Students' Use and Misuse of Mathematical Theorems: The Case of Lagrange's Theorem

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1. Introduction

Consider the following two questions from introductory group theory[1]:

What is the converse of Lagrange's Theorem?

and

Is Z_3 a subgroup of Z_6?

Do you see any substantial connection between them?

Well, if you were inclined to answer in the negative, then you are up for some surprises, just like we were. As it turned out, our students did find a connection. In fact, 20 students out of 113 computer science majors in a top-rank Israeli university answered the second question above with (some slight variation of) the following statement:

Z_3 is a subgroup of Z_6 by Lagrange's Theorem, because 3 divides 6.

(A total of 73 students answered the second question incorrectly[2].)

Examining this amazing answer seriously, turns out to yield some interesting observations on students' ways of using theorems in problem-solving situations. This will be elaborated further in Section 3. Specifically, students tend to:

— use theorems as "slogans", as a way of answering test questions while avoiding the need for understanding or for making other kinds of excessive mental effort;
— in particular, use Lagrange's Theorem or some version of its converse in situations where such use is quite irrelevant to the problem at hand;
— use a theorem and its converse indistinguishably.

Further analysis of students' ways of using Lagrange's Theorem, reveals in addition that determining the "correct" converse involves more subtlety than might be expected. It turns out that theorems (and other mathematical creatures) may have deep meaning which is not apparent from their formulation. Moreover, this deep meaning may in fact be in conflict with the theorem's surface meaning. Since students tend to relate to the surface meaning of the theorem, while mathematicians mostly relate to its deep meaning, a significantly different formulation of the converse of Lagrange's Theorem is obtained by the two groups.

The above quasi-psychological observations raise intriguing mathematical questions: What *really* is the converse of Lagrange's Theorem, and how can one tell which of several formulations is "the correct" one? Also, what *is* the "deep meaning" of Lagrange's Theorem, and why is this deep meaning obscured by the usual formulation? Can we find a formulation of Lagrange's Theorem that will make the deep meaning more explicit? These questions will be discussed in Section 2.

We would like to stress that nothing we say in this paper is intended to doubt the reasoning powers of our students, who have in fact demonstrated excellent reasoning powers in other disciplines. As we discuss elsewhere [Leron & Hazzan, preprint], their behavior is a quite natural reaction of individuals trying to make sense and to cope in what often appears to them a strange, unnatural, even intimidating environment.

2. What is the converse of Lagrange's Theorem?

The answer to the question in the title is not as straightforward as it might seem. We were surprised to find out that obtaining the converse of Lagrange's Theorem was not merely a matter of performing the appropriate syntactical transformation on the statement of the theorem. As it turns out, experts and students obtain different "converses" because the experts relate to a deeper meaning of the theorem, while students relate to its surface formulation. We proceed to examine this phenomenon in more detail.

A typical formulation of Lagrange's Theorem is:

Typical Formulation (TF):
Let G be a finite group. If H is a subgroup of G, then o(H) divides o(G)
[Cf., for example, Fraleigh, 1977; Herstein, 1986; Gallian, 1990]

As we have noted above, people usually come up with two different kinds of answers when asked to find the converse of this theorem: we will call them "naive" and "sophisticated". The "naive converse" is the one formulated by most students and even some mature mathematicians who do not specialize in abstract algebra[3]:

Naive Converse (NC):
If o(H) divides o(G) then H is a subgroup of G

In a questionnaire we passed to one of our abstract algebra classes, 12 students out of 21 gave essentially this answer (the rest gave some other incorrect version). Similarly, in an informal questionnaire we passed among college instructors, 19 out of 25 also gave this answer[4]. Indeed, if the converse of the statement $p \rightarrow q$ is the statement $q \rightarrow p$, then NC is quite a reasonable answer. However, NC is *not* the converse of Lagrange's Theorem! It turns out that there is a "sophisticated converse", which is the one chosen by algebraists and the one referred to in textbooks [Fraleigh, 1977; Herstein, 1986; Gallian, 1990].

> Sophisticated Converse (SC):
> *If k divides o(G), then there exists in G a subgroup of order k*

The discrepancy between NC and SC highlights a dilemma: If the converse of the statement $p \to q$ is the statement $q \to p$, why isn't NC the right converse? And how do experts glean the right converse SC?

Our answer to this question, as hinted above, is that Lagrange's Theorem has a "deep meaning" which is known to algebraists but not to most students. This deep meaning can also be captured in a statement (we shall call it DM) of the form $p' \to q'$, but with p', q' different from the original p, q of TF. Algebraists intuitively use DM when formulating the converse, while most students use the surface formulation TF. It is important to note that not only does the theorem have two different meanings (surface and deep), but the deep meaning is not at all apparent from the surface formulation. How can one discover the statement DM, which expresses the "deep meaning" of Lagrange's Theorem?

The deep meaning is revealed by the sort of comments instructors and authors tell students when discussing Lagrange's Theorem informally. A typical comment, in one form or another, is that Lagrange's Theorem is a *non-existence theorem*: it specifies what orders of subgroups are impossible. (For example, Mac Lane and Birkhoff, [1979]: *Lagrange's Theorem sharply restricts the possible orders of subgroups* [p. 73]. Cf. also Dubinsky & Leron, [1944] page 111.)

This observation raises two questions: One, how do experts know that Lagrange's Theorem is a non-existence theorem? and, two, how can the theorem be formulated so that its non-existential nature stands out explicitly? We answer the second question first. It seems to us that the nature of Lagrange's Theorem as a non-existence theorem is best captured in its contra-positive form:

> Non-Existence Formulation (NEF):
> *If k doesn't divide o(G) then there doesn't exist a subgroup of G of order k.*

This statement also reveals the important-but-usually-ignored fact that Lagrange's Theorem (in its direct formulation) actually contains an existential quantifier. This quantifier is almost always suppressed in the statement of the theorem (for example in TF). Taking the contra-positive of the contra-positive NEF, we get not the original formulation TF but one explicitly containing the existential quantifier. This is the DM we are after:

> Deep Meaning (DM):
> *If there exists in G a subgroup of order k, then k divides o(G).*

We now return to our first question: How *do* mathematicians come to know the deep meaning of Lagrange's Theorem, i.e., its nature as a non-existence theorem? Surely they do not follow anything like our syntactical analysis above. Instead, this knowledge seems to come from accumulated experience in group theory, mainly the way the theorem is most commonly *used* in applications and proofs. It is not surprising, then, that some quite sophisticated but non-algebraist mathematicians also get the converse wrong—they simply lack this accumulated experience.

Finally we return to our original question: What is the converse of Lagrange's Theorem? However, this time we shall convert the "deep meaning" formulation DM:

> *If k divides o(G), then there exists in G a subgroup of order k.*

This is precisely our earlier "sophisticated converse" SC!

Thus the source of the discrepancy lies in the suppression of the existential quantifier. The next question is, therefore, why do many instructors and textbook authors intuitively prefer a quantifier-free formulation, even at the price of hiding the deep meaning of the theorem? The pedagogical impulse of making the initial contact of the student with the theorem as simple and natural as possible may account for this. Instructors know intuitively that quantifiers in general and the formulation DM in particular are "messy" (looked at with the eye of the student who first encounters them) and so prefer to avoid them when possible. [This assumption gets further support from Dubinsky, 1989; Harnik, 1986; Leron, Hazzan and Zazkis, in press.] Even when all the subtle points discussed here are eventually discussed with the students, sensitive instructors often prefer to first present Lagrange's Theorem in its simplest formulation, gradually exposing its intricacies later.

3. How students (mis)use Lagrange's Theorem

The analysis in the previous section was prompted by students' answers to some written questions. We now look in more detail at some of the actual data. It turns out that the misuses of Lagrange's Theorem come mainly in three flavors.

(a) Behaving as if the converse of Lagrange's Theorem (in fact of *any* true theorem) is true;
(b) Using a "naive" converse of Lagrange's Theorem (or another incorrect formulation of the converse):
(c) Applying Lagrange's Theorem or its converse inappropriately: In none of the examples that follow was it expected or helpful to invoke Lagrange's Theorem.

We illustrate these phenomena with three examples, then suggest some brief explanations. All the students in the following examples were computer science majors in the second semester of their studies in an Israeli university. This means that in general they were very strong students. They had previously completed courses in calculus and linear algebra (an abstract approach). The questions in Examples 1 and 2 below were given to the students in their fourth week of the semester (about one-third of the way into the course), without any prior announcement. The question in Example 3 was included in a repeat of the final exam.

Example 1:
113 abstract algebra students were presented with the following question:

> A student wrote in an exam, "Z_3 *is a subgroup of* Z_6".
> In your opinion, is this statement true, partially true, or false?
> Please explain your answer

An incorrect answer was given by 73 students, 20 of which invoked Lagrange's Theorem, in essentially the following manner:

> Z_3 is a subgroup of Z_6 by Lagrange's Theorem, because 3 divides 6

In this example, as well as in the next one, students exhibited a mixture of all three misuses mentioned above: failing to distinguish between Lagrange's Theorem and its (naive!) converse, and in fact applying Lagrange's Theorem (or its converse) where such application does not help in solving the given problem

Example 2:
108 abstract algebra students were presented with the following question:

> A student wrote in an exam, "S_4 *is a subgroup of* S_5".
> In your opinion, is this statement true, partially true, or false?
> Please explain your answer.

An incorrect answer was given by 51 students, 25 of whom invoked Lagrange's Theorem, in one of the following 3 versions:

> S_4 is not a subgroup of S_5 because 4 doesn't divide 5 (15 students)
> S_4 is a subgroup of S_5 because 24 divides 120 (6 students)
> S_4 is not a subgroup of S_5 because 24 doesn't divide 120 (4 students).

Example 3:
23 abstract algebra students were given the following question[5]:

> True or false? Please justify your answer.
> "*In* S_7 *there is no element of order 8*"

In selecting this problem, our (naive) idea was to have students work with the decomposition of permutations into disjoint cycles. Thus, they would have to check whether there exists a permutation in S_7 so that the least common multiple of the lengths of its cycles would be 8. As can be seen, this is not even remotely connected to Lagrange's Theorem

An incorrect answer was given by 16 students, from among whom 12 students invoked Lagrange's Theorem in the following three versions:

> False. There is such an element because 8 divides 5040 (7 students).
> Since for every a, $a^7 = e$, it follows that $a^8 = e$ (3 students)
> The statement is true, because 8 doesn't divide 7 (2 students)

Discussion:
The examples in this section can be seen as special cases of two very general kinds of behaviors One, obtaining the solution to exam problems by clinging to clues in the surface representation (the text) of the problem; two, using the theorem and its converse indistinguishably, and other converse-related bugs In some of the examples it may be clear which of the two general behaviors is in operation, in some others it may be unclear, and in yet some others a mixture of the two may be involved. Looking with the student's eye, both these behaviors may be seen as representing the same state of mind: groping in the dark in an attempt to make some sense of a situation which hardly seems to be making any sense.

The first kind of behavior has been amply documented in the context of story problems in the elementary school. Typically, kids look for two numbers and for a word that will clue them as to whether the operation involved is addition or subtraction [Nesher and Teubal, 1975; Nesher, 1980; Schoenfeld, 1985]. Then they proceed to give the solution by performing the operation on the numbers, without any further thought or further look at the text of the story. Similarly here, students are clued to Lagrange's Theorem by a question about subgroups, and by the appearance of two natural numbers which are the orders of groups. They then answer true or false according to whether the numbers divide or not, without further thought, careful reading of the problem or checking of their solution

The second behavior appears in several variants [cf Selden and Selden, 1987]:

— incorrect formulation of the converse;
— careless invocation of the converse or of the theorem itself;
— failing to distinguish between the theorem and its converse;
— making the hidden assumption that if the theorem is true, so is its converse.

4. Conclusion

What prompts students to invoke a theorem?

There seem to be two opposite phenomena at work here: one, students are reluctant to use theorems as tools in solving problems [Schoenfeld, 1989; Leron, Hazzan and Zazkis, in press]; two, an almost automatic invocation of some version of some theorem, to solve a problem required in an exam. In such cases the students seem to cling to whatever theorem appears to be connected to the present problem, often overlooking, in their stress, fine detail such as the conditions of the theorem, the difference between a theorem and its converse, etc. Both phenomena contribute to the ineffectiveness of theorems-as-tools in students' work

It is possible that these phenomena occur mainly with a certain type of theorem: perhaps one which has a name, or one which is particularly memorable for other reasons, e g. a specially simple formulation involving natural numbers. If, as in the case of Lagrange's Theorem, the theorem can be memorized as a "slogan", then it can easily be retrieved from memory under the hypnotic effect of a magic incantation However, using a theorem as a magic incantation may increase the tendency to use it carelessly, with no regard to the situation or to the details of its applicability.

To sum up, we have been talking about the ways students use mathematical theorems, in particular, Lagrange's Theorem and is converse. The converse was instrumental in helping to reveal what Lagrange's Theorem *really* is One problem concerning the use of Lagrange's Theorem turned out to be that students think it is an existence theorem while in fact it is a non-existence theorem But none of this is apparent from the statement of the theorem. Students' strong tendency to invoke Lagrange's Theorem can be explained by their feeling of insecurity and by the sort of questions they have to answer They have to show that something is or is not a subgroup of something else; and Lagrange's Theorem *appears* to supply them with a ready-made tool which, moreover, is related to numbers about which they do feel secure: if it divides—then it's a subgroup; if it doesn't divide—then it ain't.

Notes

1 The following terminology and notation are used in this article:

Z_n is the group {0, 1 .. $n-1$} with addition mod n

S_n is the group of all the permutations on {1 . . n} with the operation of function composition

$o(G)$ is the order of a group G, i.e. the number of its elements

A finite group is a group with a finite order

We recall that the order of S_n is $n!$

We will use the following formulation:

Lagrange's Theorem: Let G be finite group If H is a subgroup of G, then $o(H)$ divides $o(G)$

2 This question was taken from Dubinsky *et al.*, in press

Actually, the full question was:

A student wrote in an exam, "Z_3 is a subgroup of Z_6" In your opinion, is this statement true, partially true, or false?

Please explain your answer

The question is intentionally vague to help stir students' discussion In particular, the group operation is left unspecified However, no matter how the group operation is chosen, the correct answer would still be "false".

3 For simplicity, we omit the "fixed" assumption "Let G be a finite group, and let H be a subset of G". We also mention in passing that the naive converse, as well as all the others we will discuss, are false.

4. In a discussion following the questionnaire, however, the possibility was raised that the choice of the converse might have been influenced by the specific formulation we chose for Lagrange's Theorem This in itself is an interesting issue for investigation, since one might expect that the converse of a theorem would be a well-defined entity, depending only on the theorem itself (whatever this might mean) and not on a particular wording of its statement

5 The order of an element a in a finite group is the least positive integer n such that $a^n = e$ (the identity element).

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