# REMEMBERING STANLEY ERLWANGER

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On June 18th, 2003 Stanley Erlwanger died. His pioneering case studies of children's mathematical thinking, from the 1970s, still provoke fresh insight. To honor Erlwanger's memory, and to draw new attention to his work, it is interesting to revisit Benny (Erlwanger, 1973; 1975), the twelve-year-old student whose own work and thinking helped to anchor a powerful critique, by researchers, of prevailing practice thirty years ago. We consider Benny here, partly because his case seems relevant to current debates, but also because certain details of his case have helped us to address some cognitive and social issues we consider in our present work (see Speiser, Walter and Lewis, 2004). Here is how Benny converts fractions to decimals:

Erlwanger: How would you write 2/10 as a decimal or decimal fraction?

Benny: One point two. [writes 1.2]

E: And 5/10?

B: 1.5. (Erlwanger, 1973, p. 8; 1975, p. 201)

Benny implements a rote procedure. He first adds the numerator and the denominator, and then places the decimal point according to a simple rule. When questioned, Benny offers detailed explanations, interpreting explicitly the symbols he manipulates. Further, Benny makes clear that he, too, works against a background of received belief. In the following exchange, Benny suggests what might have happened, for example, if he had written 2/4, or "2 over 4":

B: Then I get it wrong because they [aide and teacher] expect me to put 1/2. Or that's one way. There's another way; 2/4 to me is also 1/4 and 1/4. But if I did that also, I get it wrong. But all of them are right!

E: Why don't you tell them?

B: Because they have to go by the key... what the key says. (Erlwanger, 1973, p. 14; 1975, pp. 213-214)

Despite success in his school's mathematics program, Benny found school mathematics highly problematic. For him, school mathematics was nothing more than arbitrary rules and processes. Such rules and processes, taken individually, might (or might not) make any sense, or even, taken together, be consistent. Hence, in Benny's mathematics, no sense ultimately needed to be made. Benny did, however, try to make sense of the kind of knowledge he was building and perhaps especially, the social practice that he had to navigate. Here is another conversation illustrating how Benny saw the mathematics he had built and used:

E: What about the rules. Do they change or remain the same?

B: Remain the same.

E: Do you think a rule can change as you go from one level to another?

B: Could, but it doesn't. Really, if you change the rule in fractions it would come out different.

E: Would that be wrong?

B: Yes. It would be wrong to make our own rules; but it would be right. It would not be right to others because, if they are not used to it and try to figure out what we meant by the rule, it wouldn't work out. (Erlwanger, 1973, p. 18; 1975, p. 221)

In view of such data, it might seem especially unfortunate that Benny arrived at his particular interpretation of "official" mathematics through detailed, perhaps even intense reflection on experience.

In the mathematics program used at Benny's school (Individually Programmed Instruction (IPI) Mathematics, designed "in order to assist pupils who required remedial instruction") based on then-current research, the rules that constituted the "official" understanding often had to be inferred or guessed from worked examples (at the tops of printed worksheets) or from test results, with little interaction either with peers or with a teacher. Ironically Benny did well according to his program's norms.

Benny's case can haunt us, even now, for several good reasons. A central impact of Erlwanger's studies, when they first appeared, was the realization that for many students mathematics was not experienced even as *needing* to make sense, even though it could involve intensive thinking and decision-making. In his interviews with Erlwanger, Benny offered detailed explanations, but he did not seem concerned about how well his explanations might convince a listener. To what extent did sense or explanation matter? Through such findings, Erlwanger's papers have widely been called *disaster studies*.

The IPI curriculum that Benny experienced reflects a particularly well-entrenched behaviorist approach. Erlwanger's research helped expose a fundamental weakness of the resulting educational practice, and, by implication, of behaviorism: that it failed to engage learners as thinkers, and in particular ignored the need for learners' mathematics to make sense, both internally and as a way to work with realistic situations.

In our own work, we have emphasized a further problem raised in part by Benny. What might be the implications of Benny's tacit suggestion, in his last response above, that perhaps one *could* construct rules just as one wished – in effect concoct the whole of mathematics – provided only that one had sufficient influence? If the resulting mathematics also had to *make some kind of sense*, not just locally but also in the large, then one's freedom for invention might, as a result, be limited. Is it true, as many of us have been taught, that mathematics offers few worthwhile alternatives for how one thinks about or solves a problem?

For Benny, a wise choice in mathematics is one that helps him pass a post-test. Such choices, in practice may not be simple, given the information and experience that he and students like him have to work with. Hence, in his interviews with Erlwanger, we sometimes see Benny reflect carefully about what mathematics *does* within his social context. At one point, for example, Benny stepped beyond the question posed to him in order to describe another child's experience. Just after he explained that teachers and aides, in the official classroom process, "have to go by the key" when grading student work, Benny offered his own perspective on the social practice that surrounded him:

> *I don't care what the key says; it's what [how] you look [at] it.*That's w*hy kids nowadays have to take post-tests. That's why nowadays we kids get fractions wrong.* Gary ain't good at figuring fractions. For example, three months ago he was doing stuff like that and they kept on marking it wrong; and *he got mad and stuff. Really he tried everything*. The third time he got the post-test. *He went like this, but it was a different one*. He *went like this*, then he *went like this*, and then the third time he went like that and he mastered it. (Erlwanger, 1975, p. 214, original emphasis)

In a word, Gary *experiments*. In doing so, he implicitly *rejects* the social practice (mathematics as a fixed set of behaviors to be learned by imitation) that his classroom was expected to enact, much as Benny dismissed mathematics as a subject that could have important content. Gary's choices (and, by extension, Benny's) appear to have aimed simply at survival. As a research subject, outside the classroom, Benny could at times take on the role of open dissident, while appearing to cooperate within his classroom's social practice – and indeed, according to that classroom's standards, Benny did reasonably well. In the relative privacy of a research interview, Benny can tell Erlwanger (who cares about his thinking) that the emperor is naked, as the story goes, but Benny evidently did not find himself empowered to say so in his classroom.

If (for the moment) we regard the prevalent "official" mathematics as a coherent social discourse, then it *needs* to be rejected. Consider for a moment what rejection, for someone in Benny or Gary's situation, might entail. It might actually help such students *not* to connect the pseudo-subject they are called to replicate in classrooms to their personal experience outside the school domain, where they might well function as productive, independent thinkers. Or perhaps, stopping short of full rejection, merely viewing large parts of their school experience as provisional or hypothetical could leave such students freer later to develop critical perspectives on the social practices around them. (In a case study (Speiser *et al.*, 2004) of five pre-service elementary teachers, several students do precisely this, building important mathematics in the process.)

Erlwanger's interviews with Benny, who worked in a classroom guided by a carefully designed (and even research-based) "official" practice, suggest clearly how learners under such a practice can come to *dismiss* mathematics as a meaningful way of knowing anything, very likely to protect whatever sense they may be making on their own. From this perspective, the "official" practice seems to fail in two important ways:

- by leaving so many central questions open that the knowledge to be replicated can come deeply into question
- by forcing students to guess blindly, so that some learners may reject school mathematics as an outright sham.

In their responses to both failures, students may be acting sensibly.

*There follows a substantial extract from pages 7 to 20 of the 1973 article by Stanley Erlwanger referred to above. It is reproduced here in FLM style (i.e., lightly edited) with the permission of The Board of Trustees of the University of Illinois (Champaign-Urbana) [1]. Even though reviewers of academic articles sometimes comment on the necessity of authors using current references, this body of work illustrates the enduring importance of some older studies. (ed.)*

## Benny's conception of rules and answers in IPI mathematics [1]

This study arose from visits made to a sixth-grade class [twelve- and thirteen-year-old students] using Individually Prescribed Instruction (IPI ) Mathematics in order to assist pupils who required remedial instruction and discover the nature of their trouble. In these terms, a twelve-year-old boy named Benny did not seem a likely subject for the study. He was making much better than average progress through the IPI program, and his teacher regarded him as one of her best pupils in mathematics. In a structured program like IPI, it was expected by the teacher that Benny could not have progressed so far without an adequate understanding and mastery of previous work.

Benny was willing to talk to me, and I was eager to get started, so we began to discuss his current work. I soon discovered that Benny understood incorrectly some of the previous work. He could add fractions and multiply decimals correctly in most of the exercises, but he said that 2/1 + 1/2 was equal to 1, and 2/10 as a decimal was 1.2. Subsequent discussion and interviews with Benny led me to an understanding of his concept of decimals and fractions, and his views about rules, relationships, and answers in mathematics.

This article attempts to show that the overall goal of IPI, namely, "to develop an educational program which is maximally adaptive to the requirements of the individual" (Lindvall and Cox, 1970, p. 34) has not been a total success with Benny. Specifically, the paper shows that the disadvantages of IPI mathematics for Benny arise from its behaviorist approach to mathematics, its concept of individualization, and its mode of instruction.

We begin by examining Benny's concept of decimals and fractions.

## Conversions between decimals and fractions

Benny converted fractions into decimals by finding the sum of the numerator and denominator of the fraction and then deciding on the position of the decimal point from the number obtained:

Erlwanger: How would you write 2/10 as a decimal or decimal fraction?

Benny: One point two. [writes 1.2].

E: And 5/10

B: 1.5

Benny was able to explain his procedure; *e.g.*, for 5/10 = 1.5, he said:

> The one stands for 10; the decimal; then there's 5 … shows how many ones.

In another example, 400/400 = 8.00 because:

> The numbers are the same [number of digits] … say like 4000 over 5000. All you do is add them up; put the answer down; then put your decimal in the right place … in front of the [last] three numbers.

His explanation of the decimal point is just as strange though even more cryptic. Thus, in discussing the example, 9/10 = 1.9, he said that the decimal point:

> means it's dividing [*i.e.*, separated into two parts which] you can get [the] one nine, that [would] be 19, and [in] that 1.9, the decimal [part, *i.e.*, the 9] shows … how many tens and how many hundreds or whatever.

This method enabled Benny to convert any fraction to a decimal. Some of the answers he gave were:

> 429/100 = 5.29, 3/1000 = 1.003, 27/15 = 4.2, 1/8 = .9, 1/9 = 1.0 and 4/6 = 1.0.

Benny applied this method consistently. Moreover, he was fully aware of the fact that it will give equivalent results for many different fractions, but he did not appear to think that there was anything wrong with that:

E: And 4/11?

B: 1.5

E: Now does it matter if we change this [4/11] and say that is eleven fourths? [E writes 11/4].

B: It won't change at all; it will be the same thing … 1.5.

E: How does this work? 4/11 is the same as 11/4?

B: Ya … because there's a ten at the top. So you have to drop that 10 …take away the 10; put it down at the bottom. [Shows 11/4 become 1/14]. Then there will be a 1 and a 4. So really it will be 1/14. So you have to add these numbers up which will be 5; then 10 … so 1.5.

His two equivalent algorithms can be illustrated as follows (where a, b, and c refer to digits):

> ab/c = a.(b + c) or ab/c = b/ac = a.(b + c).

Benny employed a similar procedure for converting decimals to fractions, namely:

> .*x* = .(a + b) = a/b or b/a.

E: How would you write .5 as an ordinary fraction?

B: .5 … it will be like this … 3/2 or 2/3 or anything as long as it comes out with the answer 5, because you're adding them.

We see from these examples that for Benny a decimal is formed by fitting together symbols – two or more digits and a point – into a pattern of the form a.bc … (where again a, b, and c stand for digits). Converting a fraction to a decimal gives a unique answer, *e.g.*, 3/2 = .5; but converting a decimal, *e.g.*, .5, to a fraction leads to any answer from the set of number pairs whose sum is the required digit, for .5, the solution set is {3/2, 2/3, 1/4, 4/1 …}.

## Addition and multiplication of decimals

In operations with decimals Benny works with digits as whole numbers first. Then he decides on the placement of the decimal point from the total number of decimal places in the problem. His procedure for addition is shown below:

E: Like, what would you get if you add .3 + .4?

B: That would be … oh seven [07] … .07.

E: How do you decide where to put the point?

B: Because there's two points; at the front of the 4 and the front of the 3. So you have to have two numbers after the decimal, because … you know … two decimals. Now like if I had .44, .44 [*i.e.*, .44 + .44], I have to have four numbers after the decimal [<*i.e.*, .0088].

He employs a corresponding procedure for multiplication of decimals.

E: What about .7 × .5?

B: That would be .35.

E: And how do you decide on the point?

B: Because there's two points, one in both … in front of each number; so you have to add both of the numbers left … 1 and 1 is 2; so there has to be two numbers left for the decimal.

These methods lead to answers such as:

> 4 + 1.6 = 2.0, 7.48 – 7 = 7.41, 8 × .4 = 3.2, and .2 × .3 × .4 = .024.

In all this work Benny appears confident. He is unaware of his errors. In interviewing him at this stage, I did not attempt to teach him or to even hint as to which answers were correct. He did not ask for that either.

### Addition of fractions

Benny had already completed work on equivalent fractions and addition of fractions with common denominators for 1/2 through 1/12. He appeared to understand halves and fourths, *e.g.*, he knew that 1/2 + 1/4 = 3/4. Benny believed that there were rules for different types of fractions:

B: In fractions we have 100 different kinds of rules …

E: Would you be able to say the 100 rules?

B: Ya … maybe, but not all of them.

He was able to state addition rules for fractions clearly enough for me to judge that they depended upon the denominators of the fractions and were equivalent to the following:

> a/b + c/b = (a + c)/b, *e.g.*, 3/10 + 4/10 = 7/10;
>
> a/b + c/d = (a + c)/(b + d), *e.g.*, 4/3 + 3/4 = 1;
>
> a/b + c/c = 1 a/b, *e.g.*, 2/3 + 4/4 = 1 2/3;
>
> a/10 + b/100 = (a + b)/110, *e.g.*, 6/10 + 20/100 = 26/110.

Benny had also used fraction discs … when he showed me how he used them, he arrived at an incorrect result, as shown below:

E: Now when you simplify 3/6 what do you get?

B: It should be 1/2 because we got these fraction discs. [But then he goes on to say] When you add, 1/4 and 1/3 and 1/8 equals 1/2 [instead of 3/15, as his rule for adding fractions, above, should give].

But fractions, to Benny, are mostly just symbols of the form a/b added according to certain rules. This concept of fractions and rules leads to errors such as 2/1 + 1/2 = 3/3 = 1. Further, 2/1 + 1/2 is:

> just like saying 1/2 + 1/2 because 2/1, reverse that, 1/2. So it will come out one whole no matter which way. 1 is 1.

### Mastery and understanding in IPI

How is it that Benny, with this kind of understanding of decimals and fractions, had made so much progress in IPI mathematics? The advocates of IPI claim that its unique features are its sequentially ordered instructional objectives and its testing program. Lindvall and Cox (1970, p. 86) state:

> A basic assumption in the IPI program is that pupils can make progress in individualized learning most effectively if they proceed through sequences of objectives that are arranged in a hierarchical order so that what a student studies in any given lesson is based on prerequisite abilities that he has mastered in preceding lessons.

Another report on IPI by *Research for Better Schools, Inc* and *The Learning Research and Development Centre* (undated) states:

> Each objective should tell exactly what a pupil should be able to do to exhibit mastery of the given content and skill. This should typically be something that the average student can master […].

Furthermore:

> The validity of the content-reference tests used in IPI depends upon the correspondence of the test items and the behavioural objectives. (Lindvall and Cox, 1970, p. 24)

Glaser (1969) argues in favor of the IPI testing program:

> An effective technology of instruction relies heavily upon the effective measurement of subject matter competence at the beginning, during and at the end of the educational process. (p. 189)

IPI mathematics emphasizes continuous diagnosis and assessment through pre-tests, curriculum-embedded-tests and post-tests. Lindvall and Cox (1970) stress that:

> The tests are the basic instrument for monitoring [a pupil's] progress and diagnosing his exact needs [… and state that:] A proficiency level of 80- 85 percent has been established for all tests in the IPI program. (p. 21)

Clearly, then, "making good progress" in IPI means something other than what we had thought. Benny was in a small group of pupils who has completed more units (with a score of 80 percent or better) than any other child in the class. He worked very quickly. When he failed to get 80 percent marked right by the IPI aide, he tried to grasp the pattern of the correct answers; he then quickly changed his answers in ways that he hoped would better agree with the key.

Benny's case indicates that a "mastery of content and skill" does not imply understanding. This suggests that an emphasis on instructional objectives and assessment procedures alone may not guarantee an appropriate learning experience for some pupils.

The argument that Benny may have forgotten previous work and is merely guessing approaching new exercises does not hold. He has developed consistent methods for different operations, which he can explain and justify to is own satisfaction. He does not alter his answers or his methods under pressure.

## The role conflict of the IPI teacher

One could argue that the effectiveness of IPI depends on the role played by the teacher. Since IPI provides material for individual work and there is a teacher-aide to check pupils' work and record results, the teacher has considerable free time for assistance to individuals. Lindvall and Cox (1970) observe:

> As a result of continuing day-by-day exposure to the study habits, the interests, the learning styles, and the relevant personal qualities of individual students, the teacher gathers a wealth of information that should be employed in developing prescriptions and in determining the instructional techniques that can best be used with a particular child […] IPI requires frequent personal conferences between student and teacher […]. (p. 25)

But on the other hand, a basic goal of IPI is pupil independence, self-direction, and self-study:

> Instructional material are used by pupils largely by individual independent study [and] require a minimum of direct teacher help to pupils. (*op. cit.*, p. 49)

These are conflicting roles for teacher and pupil, and, in different cases, the conflict may be resolved differently. Benny has used IPI material since the second grade and is familiar with the system and seems to have accepted the responsibility for his own work. He works independently in the classroom, speaking to his teacher only when he wants to take a test, to obtain a new assignment, or when he needs assistance. He initiates these discussions with his teacher. He does not discuss his work with his peers, most of whom are working on different skills. Therefore, individualized instruction for Benny implies self-study within the prescribed limits of IPI mathematics, and there is never any reason for Benny to participate in a discussion with either his teacher or his peers about what he has learned and what his views are about mathematics. Nevertheless Benny has his own views about mathematics – its rules and its answers.

## Benny's view of the restricted nature of the answers in IPI

Benny determines his rate of progress through the material his teacher prescribes, and he decides when he is ready to take tests. He knows that his progress depends on his mastery of the material – he has to score 80 percent or better in order to pass a skill. But since the answer key in IPI has only one answer for each problem, this implies that at least 80 percent of his answers have to be identical with those in he key. He knows that an answer can be expressed in different ways as the following excerpt illustrates:

> E: Can you give me an example where I would think they're different but the answers were really the same.
>
> B: O. K. Like, what do you think of when I write 1/2 + 2/4? What's the first thing you think up?
>
> E: 1.
>
> B: O. K. If I write 2/4, what does that equal to you?
>
> E: 1/2.
>
> B: O. K. Now like to me, over here [*e.g.,* 1/2 + 2/4], it seems that's 4/4. Over here [*i.e.*, 2/4], to me it seems just like writing two quarters … for money, 50 cents … whatever.
>
> E: How does that differ from what I said?
>
> B: Nothing! They're the same, but different answers. 4/4 is one, while 2/4 is a half.

One implication of this discussion is that some answers, which he knew were correct, were marked wrong because they differed from those in the key. The excerpt below shows what happens if he had a problem like 2 over 4 and he wrote the answer as 2/4.

> B: Then I get it wrong because they [aide and teacher] expect me to put 1/2. Or that's one way. There's another way; 2/4 to me is also 1/4 and 1/4. But if I did that also, I get it wrong. But all of them are right!
>
> E: Why don't you tell them?
>
> B: Because they have to go by the key … what the key says. I don't care what the key says; it's what you look on it. That's why kids nowadays have to take post-tests. That's why nowadays we kids get fractions wrong.

However, from this valid argument, Benny makes an incorrect generalisation about answers. For example, he had solved two problems as follows: 2 + .8 = 1.0 and 2 + 8/10 = 2 8/10. The following excerpt illustrates what Benny thought would happen if he interchanged the answers:

> B: … Wait. I'll show you something. This is a key. If I ever had this one [*i.e.*, 2 + .8] … actually, if I put 2 8/10, I get it wrong. Now down here, if I had this example [*i.e.*, 2 + 8/10], and I put 1.0, I get it wrong. But really they're the same, no matter what the key says.

This view about answers leads him to commit errors like the following:

E: You see, if you add 2 + 3, that gives you 5 …

B: [Interrupting] 2 + 3, that's 5. If I did 2 + .3, that will give me a decimal; that will be .5. If I did it in pictures [*i.e.*, physical models] that will give me 2.3. If I did it in fractions like this [*i.e.*, 2 + 3/10], that will give me 2 3/10.

We now examine how the IPI creates a learning environment that fosters this behavior. First, because a large segment of the material in IPI is presented in programmed form, the questions often require filling in blanks or selecting a correct answer. Therefore, this mode of instruction places an emphasis on answers rather than on the mathematical processes involved. We have already noted that the IPI program relies heavily on its testing program to monitor a pupil's progress. Benny is aware of this. He also knows that the key is used to check his answers. Therefore the key determines his rate of progress. But the key only has one right answer, whereas he knows that an answer can be expressed in different ways. This allows him to believe that all his answers are correct "no matter what the key says".

Second, the programmed form of IPI was forcing Benny into the passive role of writing particular answers in order to get them marked right. This is illustrated in the following excerpt:

E: It [*i.e.*, finding answers] seems to be like a game.

B: [Emotionally] Yes! It's like a wild goose chase.

E: So you're chasing answers the teacher wants?

B: Ya, ya.

E: Which answers would you like to put down?

B: [Shouting] Any! As long as I knew it could be the right answer. You see, I am used to check my own work; and I am used to the key. So I just put down 1/2 because I don't want to get it wrong.

E: Mm …

B: Because if I put 1/4 and 1/4, they'll mark it wrong. But it would be right. You agree with me there, o.k. If I put 2/4, you agree there. If I put 1/2, you agree there too. They're all right!

Through using IPI, learning mathematics has become a "wild goose chase" in which he is chasing particular answers. Mathematics is not a rational and logical subject in which he can verify his answers by an independent process.

One could argue that Benny's problem with answers is a result of marking procedures rather than a weakness of IPI. This argument is not allowed by the teacher's perception of her role. First, the aide's responsibility is to check Benny's answers against those in the key *as quickly as possible*. Second, his work does not go from the aide to his teacher. It is returned directly to him. Therefore, his teacher can only become aware of his problems if Benny chooses to discuss them with her.

Benny directs some of his criticism at his teacher and the aide when he says, "they have to go by the key … what the key says". He illustrates this vividly in the following excerpt:

B: … They mark it wrong because they just go by the key. They don't go by if the answer is true or not. They go by the key. It's like if I had 2/4; they wanted to know what it was, and I wrote down one whole number, and the key said a whole number, it would be right; no matter [if] it was wrong.

[…] We noted earlier that the IPI system, by using independent study as the only mode of learning, decreases the opportunity for discussions between Benny and his teacher. And now, through an emphasis on answers in the IPI testing program, the key appears as the link that associates Benny's teacher with his frustrations. It appears then that, in IPI, teachers are prevented by their rope perception from understanding the pupil's conception of what he is doing. His teacher could encourage him to inquire, to discuss and to reflect upon his experiences in mathematics only if she has a close personal relationship with him and understands his ideas and feelings about mathematics.

## Benny's conception of rules in mathematics

Benny's view about answers is associated with his understanding of operations in mathematics. He regards operations as merely rules; for example, to add 2 + .8, he says: "I look at it like this: 2 + 8 is 10; put my 10 down; put my decimal in front of the zero." However, rules are necessary in mathematics, "because if all we did was to put any answer down, [we would get] 100 every time. We must have rules to get the answer right." He believes that there are rules for every type of problem […].

However, as we have seen, Benny has also discovered, these rules aside, that answers can be expressed in different ways. ("1/2 + 2/4 can be written as 4/4 or 1." This leads him to believe that the answers work like:

> magic, because really they're just different answers *which we think they're different*, but really they're the same.

He expresses this view, that you can't go by reason, in adding 2 + 8/10 as follows:

B: … Say this was magic paper; you know, with the answers written here [*i.e.*, at the top] … hidden. I put 1.0, you know, right up here; hidden … until I press own here [*i.e.*, at the bottom]; and this comes up [in the middle of the paper] an equal sign, two whole and 8/10; or in place of the equal sign the work 'or', and the same down here.

Benny also believes that the rules are universal and cannot be changed:

E: What about the rules. Do they change or remain the same?

B: Remain the same.

E: Do you think a rule can change as you go from one level to another? [*i.e.*, levels in IPI mathematics.]

B: Could, but it doesn't. Really, if you change the rule in fractions it would come out different.

E: Would that be wrong?

B: Yes. It would be wrong to make our own rules; but it would be right. It would not be right to others because, if they are not used to it and try to figure out what we meant by the rule, it wouldn't work out.

Benny's view about rules and answers reveal how he learns mathematics. Mathematics consists of different rules for different types of problems. These rules have all been invented. But they work like magic because the answers one gets from applying these rules can be expressed in different ways, "which we think they're different but really the same".

Therefore, mathematics is not a rational and logical subject in which one has to reason, analyze, seek relationships, make generalizations, and verify answers. His purpose in learning mathematics is to discover the rules and to use them to solve problems. There is only one rule for each type of problem, and he does not consider the possibility that there could be other ways of solving the same problem. Since the rules have already been invented, changing a rule was wrong because the answer "would come out different".

This emphasis on rules can be seen in his approach to decimals and fractions. Decimals and fractions are formed according to certain rules, *e.g.*, a.bc and a/b, $0 < a < 10$. The conversions between decimals and fractions depend on rules, *e.g.*, a/b = .(a + b) or b/a = .(a + b), provided $a + b \geq 10$, otherwise b/a = .0(a + b). There are rules for operations, *e.g.*, 2 + 3 = 3 + 2 because "they're reversed" or "they're switched". Therefore, "2/1, reverse that [gives] ?". There are rules for decimals, *e.g.*, a + .b = .(a + b), as in 2 + .8 = 1.0 and 7.48 – 7 =7.41. In multiplication, a × .b = .(a × b) as in 8 × .4 = 3.2. There are rules for adding fractions. […p. 19 decide whether this is necessary] When thinking of rules, Benny seems to be unaware of mathematical relationships and the principles which underlie the rules. His rules seem to emphasize patterns. Yet, occasionally, he shows signs of being dissatisfied with the rules:

E: Let's take your first example, where you said 2 + .3 = .5. 2 is a whole number. What happens to it when you add it to a decimal?

B: It becomes a decimal.

E: You mean it happens just like that?

B: No! Mm … I would really like to know what happens. You know what I'll do today? I'll go down to the library … I am going to look up fractions, and I am going to find out who did the rules, and how they are kept.

The above examples demonstrate that although Benny does not understand decimals and fractions, he has rules that enable him to perform operations. When he uses these rules however, many of his answers are incorrect. He believes that his answers are correct, and the key has only one of the answers. His task then becomes that of chasing answers that agree with the key.

## Notes

[1] Reprinted by arrangement with the University of Illinois. Originally published in the *Journal of Children's Mathematical Behavior* **1**(2), Autumn, 1973. 

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