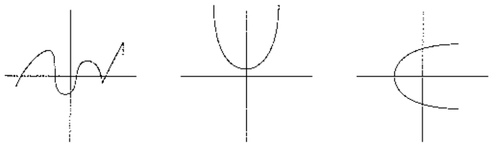

*Figure 1* Graphs of functions and non-functions

Let us briefly consider why we need functions to be single-valued in the school curriculum. A common explanation for this (the circle, parabola and other such 'nice' graphs notwithstanding) is that most correspondences met, either in mathematics or in everyday and professional life, are univalent. But is it really the case, or is it that univalent correspondences are more noticeable because we choose to deal with them and not with multi-valued ones, in order to avoid complications? The use of contemporary technological tools enables and encourages students to explore instances that their teachers or curriculum developers may not have anticipated.

For example, a colleague of ours, Nurit Hadas, has called our attention to an interesting investigation:

> Given a triangle ABC where AB and AC are 4 and 5 units in length (respectively), investigate the behavior of the area of the triangle as the altitude AD (D the intersection of the altitude with the side BC) changes.

A useful tool for such an investigation is a piece of software which allows students to explore interactively how geometrical figures behave when they are 'dragged', and also allows figures to be linked to tables and graphs that simultaneously update and display information. Thus, here, the student can display tabular and graphic representations of the connection between the length of AD and the corresponding area of the triangle. Quite surprisingly, the investigation reveals that for 'many' altitudes there are *two* corresponding triangles whose areas are *not the same* (the same conclusion can, of course, be reached algebraically). In other words, the 'function' that assigns to each altitude the area of its triangle is bi-valued. This characteristic is displayed clearly in the resulting graph (see Figure 2).

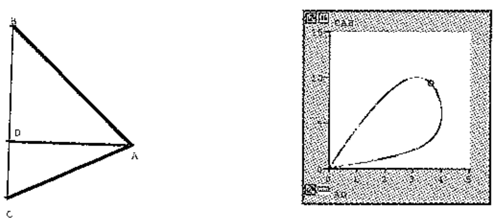


*Figure 2* The 'function' that assigns to the length of the altitude of this triangle, the corresponding area of the triangle

Should we then dismiss this 'function', just because it is not included in our current definition? The situation is mathematically rich; it connects investigations in geometry with functional representations and surprising results can be obtained using common functional tools. There is not a genuine mathematical difference (in the presentation of the problem, in its solution processes or in the presentation of findings) between problem situations like this (and many others that might occur when students are given powerful tools and allowed to investigate) and similar problems that happen to result in single-valued functions.

The example above might suggest that we propose to go back in history to the times when functions were multi-valued. Well, not quite. But we do suggest that there may be didactical advantages to following to some extent the historical development of functions, to adopt the older textbooks' approach and postpone the introduction of univalence of functions until it is beneficial and 'reasonable', which may be when analysis of functions is reached (by a minority of the students who are today introduced to functions). Until then, we suggest that students explore problem situations in which multi-valued as well as single-valued functions may arise and that no emphasis be put on the univalence requirement. We do not wish to imply that we advocate the idea that instruction should necessarily (or even often) follow the historical development. Rather, in this particular instance, we think that postponing the discussion of univalence until it is needed may contribute to student learning in several ways.

First, our suggestion may be a small contribution to making mathematics more meaningful for students and less an arbitrary collection of rules and definitions. Mathematical definitions are seen to be human-made for convenience and are changeable according to need. Second, a number of studies indicate that students have difficulties in understanding and using the function concept. Consequently, it seems more reasonable to focus student attention on essential aspects of functions, such as how they describe relationships between variables, their power to mathematise situations, how representing a function in different ways can expand understanding about their behaviour and how approaching them in different ways (e.g. object vs process, point-wise vs. globally) may be useful in different situations. Third, it may assist in learning about sub-concepts. For example, when learning about inverse functions one can focus on the essence of the concept - namely, undoing - and not on the unnecessary and often complicated examination whether the inverse function even exists, as many textbooks tend to do.

Some may claim that if our suggestion is followed, students will think that functions are multi-valued, and that, at a later stage, the 'harm' we caused will have to be corrected. To those we would reply that since so many students today already consider a circle or an ellipse, for example, to be functions, even though their teachers and textbooks emphasize that functions must be single-valued, we do not think that our suggestion would make the current situation 'worse'. It might just be the other way round. If the requirement for univalence is introduced when the situation requires, i.e. with a sound reason, then it may make it easier for the students to accept and remember. More than that, we do not think that the situation where students conceive functions as multi-valued is a situation which needs 'to be corrected'. Rather, it should be refined. After all, when the text or teacher so chooses, all functions thereafter are continuous, or differentiable, etc. - because there is a good reason for the added restriction.

## References

Even, R. (1993) 'Subject-matter knowledge and pedagogical content knowledge: prospective secondary teachers and the function concept', Journal for Research in Mathematics Education 24(2), 94-116.

Freudenthal, H. (1983) Didactical Phenomenology of Mathematical Structures, Dordrecht, D. Reidel.

Swafford, J. O. and Brown C. A. (1989) 'Variables and relations', in Lindquist, M. M. (ed.), Results from the Fourth Mathematics Assessment of the NAEP, Reston, VA, NCTM.