Conclusions

In the design of the tasks and their use in the exploratory interviews we have demonstrated the existence of a range of skills of wide importance in applications of mathematics but generally neglected in mathematics curricula. We have shown something of the nature of graphical interpretation as a progressive interaction between graph and situation, and some of the key aspects of difficulty — pictorial and situational distractors, the confusion of greatest increase and greatest value, the reading of intervals and gradients as compared with points, and the value of grid-reading.

The teaching experiment has shown the value of the discussion approach based both on graphs and on tables and suggests that a combination of these would probably be the most effective. It also appears important to ensure that all the aspects of difficulty revealed in the interviews (and no doubt others which we have missed) are represented in the teaching programme, since rather little spread of learning is likely to occur.

We should be interested to see this work followed up by a more developed teaching experiment which might examine more deeply the contributions of the graph and the table aspects of the approach. We also mention that the South Nottinghamshire Project material referred to above has been extended by the production of further material inspired by the work described here [Swan, 1981]

We gratefully acknowledge the help of members of Toot Hill School, Bingham, where the experiments were carried out.

References

Bell, A W, Wigley, A and Rooke, D (1978) *Journey into Maths The South Nottinghamshire Project, Stages 1 and 2*. Blackie.

Bettis, N C and Brown, L (1976) Graph Reading Competencies among Elementary Pupils Involved in Inquiry-Based Science and Social Studies Programs. Illinois School Research and Development, 12-2.

Foxman, D D *et al* (1980) *Mathematical Development The APU Primary Survey Report No 1* London: HMSO.

Hart, K. (1980) Secondary School Children's Understanding of Mathematics. Centre for Science Education, Chelsea College, University of London

Janvier, C. (1978a) The Interpretation of Complex Cartesian Graphs Representing Situations — Studies and Teaching Experiments Shell Centre for Mathematical Education, University of Nottingham.

Janvier, C. (1978b) The teaching of graphs: a language approach Proceedings of the XXXth Conference of CIEAEM.

Janvier, C. (1981) The use of situations in mathematics education *Ed Stud Math* XXX-XXX.

Kerslake, D. (1977) The understanding of graphs. *Mathematics in Schools*. 6(2).

Ling, J.F. (1977) *Mathematics across the Curriculum* Blackie

Read, C W W (1970) Guidelines in Mathematics for Teachers West Sussex County Council.

Swan, M (1981) Sketching and Interpreting Graphs. Pupils' booklet accompanying Functions and Graphs South Notts Project Shell Centre for Mathematical Education, University of Nottingham

Ward, M (1979) *Mathematics and the 10 Year Old*. Schools Council Working Paper No 61, Evans/Methuen Educational.

Communications

A seminar on problem solving

CALEB GATTEGNO, ARTHUR POWELL, STEVE SHULLER, DICK TAHTA

After reading various articles in the November 1980 issue of this journal, all of which referred in some way to problem solving, a group of readers met in New York City for a seminar on the topic. We were not sure how best to present the results of our discussions, but we wanted to send a contribution — partly to share what we did, and partly in the hope our story might encourage other groups to meet and work on questions raised in the journal. The following is, first, a transcription of a tape-recorded summary by the first author, supplemented by written statements from the others.

CG: Problem solving is a little up in the air and requires a little more definition. And when I say "definition" I use the word in its optical sense: where there is better definition, where you can see more, can see more clearly, not that you have the words for it. That's an important aspect of how the mind works in terms of definitions. I want a better definition of the problem, not a better phraseology for it.

The first point I raised is that problem solving does not belong to mathematics only. We have to know that it exists in the wider framework of life. When we come to mathematical problem solving we may be better prepared if we have handled matters about problem solving in other areas as well. We touched on the fact that if we give people problems that we already know how to solve, we never discover what are the difficulties in problem solving encountered by others. To find that out we have to work with others on problems that are not known to us, that we haven't yet solved That is as true of problems in elementary mathematics as of other problem that arise

We also discussed "solutions". We agreed that we would not consider that solving a problem is giving an answer, but that we have solved a problem when we see that is has generated many more problems which are all sufficiently attractive to engage us So problem solving is not an end in itself, but a method by which we generate challenges for ourselves This requires that we recognise that problem solving is not essentially an intellectual activity. The writing of it — the writing of a solution — is a disciplined intellectual activity, but the finding of the solution is a much more complex travail in which affectivity is the force that keeps us working on the problem and gives us the stamina to continue working on it. "Staying with the problem" is the most important feature of problem solving, and yet one that is almost never discussed.

Another aspect we felt is entirely neglected (although the story of Poincaré and the solution of the Fuchsian functions is often told) is that problem solving involves the whole of oneself and can require that we use a great deal of so-called subliminal know-how — a know-how that often seems unavailable in the waking state, although we can notice that, even when awake, abandoning a problem to go and have a drink, say, and then coming back gets us started on a new tack. This says that the way the mind is involved in problem solving is not already charted; we cannot say, "If you do this and this and this you will solve this problem" To be really concerned with the teaching of problem solving or heuristics is a different challenge from the one we usually read about; we have to be prepared to renew ourselves, we have to be open to items from other horizons if we are to see what we actually do in solving problems

One way of solving a problem is to surrender to it. This means that our will cooperates with the challenge, it doesn't mean that we become passive We cooperate with the challenge and we try to let it tell us what to do. And what it tells us can be anything, so there is no list of instructions for it. Nobody can give such a list. One thing the challenge tells us is, "Stop being concerned with it." That moment, the moment when we stop being concerned, we find a light and can enter the problem. We can provide an answer to the question that was there, and this may be simultaneously associated with seeing that we can have lots of other problems connected with it

This sort of answer isn't a numeral or a particular quantity as it is in some of the classroom exercises we call problems and where the student gives an answer by saying "7" or "60 km/hr", and so on, whichever is proper. When we look back at all the people who have contributed to science we see that they were in contact with problems, but in contact with problems that nagged them, that challenged them, that kept telling them things that they sometimes accepted to test and at other times refused. When Tall in his article (FLM Vol 1, No. 2) tells of the route that he took and then found that it wasn't right, his account and history tell us that this happens so many times that problem solving may *have* to mean engaging in routes that seem promising, that give results, but when you come to the end you find it is a dead end. And that's part of educating people in problem solving — that there isn't always a royal route. If you knew what it was you would take it. Who was it said there are no royal routes in mathematics?

So, unless there is a tension in the mind of the searcher we cannot say that there is a problem, and unless the tension is dissolved we cannot say that a solution has been reached. When we say people "want" to solve problems, we are recognising the involvement of persons If we are to give people problems we have to give them in a shape that can be gotten into, that will challenge, so that they will want to see what they can do with them. If we give a sixteen-month old child a box, and if he has already played with boxes, we see that the first thing he does is to apply what is already available, what he already knows how to do. And if the box doesn't respond the child doesn't throw the box away. Well, he might But generally he will get involved and ask some questions — questions which an observer might interpret as being concerned with the box, but which could equally well be interpreted as the child's questions of himself. Can I hold it in this way? Can I do something like this? And this may lead to a new approach and the box will open; then the child will close the box and test it again to see if the solution is valid and when it is valid This is how experience accumulates. "When you give me a third box I'll have two possibilities at my disposal, and if neither works I'll try another one. And if you give me a trick box, and I try and try and try and can't open it, then I'll throw it away But not because I have contempt for it, but because I am enraged and must discharge my energy, or because I have met my limitations and I am now recognising that that's where I am. "

Problems which don't generate tension are not educational and we cannot say that they are worth giving to our students.

Looking at what scientists do, we see that anything goes when we start to work on a problem We can use any approach because the first requirement is to get better acquainted with the challenge. And who knows what may lie behind the appearance? Who knows what it will bring, or what we will have to bring to it? This period of trial has been called "trial-and-error" in the literature, and this is an adequate name for an exploration in which one is at peace with one's errors. But this component often gets overlooked and it is easily forgotten that when meeting the unknown it is necessary to be at peace with whatever happens Who can say that when one route is tried and doesn't produce an answer it will not be an important learning at a later stage? All those paths which lead nowhere are as much a legitimate part of our experience as all those that go somewhere, so that in the arsenal of our knowledge about solving problems are not only the successful actions but all those we have given ourselves to. We learn from all this and prepare ourselves for further adventures. When a challenge of a certain kind appears we know what sort of response is proper, what kind of action belongs to it. That doesn't mean we always solve the problem, but we know what is connected with it. We reduce the number of wasted and useless trials that are just random attempts, shots in the dark, and we develop a kind of knack, a kind of sense that a problem of this kind should be tried in this way and a problem of that kind in that way.

Once we have taken to heart the lessons of history — or as much as we can absorb, and each of us will choose a different set of examples — we see that the most daring minds asked questions that in the beginning meant nothing to anybody, including the person who asked them, but the questions lingered and stayed there and came around again and again, asking "What *is* it that you are working on?"

I know that when we talk about problems in mathematics it is expected that we talk about mathematical examples — from history or elsewhere — but I don't think it's necessary to do that in order to convey the state we want to put people into so that they can be educated in the approach to problems. And I would like to tackle a question which is not about mathematics at all but which is particularly clear in showing how remaining with the problem, surrendering to the problem, one can develop an approach to it which ultimately provides something that can be called a solution.

I want to show how one can handle the problem of spelling (specialized to the spelling of English). I will summarize the question — I have already written about it and you may have read the account — so that it may serve as an illustration of solving a problem.

To be good at spelling is to write a word that conforms to the contents of the dictionary. Therefore one can say that if one has a mental store of accurate "photographs" of all the words in the dictionary, one is good at spelling. But of course nobody goes about looking at all the words in the dictionary making careful photographs of every word And in particular, young children cannot apply this to the first words they learn since they can't look in the dictionary until they have learned a quite complicated device for finding words in it. So if we say that it is necessary to have the right images in the mind, we are saying that the first requirement for being good at spelling is to look at words in order to see them. But since when we are young children the words we look at are printed, and the words we already know are spoken, in order to make sense of a written word it is necessary to make a connection, to recognise that the printed word seems to say what we say if we allow the design to trigger certain sounds. So we have to inject into our perception of the printed word the sounds that the word triggers A design such as *it* must make us utter the sound "it" that we already know so that we recognise that the spoken word "it" looks like *it* when we write it down. (We notice that this has nothing to do with the names of the *letters*, which is indeed a distraction that we want to eliminate.)

Now, we have to look at words in order to see them, but when we have seen them we have to be able to form an image which can be called back or evoked Therefore each image of a word must have this additional property of being evocable And when it has been evoked one must be sure that it is correct, that it is the word one wanted So the word one writes, or the word one is looking at, must match the design that has been photographed. They must be conformal. And to be able to read or write one must recognise the match with such certainty that one knows one can go on immediately to the next word.

So we can say that in order to be good at spelling we must put together four workings of the mind — the capacity to perceive, the capacity to evoke, the capacity to recognise, and the capacity to check and be sure. But all of us, even young children, have these powers of the mind. So the study of spelling can begin.

Suppose we have made a start on this study and meet a complicated word, perhaps one that we have not incorporated into our spoken vocabulary. If we hear someone speak it, what can we do? As speakers we have already made a study of the sounds of the language (English, in this case) so we know the component sounds in the new word and we know how many there are Let us say the word is "permanence". Then for each of the components we judge whether there is some writing for it that we already know conforms to the sounds we hear. And because English has many ways of writing many of its sounds, we have to cope with ambiguities. Shall we write "per" as *per*, *pir* or *pur(r)*? Is "man" (an unstressed syllable) to be written *man*, *men* or *mun*? And is the sibillant at the end of the word *s*, *ss* or *ce*? So the word presents quite new challenges, which we usually call the problems of decoding and encoding. Encoding is putting the sounds into the right shape; decoding is getting the right sounds from the design.

Now this requirement depends on history. It depends on other people. It depends on who chose what sort of script to write down, and what conventions to govern it. To be a good speller one has to know the sounds of the English language and to know all the signs that someone else decided go with each sound But this is still not sufficient, even though it goes a long way to solving the problem of encoding/decoding. We must still know which particular group of signs in what order conforms to the "proper" spelling, so we must fall back again on the first of the four mental capacities mentioned above.

Nevertheless, in spite of the ambiguities, there are other factors which can help one settle the "right" shape of a word. Etymology, which connects with meaning, and awareness of the way in which the context in which a word is heard can tell one which shape to use (so that we know whether to write *pair*, *pear* or *pare* from the other sounds that accompany it)

So a solution to the problem of spelling requires getting people to engage in a self-disciplined way with their powers of perception, evocation, recognition and verification, giving them an instrument which displays all the signs of all the sounds of English, and making them sensitive to context and the influence of meaning

In what sense is this a solution? It is not yet a solution for everyone learning to spell But it lights up the problem so that we can see what must be learned and what practiced, what exercises and what activities will help; it takes care of the idiosyncrasies of the language and it shows what will be required in terms of flexibility by the person learning to spell and the person trying to help others to spell

I find the example valuable because it is not too large and difficult and because it isn't finished. There is no final solution but a number of solutions. The description I've given is an aspect of a solution. It is an instrument I can use in fashioning a solution for anyone who comes to me for help in solving the problem of spelling

I suppose there will be no consolation for mathematics teachers in knowing that the problem of spelling can be solved. They may ask, how can I apply this to making my students know how to solve problems? Well, I have produced a better definition of the problem of spelling and how it could be attacked, so I think I am illustrating a number of things that are important when we connect with a problem.

If it is true that every problem — every real problem — challenges us differently, it is part of the advice we get from life that problem solving is not a discipline, like heuristics, but something requiring that we change ourselves in certain ways. To be educated in problem solving may be to have experienced in a variety of situations that we must do something with ourselves to get into a problem, to stay with it, and to come away with a solution.

AP: There is a challenge for mathematics educators and others concerned with the teaching of problem solving that is seldom addressed. It is to examine the process of solving problems in other fields in order to abstract and identify the components of the self's activity that are involved in the process. One of the components, which was recognized and explicated in our discussion, is the role of affectivity in solving a problem. This is in contrast with the belief that mathematical problem solving is purely an intellectual activity which can be schematized. Indeed, the intellectual component is required at some stage, but as the spelling example illustrates, other aspects of the self are demanded

Gattegno's approach to solving the problem of spelling appears to be to keep it open He remains with the problem and maintains its full complexity. It is this attitude during the period of getting acquainted with the question that yields an entry to it in depth. He turns the problem around and around while he considers that retaining the accepted concatenation of signs for a particular word requires the formation of a mental image of it. He recognizes that the common names of the letters forming the word offer no clues to the sounds associated with the word and that, in English, there may exist more than one sound associated to a sign and that a sign is usually represented by more than one sound.

Remaining with a problem and surrendering to it are part of a process which is necessarily held together by affectivity This process leads to a solution of the problem of spelling which is quite unlike the usual "solutions" — i.e. learning the "rules" of orthography, which seem to have many exceptions, and memorizing the concatenations of signs for each word.

It is necessary to distinguish between defining a problem and solving a problem, i.e. dissolving the tension or doing away with the perplexity induced by it A problem is a challenge which an individual becomes involved in and connected with through the mechanisms of affectivity. Once it is solved, however, further challenges are generated The dissolution of tension is therefore present only momentarily; then a transformation of energy occurs which, upon reflection, leads to further tension. In this process intellectual growth and self-esteem are enhanced Both help to develop an individual's problem solving capacity. It is the conscious attention to this dialectic that is pedagogically important in problem solving activities in classrooms.

SS: The more mathematics educators discuss problem solving, the more difficult it is to take a fresh look at this topic, or even to recognise promising insights amid the clutter of oft-repeated beliefs Even though we sometimes acknowledge that problem solving is a pervasive human activity we have tended to narrow our focus to specific classes of mathematical questions which are not really problems for the investigators This orientation forces upon us a perspective which cannot take into account the inner workings of people engaged in problem solving.

The example of spelling that came up in our discussions is appropriate because it is likely to be new, therefore it can be a genuine problem for us, yet it is accessible. The following notes summarize some of what I learned about problem solving by watching myself work on the problem of spelling.

1. I said that spelling *can* be a genuine problem for us because I know that when we started our investigation it was not a problem for me — because I was not interested. I knew that I did not know how one became a good speller, but I did not much care, so there was no inner tension, no problem Only when our initial exploration made me realize that there were some unexpected complexities to consider — the need to account for both oral and written forms of words, for example — was there an inner tension that made spelling a problem for me.

2. Familiarity with spelling made it easy to find superficial explanations — like, spelling is "taking good photographs." For me, this was a way to get rid of someone else's problem. Familiarity *interfered* with surrendering to the problem; it took discipline (will) to begin asking questions of myself which made me begin to see spelling in its complexity This inner dialogue produced the inner tension which made spelling a problem for me.

3. I might not have "gotten into" the problem of spelling at all had not others around me been asking themselves (and each other) questions which kept us with the problem, and providing counterexamples to dislodge us from false solutions.

4. The inner tension that produces the problem can be dissolved by a solution, to be sure, but also by distractions, false solutions, or simply giving up. If one remains in touch with the problem, however, the tension eventually recurs. In me this usually begins with a vague uneasy feeling which in time becomes focused on an aspect of the problem that needs further work

5. Viewing solutions as instruments for further work provides a useful criterion for recognising genuine solutions, and especially for distinguishing them from other tension-reducing inner movements.

6. When working on the problem, there are moments of clarification when the problem is transformed into another problem which seems more accessible. For example, part of the problem of spelling was transformed into that of classifying all the sounds and spellings of English So, problem solving can be seen as a pattern of recurring tensions and resolutions as one problem is transformed into another, and another. If the initial problem is a fruitful one, this series of transformations is endless, although we can stop after any transformation point and call that transformation a solution.

7. None of the familiar heuristics helped to solve this problem, and only one heuristic principle emerges from the experience: to "surrender" to the problem. This is disturbing, because "surrendering" sounds mystical. I am sure, however, that working on this problem taught me something about how to "surrender" to problems, and the fact that I was working with people who had developed this skill facilitated my learning (see #3 above).

8 I am beginning to see that problem solving is, in its broadest sense, preparation to meet the unknown What usually passes for problem solving is, in contrast, preparation to answer particular classes of questions, such as word problems or algebraic identities. Working on the problem of spelling made quite clear that preparations of the latter sort are not adequate for the former purpose

DT: To consider spelling as an example of problem solving might seem at first to be unnecessarily removed from the immediate concerns of mathematics teachers. The transcripts of the seminar discussions certainly indicate some resistance at this point. Yet, ironically enough, the need to relax such resistance is one of the central points at issue It seems easy enough to assent to the proposition that one needs to surrender to a problem, needs to remain with it and not be over-concerned with a solution. In this case the problem was, I suppose, the problem of problem solving. Yet when it came to considering an example that could elucidate this, there was suddenly the strong feeling that spelling was not particularly relevant — at any rate to mathematical problem solving. Perhaps some readers will have felt something similar during their first reading of the above account.

The example certainly turned out to be a good illustration of the point that to learn something about problem solving it is as well to embark on something that is really unknown. Trying to stay with the problem of spelling was a worthwhile discipline. What was gained by doing that? Well, it seems appropriate here to give a personal reaction and I would like to mention briefly two things that struck me in particular

Firstly, I noticed that my initial approach was to refer the problem to what I already knew. This is, after all, a regularly recommended heuristic suggestion. But I happen to be a relatively good speller and all I could do out of my own experience was to think in terms of "photographic memory". When I deliberately tried to move away from my own experience — trying to be with the problem rather than with myself — I realised more clearly that spelling initially demands a bringing together of a visual component and an aural component The problem lies in the link between the visual and the aural. The implications of this are followed up in the above account

The second issue that was, for me, particularly striking was the notion that a solution to a problem is an "instrument" for generating further problems and solutions In this case, it turns out there is a dissolving of the epistemological tension. To be able to spell is to be able to do such and such — the details presented are convincing But the existence of other minds to whom this has to be brought reminds us clearly that there are never any absolute or permanent pedagogical solutions To consider *teaching* spelling is to give oneself a series of further as-yet-unknown challenges.

I do not know whether the reader coming to the example through the printed word will feel anything like the same tensions and dissolvings It may be worth reading again some of the general points about problem solving that were discussed at the beginning of the seminar with the example of spelling now in mind. It does seem worth asserting again here that we have something to learn from other than mathematical problems. Our experiences of these reminds us, more clearly than those from a more familiar field, that for there to be a problem someone has to be affectively involved, that this person has to work on himself/herself and surrender to the problem, that in solving it he or she generates further challenges These may seem obvious maxims but they do not seem to be clearly stated when people propose lists of procedures that purport to help others solve problems. Certainly no one suggests sleeping on a problem though many quote the Poincaré story. Sleeping seemed to be an important (though not perhaps sufficiently stressed) feature in David Tall's account of a mathemtical discovery in the second issue of this journal All of us in fact sleep on problems that are *really* problems for us. It might be helpful if further discussions of problem solving brought such known, but often ignored, personal factors into account

Reflections on a Letter of Acceptance: An Effort at Communication

STEPHEN BROWN

I have on occasion learned a lot through correspondence with editors who have rejected my articles. I am not used to being provoked by an editor who accepts what I have said with essentially no requirements for revision. Experience with my article ("Ye Shall be Known By Your Generations") in the last issue of *For the Learning of Mathematics* was an exception, however. As the title implies, the focus of the article is on the nature of problem generation, a part of which addresses the issue of *understanding* in relationship to that enterprise.

David Wheeler, in his acceptance letter, responded to my comments on understanding with the following remarks:

> *It's talk about understanding that makes my hair curl. I see your reason for starting with it, for historical/new math reasons, but I can't help thinking you invest too much of your own values in it And I notice that you implicitly drop understanding when you come to talk about problem solving. Of course — understanding has almost nothing to do with problem solving. What (I think) happens is that, to my taste, you do not sufficiently distinguish between the teacher's and the learner's interests (and I don't want to create a rift here, since every teacher has to be a learner sometimes too, but at times the teaching function is paramount). The student's task is not to understand mathematics but to function in it — to be able to think in it, to solve problems in it, to prove things, and so on Mathematics as an activity* per se *is not something*

> *to be understood but something to be done (lived, if you will).*

There was much more Wheeler said in that letter (for example suggesting that philosophical analysis is more of a distraction than a form of enlightenment both for teacher and student if they are concerned with "getting on" in their mathematical activity, and raising questions about the value of perceiving every mathematical situation from a totally fresh perspective), but I would like to focus here on questions raised for me regarding possible differences in our attitude towards understanding in mathematics.

Before attempting to raise some of these questions, it would perhaps help to clarify his position if I excerpt a portion of a later correspondence from David Wheeler regarding the same issue:

> *"Understanding" is for me a treacherous concept since it begs too many questions about* whose *understanding is sought. Most people who talk about understanding clearly mean their own, which they would like to be shared. I find the concept of "awareness" — just as broad a construct — better for my purposes since there is less possibility that it can be regarded as anything other than personal to the one who acquires it. (The trap of this emphasis, as I am aware, is that it can tend to individualism, laissez-faire-ism and solipsism. I shudder at what the Californian psychiatrists manage to make of it.)*
>
> *The balance is a healthy (sic) streak of pragmatism. I see the horrors of institutionalized mass education but I don't want to run from them in despair. I'm not a romantic about education, whatever my tendencies in other directions. I want proposals to be workable in the situation as it is."*

I have misplaced and forgotten a great deal of my responses, and though I could certainly retrieve most of my remarks by phoning several students last semester who reviewed the correspondence (since I was concerned over being misunderstood — the word is revealing), I think we might find it valuable to capitalize on my state of amnesia. That is I will try very hard to minimize a defensive posture and replace it with as receptive a one as I can muster. In fact Wheeler has expressed extremely well a point of view with regard to understanding. For the purpose of provoking colleagues as I was provoked myself, I might even end correspondence at this point and ask readers to review his remarks with the intention of evaluating them in terms of validity of assumptions, implications and so forth. Anyone so inclined might stop at this point and ignore the "Brownian motion" that follows. In order to enlighten myself however, I proceed — with the hope that at least I will be surprised by some of what follows.

(1) *Understanding and awareness*

How do understanding and awareness relate to each other? Though one can certainly be led astray by an essentially *aseptic* ordinary language analysis, there are times when such a philosophical orientation yields pay-dirt. Let us try it out here. How do we use "understanding" in the English language (and comparison with other languages may be revealing)?

Among possible forms are:

(a) X understands *that* Y (where X is a person and Y a proposition).
(b) X has an understanding *of* Y.
(c) X understands *how to* Y (here Y is no longer a proposition).
(d) X understands *why* Y.
(e) X is understanding *towards* Y (as in being empathetic)
(f) X understands Y (where Y need not be a proposition or a person).

There are other forms also, but let us try out a similar analysis of "awareness."

(a′) X is aware *that* Y; (a″) X has an awareness that Y.
(b′) X is aware *of* Y.

I have trouble coming up with other forms for awareness, though I suspect they exist. Now what is there that is captured by these different constructions? What are some of the subtle and not so subtle differences among them? Though I would love to see a careful analysis of the terrain at some point, I cannot offer more than a few observations:

(i) *(a) and (a′)*:

(a) is certainly the most prevalent form of understanding in most mathematics texts — certainly, those of "modern math" vintage. Furthermore (a) seems to have built-in just about the same assumptions as (a′). Now what is potentially unappealing about (a)? David Wheeler suggests that the connotation is essentially that of a student being coerced to accept the view of another person — a teacher. I do not see (understand?) that this follows from an analysis of (a) necessarily but I think I see how one would reach that conclusion. What are some possible Y's in "X understands that Y"?

Obvious candidates might be:

"In a right triangle, $c^2 = a^2 + b^2$"
"A differentiable function is everywhere continuous"
"For all a, b, $(a + b)^2 = a^2 + 2ab + b^2$"

Now what can we say about these statements?

First and most importantly, these are propositions that are *true* statements. We for example would *not* claim that:

"X understands that for all real numbers, $(a + b)^2 = a^2$" Instead, we might say that X *thinks* he understands, or believes that for all real numbers . . .

Secondly, we would probably *not* attribute understanding to X if he or she could provide no *evidence* for belief in the propositions.

Now is there anything inherent in the search for truth and for evidence that leads to understanding as a seductive activity? This may be so, but if it is the case, it may be the result of an obsession with *passing on standard knowledge* (e.g., the kinds of propositions discussed earlier, such as the Pythagorean theorem). Is there anything inherent in the concept of understanding that precludes me as a teacher from encouraging you as a student to search out *understandings* that may not be ones I have myself explored before? When I do that, am I asking you to adopt *my* understanding if I try to persuade you to examine your emerging "understandings" for *truth* and *evidence?*

Perhaps it is the case that if the emerging "understandings'' are those of the students and not the teacher, then we ought *not* to impose *any* categories (like truth, evidence, generalizability) that might get in the way of the student ''doing'' mathematics. But isn't part of our role one of encouraging *reflection* and not just *movement?* Do we not over-romanticize considerably if we view our role as educators as one of getting out of our student's way on the grounds that any movement is forward or that intervention is harmful?

(ii): *The meaning of (b')*

(b') — X is aware of Y — appears to have a set of assumptions that might indeed be different from any of those in (a)-(f). In an expression such as $(a+b+c) - (e+f+g) = (a-e) + (b-f) + (c-g)$, we might say that X is aware of the minus signs on both sides of the equation.

Now this example is interesting for several reasons. First of all, that awareness could be achieved even if this is *not* a problem or a concept that interested X. He might in fact have been given that identity by a teacher and have been asked to learn it. So it seems possible that awareness can come about even when the focus is the subject matter of the teacher and not the student's at all — a distinction that Wheeler was apparently trying to establish in differentiating understanding from awareness.

Secondly, however, this *awareness* is a very *partial* kind of cognition that does not in itself necessarily enable one to ''move ahead'' (even if the generalization were arrived at by the student or his/her own). But perhaps that is a distinguishing feature of awareness (b') that we as teachers might profit from. Perhaps we do not attend enough to the isolated and even possibly ''irrelevant'' things our students ''see.'' *Understanding that* has a kind of ''push'' to it (even if it is *not* the teacher's understanding to begin with) that awareness (b') appears not to have. We are less prone to ask for validity or justification of a (b') type of awareness. What is there — if anything — that we might be obliged to encourage in our students with the emergence of awarenessess? I am suggesting an analysis of that question that hinges not only on pedagogical issues but on the logic of the concept of awareness. That is, if Y turns out to be a false proposition, then I as a teacher can claim that regardless of my student's inner state of mind, he or she cannot understand that Y. Are there analogous kinds of criteria for the concept of awareness? This is perhaps a good point to drop category (1) and to move on to other provocative issues raised by David Wheeler's comments.

(2) *Understanding and problem solving*

Wheeler has claimed that ''the student's task is not to understand mathematics but to function in it, to be able to think in it, to solve problems in it, to prove things and so on.'' I think he is getting at something important, but the issue may be obscured by pitting ''understanding mathematics'' against ''solving problems'' and so forth. The difficulty arises I think as a result of comparing macro *vs.* micro tasks. Understanding *mathematics* implies achieving something with regard to a *large* body of knowledge (and such monsters as the structure of the discipline rise up). Problems on the other hand are solved one by one.

Instead of talking about understanding *mathematics*, talk about understanding *problems* that one is working on. Is it the case that a student can solve a *problem* without *understanding the problem?* And what does it mean to say that the student *understanding what he has proven?* I think that understanding is in fact built into the concept of problem solving in the sense, for example, that one has to at least understand what is being asked for in a problem. Furthermore, as I tried to argue in my article, I believe that such understanding, though necessary is not acquired cheaply. That is, one cannot *understand* a problem without generating other problems along the way — and thus ultimately understanding is not a teacher-pleasing behavior but on the contrary a creative and solitary act.

In the previous paragraph, I implied that it is reasonable to ask what one understands when he has proven something. The question is important because we have all had the experience of *proving* something and not understanding the significance of what we have proven.

This analysis may seem at this point like an academic arid exercise, and perhaps I have missed the mark in trying to understand David Wheeler's criticisms of my preoccupation with understanding. If I am correct, however in claiming that understanding is embedded in problem solving and frequently lacking in alleged proofs, then it would seem to be a concept that is worth attending to more carefully. I would like to end my unreasonably long communication with one more category prompted by David Wheeler's remarks.

(3) *The role of dialogue, conversation and history*

The earlier discussion of understanding in relationship to awareness hit upon a category that Wheeler raised in his correspondence that is worth thinking about regardless of the kind of ordinary language analysis suggested in (1). That is, in suggesting that the student's job is to ''do'' mathematics, the point seems to be that s/he ought to be on his or her ''own'' to some extent rather that attached to a set of strings that I as teacher am pulling. The above metaphor is of course in need of ''unpacking'', but rather than doing so, I would like to suggest some issues on a global level that are in need of exploration.

First of all, much of my professional life has been tied up in trying to support that point of view. In fact much of the driving force behind my plea for problem generation in the curriculum is an effort at disengaging the student from a ''taken-for-granted'' reality that we tend to foist upon him. But I would like to pause at this point and question what we give up when we take that position in its extreme form.

(i) *The role of history*

Is it not part of our task as educators to have students merely ''follow'' the evolution of an idea over time? Surely we must do a disservice to our students by giving them the impression that problems have no history. Now we certainly can be clever pedagogically and might even want to give students problems to work on that reflect the unfolding of a concept over time, but do we *not* also want to have them become aware of how these problems were actually perceived by others? Do we not want them to see how assumptions of the culture of the time affected the way that mathematicians viewed their work as well? Do we not want them to try to understand what might have been responsible for the inability of Euclid to see the concept of ratio as we see it today despite the fact that he could deal with the ostensibly more sophisticated concept of proportion?

It is perhaps worth thinking through *variations* on the role of history For example, for the purpose of trying to see the *significance(s)* of any idea (and not just understanding it) in relationship to other ideas it might be valuable to encourage students to do some "pseudo-history" on the idea. That is, students might be asked to consider not how the idea/concept, etc., *actually* evolved, but to imagine how it *might have* evolved, i e. what *might* have been the germ of the idea 500 years ago, 2,000 years ago?

(ii) *Someone else's world*

Isn't part of being educated trying to "see" the world from someone else's perspective? Not only from a historical but from a cohort point of view, do we not want students to try to hear and see how their fellow classmates view the world as well? Don't I want to learn to listen to someone else's argument as well as make up my own? Don't I also want not only to solve and to prove, but to persuade someone else of what I have done? Certainly the discourse of proof is different from that of explanation and persuasion at least on some occasions.

(iii) *The role of dialogue*

Most importantly, there are many things of a mathematical nature that I am unsure of, that my students are unclear about and that are worth discussing even if such dialogue turns out not to advance my mathematizing. What is a variable? What is a proof? Why do we prove things? What is a problem? Why are there unsolved problems? Who should select the problems that I eventually work on? How does solving math problems relate to solving problems in science? in the humanities? in real life? To what extent do I seek generalizability and to what extent uniqueness as I solve problems in math? in "real life"?

These are all problems that plague me Yet they are problems that I have made some headway in answering by engaging in dialogue with others Is there no educational value in encouraging our students to so engage themselves as well?

Some things may be relinquished by engaging in such dialogue, just as some things may be relinquished by doing history or by learning to see the world from a colleague's point of view. In particular the activities of section (3) are linked in that they may be at odds for the most part with the activity of *mathematizing* What I am suggesting is that mathematics is a robust discipline and to the extent that we perceive our role as *educators* we ought not be committed to only *one* use of the subject. Mathematics can be a vehicle for increasing a strictly mathematical awareness, but it can be a vehicle for affecting other kinds of awareness as well, awareness that may tap the most fundamental source of our humanity.

(4) *An Epilogue*

In searching for alternatives to mathematizing ("doing" mathematics) in the curriculum, four things should be clear:

(i) I do not de-value that activity Rather I think it is enormously important — especially when seen against the backdrop of what has passed for mathematical activity in the curriculum

(ii) As educators, we are remiss if we conclude without considerable debate that the *only* reasonable kind of understanding *or* awareness that derives from the subject matter of mathematics is rooted in a mathematizing type of activity. For *some* people and on *some* occasions (where some might be quite large), we might find it valuable to *use* mathematics with *no* intention of mathematizing at all I have hardly begun to spell out the alternatives and surely there are many more options than I am even aware of at this point. As educators, we must identify the issues that need to be addressed in order to help us determine the forms of mathematical activity that ought to flow from mathematics as a discipline. Surely it is shortsighted of us if we have as our *only* goal an increase in mathematical thinking and if we neglect to ask questions about the nature of society, the purpose of education and the relationship between the two. (As mathematics *educators* we might in fact conclude that some of these goals are best served by teachers of other disciplines, but that does not, I believe, destroy the significance of my plea)

(iii) To focus on *mathematizing* as an almost *exclusive* activity of the discipline of mathematics, is to imply that the educational/social unit is the *individual*, and that his/her relationship to others in time or place is relatively unimportant A resolution of the issue is complicated, but perhaps we ought to ask if that is the implicit message we wish to convey to students.

(iv) Despite a limited scope, the alternatives I have suggested are for the most part radical ones — ones that have *not* been tried out in any systematic way. Furthermore, they are alternatives that are in need of refinement, clarification and the like. Like mathematizing, however, the alternatives seek out a respect for mind, not only the individual but the collective mind, and the mind of the past we well as that of the present What is wrong with much of what passes for curriculum in schools is *not* that mathematizing is lacking but more generally that respect for a mind in any form is lacking Students who are asked to "follow" a standard text, for example, are not invited to think through the unraveling of a problem either in time or according to other conceptual dimensions Rather they are subjected to the latest plagiarism-type activity of an author who no more is aware in any deep sense of the reason for sequencing activities than is the student who is asked to jump through the hoops.

In concluding, I confess that I had hoped to be more neutral, fair and dispassionate this time around than in my original response to Wheeler, and though I have intentionally sought not to recapture my initial defensive reaction, I am afraid I have built up to the same crescendo once more. I hope the reader will be able to take at least some of the possibly arrogant assertions found here and turn them enough into questions so that I might profit from the kind of dialogue I have called for as a necessary and missing ingredient of the mathematics classroom as well as of the mathematics education community writ large.

Imagination, among other things

SANDY DAWSON

I recently had occasion to go back over the first three issues of Volume One of *For the Learning of Mathematics*. In doing so, it struck me that the set of articles contained therein conveyed some issues, themes, and/or questions which are very important to mathematics educators. In writing to you it is my hope that other readers will refute my conclusion or, failing that, provide additional insights which I have missed in my reading of the articles

The lead article of issue one, Dick Tahta's, "About Geometry", provides the best justification I have seen for the inclusion of geometry in the school curriculum. Unfortunately, his view of geometry and how children can be assisted in teaching themselves geometry (viewpoints reinforced by Gattegno's article, also in issue one, titled, "The Foundations of Geometry") are not ones commonly held by classroom teachers. What Tahta and Gattegno seem to be suggesting is that children's learning of geometry should be based in large measure on images which the children themselves create, that these images can then trigger the words which eventually can lead to a more formalized study of geometry This is a suggestion or an issue which certainly needs further exploration Reports from teachers who have experimented with such an approach would sharpen our understanding of how geometry can be brought back to life in classrooms.

In his article, "On Virgil: My Opening Lecture to Mathematics 120," Taylor argues that the teacher is the "most essential component" of the curriculum, but I believe that his own words belie his commitment to this model. In his postcript, Taylor strongly suggests that it is the student who is central to the process, because it is the student who must do the learning. Taylor also suggests that the students must make the material under study — the mathematical problems — their own. Surely, then, Taylor's own words imply that students are the central focus of the curriculum and of all classroom activity, and not the teacher as Taylor contends in his model. Moreover, if for students to learn mathematics (or anything else for that matter) they must make it their own, then does not that imply that it is the students' images of the problems under study which are of paramount concern to their teacher? Ah! There it is again; that word — images I see imagination and children's use of their imagination as one theme pervading many of the articles in Volume One

David Tall's article in the second issue, "Anatomy of a Discovery in Mathematics Research," was exciting to read It was so refreshing to have a mathematician take the time to record and share the processes and states of being he experienced during his year-long investigation of a problem Moreover, Tall gives some insight into how he used his intuition and imagination in attempting to find solutions to his problems In doing so, he reinforced Gattegno's view (Letter to the Editor, No 2) of the positive and constructive role of errors in the generation of mathematics This latter point raises a question for mathematics education researchers as to how children's errors should be interpreted, and also what conclusions can validly be drawn from children's errors. These are *not* questions which have been given very much thoughtful study.

It was when I read one of David Henderson's three short papers in the third issue, the one entitled "Mathematics as Imagination", that I was most forcibly struck by the theme found in many of the articles; namely, "... that the meaning of mathematics can be found in (or based on) . experiences and imagination" The theme that imagination, a power of the mind which we all possess, is a crucial aspect both in the generation and the learning of mathematics is one which has only recently been addressed by mathematics educators. It was pleasing to me that so many writers in Volume One had chosen (perhaps accidentally?) to focus their attention on this theme. The relationship(s) between imagination — and, indeed, all powers of the mind — and the generation and learning of mathematics is a field of research that has sat in summer fallow far too long It is a field which needs to be ploughed, seeded, fertilized, and its potentially rich crops harvested for many years to come.

At the end of Dick Tahta's second article in Volume One, entitled "On Poetry and Mathematics", he raises what to me are some very important questions; questions which extend the theme I mentioned above, and questions which mathematics educators (with a few notable exceptions, Tahta obviously being one) have not concerned themselves with very much at all. Tahta writes:

> *The multivalent condensations that are known as images in both poetry and mathematics seem to be important ingredients of the inner life. What role do they play in the development of young children? How and why do people make images with them — especially those to do with ordinarily (sic)* and rhythm? In what way do some images link with external reality? How do condensations become "charged" with some form of energy and how is this evoked, for example, by ancient number myths?*

I have seen very few mathematics educators who have even been concerned with their student's inner life let alone try to take this into account when teaching them mathematics. Perhaps it is time that mathematics educator's concerned themselves with the questions Tahta poses.

I could certainly mention other articles which contributed to my conclusion about the issues, themes, and questions which seem to run through Volume One of FLM. However, I am writing a letter and not an article about the articles so I have made a selection of articles to comment on The selection in no way implies, however, that other writers did not also deal with at least some of the issues, or themes, or questions I identified.

Other readers of Volume One may disagree with my emphasis, or question the importance I attach to it. I hope they will convey their insights to me and other readers of FLM.

On a personal note, I must say that I have found FLM to be a most stimulating addition to my professional reading Indeed, and this may say more about me than about the

*Dick Tahta wrote "ordination", and that is what we should have printed My apologies to all. D.W.

journal, I have found FLM to be the one journal which consistently — article-to-article, issue-to-issue — extends my awareness of how mathematics is learnt, which challenges me to re-think (and, at times, to stop thinking) what I do when I learn mathematics or when I try to help others learn mathematics

The clinical interview: a comment

JOHN TRIVETT

A short comment on Ginsburg's article, "The Clinical Interview in Psychological Research on Mathematical Thinking" from FLM, number 3, March 1981? Yes, I would like to make a comment because it is a subject which touches me deeply, not so much as a researcher but as a teacher.

A teacher of mathematics is certainly as dependent on what is beneath the "discovery (and) identification of cognitive activity (structures, processes, thought patterns, etc.)" and the "evaluation of levels of competence" as is a professional researcher.

Inevitably neither teachers nor students can help but be affected in their daily classroom efforts by "naturalistic observations", "natural inclinations", "reflection", ambiguities and responses from each other that do not accurately reflect what they are thinking, and the fact that "genuine mathematical understanding is extremely complex".

Surely therefore it behoves the teacher to (a) employ tasks which channel the students' activities into particular areas, (b) demand *reflection,* (c) recognize that his/her questions are *contingent* on the students' responses, (d) employ some basic features of the *experimental method* and (e) see that, nevertheless, some degree of *standardization* may be possible (altering some words of the points of procedure itemized by Ginsburg to fit the teacher's point of view, but maintaining his italics.)

I do not, of course, suggest that all teachers are aware of this, let alone aware of all the implications or able to use those implications, other than slightly, perhaps, to benefit their own teaching and the students' learning.

For me Ginsburg succeeds in making the case for clinical interviews having the attributes he mentions: relaxing both for student and interviewer, quiet, unhurried, lacking interruption But teachers can develop similar situations in classrooms! If they have not yet perfected all that with 30 students during a lesson, they can certainly do it outside class, at lunch time, after school, even out of school. Don't please say that teachers are not paid for this. That is not my concern at the moment, only that if one wishes one can do so. If teachers are determined to improve significantly their math teaching in class there are ways these days. The required wisdom and skills are available

What ways? you ask. From the ways that stem from,

1. knowing for sure that *all* students can do math and enjoy doing so, though they may not think so from the lessons they have been through;
2. knowing that all students *want* to learn the math they meet, despite the fact that they often "don't get it!";
3. the awareness of a new role by teachers in rethinking "correction", "explanation", "the basics", the relevance of algebra in arithmetic and how children learn;
4. the acceptance that every student is continually trying seriously — even desperately — to communicate to the world outside himself what he sees inside his self and that this needs to be accepted as a priority in school, more important than thinking that children are there to absorb only what needs to be passed on by the traditions of the culture;
5. being able day by day to invite students into classrooms situations, symbolic, concrete, applicative, emotional, pleasurable, which situations are pregnant with mathematical ideas so that when the invitations are accepted insights are experienced by all, at first hand. From that the symbolisms suggested attain secure personal associations capable of long endurance;
6. a courage and a respect to interpret texts, curriculum guidelines, the wishes of parents and administrators to harmonize with a sound backing of professional responsibility so that the interpretations are finally decided by the individual teacher;
7. seeing that the teaching of arithmetic is akin to the teaching of a language which without the spoken components being owned by the students becomes at best only a book of static phrases Therefore conversations, not one-sided verbal expressions, are essential for every learner of mathematics. Conversations without fear or hesitation that error will be censured must be risked

In short, the qualities of Ginsburg's interviews can be present in lessons. It is fine that such interviews exist to help those who do not receive what is needed in the classroom. It would be far better to lessen the necessity for such help outside. "Contingency defines the clinical method", says Ginsburg. I say, "Contingency must also punctuate classroom methods."

Just one other point from the excellent article. Given the classroom *modus operandi* I have merely hinted at, might we not expect Patty (from pages 7 and 8) to have been fluent for a long time in the language on which the problems cited in her interview depend? She would surely not have had to depend on *writing* 29 + 4. Because if she had had lots of conversations in quiet, relaxed, unhurried circumstances without feeling the pressure that she had to produce the answers most teachers seem to want at high speed, then she would most likely use some substitution strategy as, for example, "Twenty-nine is one less than thirty so twenty-nine plus four is thirty plus three (the "the one up, one down" game) and that's thirty-three." In my experience and that of those who encourage mathematical conversations with children, partly based on manipulation and doodling with fingers, counters, pebbles and coloured rods (or what-have-you), the Pattys of this world would *never* say "sixty-nine" for the answer to "29 + 4". This does not of course invalidate the clinical interview. It does render a judgment on what probably happened to Patty in her classroom.