Communications

Horizon content knowledge in the work of teaching: a focus on planning

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Horizon content knowledge, one component of Ball *et al.'s* mathematical knowledge for teaching framework (*e.g.,* Ball, Thames, & Phelps, 2008), has yet to reach adequate clarity and consensus in the field. Recently, various scholars have worked to further conceptualize and describe the mathematical horizon (*e.g.,* Jakobsen, Thames & Ribeiro, 2013; Figueiras *et al.,* 2011; Zazkis & Mamolo, 2011). In this communication, we identify some limitations in the ways such knowledge has thus far been described and offer an additional form of potential impact of horizon content knowledge on the work of teaching.

The imagery of the mathematical horizon is particularly pertinent and, we believe, a powerful metaphor for teaching. Teachers need to know not just the current mathematical landscape; they also need to have a broad perspective of the discipline and a sense of what is still to come. Ball, Thames and Phelps (2008) originally described horizon content knowledge as:

> an awareness of how mathematical topics are related over the span of mathematics included in the curriculum. First-grade teachers, for example, may need to know how the mathematics they teach is related to the mathematics students will learn in third grade to be able to set the mathematical foundation for what will come later. It also includes the vision useful in seeing connections to much later mathematics ideas. (Ball, Thames, & Phelps, 2008, p. 403)

This definition appears to be relative to the grade a teacher teaches: if mathematical content can be pictured as an upcoming landscape, then the content that lies in the "horizon" necessarily differs for a 1st grade teacher and an 11th grade teacher. This description has caused some confusion about what horizon knowledge entails, since the construct could be partitioned into two aspects: (1) a *curricular mathematical horizon,* which would be school mathematics content beyond a teacher's current grade level; and (2) an *advanced mathematical horizon,* which would relate to more advanced mathematics. In their work on advanced mathematics knowledge, Zazkis and Mamolo (2011) similarly describe this distinction as the difference between the learners' (curricular mathematical) horizon and the teachers' (advanced mathematical) horizon. More recently, Jakobsen, Thames and Ribeiro (2013) attempt to clarify that horizon content knowledge "is distinctively relevant to the conversation about 'advanced' mathematics courses [...] [and] the meaning of 'related' [in the original definition from Ball, Thames, & Phelps (2008) above] was not meant to be about the curricular development of the content" (p. 5). They (along with Figueras *et al.,* 2011) also point to some other potential differences in perspective on the mathematical horizon, such as whether Klein's (1932) notion about elementary mathematics from an advanced standpoint, or an inverted version, advanced mathematics from an elementary standpoint, is more useful. Here we consider horizon content knowledge in the sense of the advanced mathematical horizon.

Fundamental to the discussion about teachers' mathematical knowledge is the assumption that such knowledge is relevant for and contributes to the work of teaching. In particular, we aim to look at *how* knowledge of the mathematical horizon can contribute to this work. We make a rudimentary distinction between the planned and the in-action work of teaching. By the *planned* work of teaching, we mean the more *proactive* considerations that typically arise during the planning process, such as the sequencing of content, activities, planned questions, *etc.* By the *in-action* work of teaching, we mean the more *reactive* aspects of instruction that inevitably arise in the moment during the complex task of teaching, such as interpreting students' solutions, responding to their questions, and exploring a student's mathematically interesting approach.

Thus far, the descriptions and depictions of horizon knowledge present in the literature focus on its influence on teachers' in-action teaching practices. For example, Jakobsen *et al.* (2012, 2013), in their definition, specifically connect horizon content knowledge with the in-action teaching role of enabling teachers to "hear" students and judge the importance of their ideas and questions. Similarly, Zazkis and Mamolo's (2011) examples of advanced mathematical knowledge "focus on the teacher and his or her response to students' work and questions [...] [which] can influence a teacher's pedagogical moves *in the moment*" (p. 10, emphasis added). Extant illustrations similarly contain this focus on in-action responses. In one vignette, Jakobsen *et al.* (2012) present a student who provided an unconventional split of a rectangle into four equal parts (see Figure 1). The student claimed that while her picture was not quite accurate, she could move the lines such that the areas were equal. In this case, the authors contend that a teacher's response could draw on a flexible understanding of continuity and an intu-

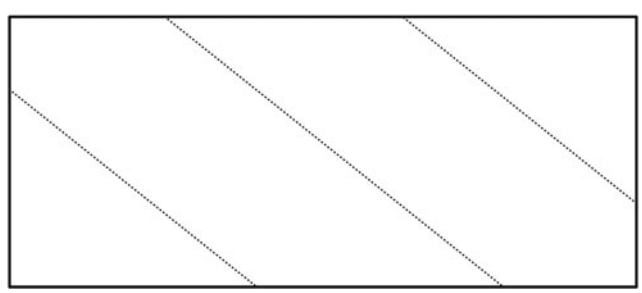

Figure 1. An unconventional split of a rectangle (adapted from Jakobsen et al., 2012, p. 4637).

itive feel for the intermediate value theorem in the context.

While these authors' intentions may be that horizon knowledge impacts other facets of teaching, the descriptions and examples thus far highlight only the contribution of horizon content knowledge to a teacher's in-action practices.

The mathematical horizon: impacting the work of planning

While the ability to react and respond to students appropriately and with mathematical integrity in the moment is certainly an important component for improving the practice of teaching, we explore the possibility of horizon content knowledge contributing to other aspects of teaching, as current examples may limit our understanding of the full impact of knowledge of the mathematical horizon. Below, we present two teaching vignettes, one each from middle and high school, that, we argue, demonstrate how teachers' advanced mathematical or horizon knowledge can also impact the decisions teachers make during planning.

Vignette 1

Mr. Reese is a high school geometry teacher and is planning an introduction to triangle properties. In earlier grades, students have learned that the interior angle sum for a triangle is 180°; a high school level geometry course, however, should expose students to more rigorous justification and proof of this property. Mr. Reese decides that it is especially important to him that students critically engage with this content and not disengage when proving something they have already been told is true. Reflecting on his own knowledge of triangles and of Euclidean and non-Euclidean geometries, Mr. Reese decides to use the fact that *not every triangle* has interior angles that sum to 180°; this is only true for planar triangles in a Euclidean geometry. Based on this knowledge, Mr. Reese plans to use a globe. He intends to: (1) ask students how to draw a triangle (based on the definition) on the globe; (2) draw a triangle on the globe representing a quarter of a hemisphere (Figure 2); (3) ask students about the interior angle sum of the triangle; and (4) have students verify that, in fact, each of the three angles are right angles, summing to 270°, not 180°.

Mr. Reese anticipates that students will think he has "cheated" in some way or another. He hopes this will help

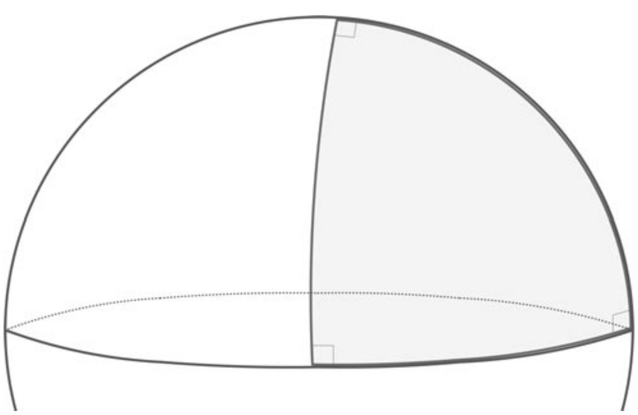

Figure 2. A triangle on a sphere.

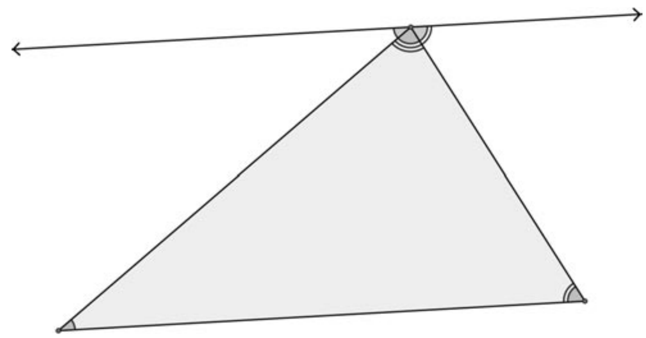

Figure 3. Planar triangles have an interior angle sum of 180°.

engage the class in more meaningful thinking about triangles on flat surfaces, because it is not the line segments themselves that are curved (in fact, they are straight), but rather the surface on which the segments are drawn. He plans to ask students what it is about *flat* surfaces that make triangles have an interior angle sum of 180°, introducing a task that requires students to justify the interior angle sum by constructing parallel lines. This proof relies on the property that if two lines are parallel then the alternate interior angles are congruent, demonstrating that the three angles in a triangle are congruent to three angles that form a linear pair, which is 180° (Figure 3).

Mr. Reese hopes that this use of cognitive conflict, which stems from the Piagetian notion of equilibration, will help engage students with the mathematics, as well as have students learn more rigorous justification and proof for the interior angle sum of planar triangles being 180°. Mr. Reese does not intend to discuss additional properties of spherical geometry, in which no lines are parallel. However, he recognizes that this fact causes the argument that is valid for planar geometry to be invalid for spherical geometry, and makes a mental note in case students raise the issue in class.

Mr. Reese's knowledge of non-Euclidean geometry allowed him to connect an example of a triangle in spherical geometry to his high school geometry classroom, informing his planning, specifically around the work of *engaging* students in critical thinking about planar triangles and the assumptions of Euclidean geometry. While the primary content of the lesson was still about planar geometry, his horizon knowledge of non-Euclidean geometry nonetheless impacted his planning in specific ways and for specific pedagogical purposes.

Vignette 2

Mrs. Billups is a middle school teacher and is planning a lesson on solving simple linear equations as an introduction to algebraic reasoning. As an experienced teacher, Mrs. Billups is aware of how students frequently abuse the idea of cancellation when solving equations, inappropriately crossing out numbers and expressions. She wants her lessons to help students make sense of the simplification process that results in cancellation. From her own coursework in mathematics, Mrs. Billups learned that the algebraic structure of a group, one of the foundations for algebraic reasoning, is defined by having four properties hold for the set and operation under consideration: closure, associativity, identity element, and inverse elements. For even a simple linear equation, Mrs. Billups recognizes the complexity of reasoning involved in which all four of these properties contribute to isolating the variable, and that the identity and inverse elements play a critical role in how simplification results in cancellation. While she has no desire to teach her students abstract algebra, Mrs. Billups does want to make sure important mathematical ideas are raised so that students will be more careful and thoughtful about when numbers and expressions cancel. After she has covered some of the basic ideas about solving simple linear equations (*e.g.,* do the same operation to both sides of the equation), she decides to pose the following task:

> Using algebraic methods, solve: $x + 5 = 12$ and $5 \cdot x = 32$

As students solve these problems, Mrs. Billups plans to have students discuss the following questions to probe their understanding: (1) Why was your first step to subtract 5 (*i.e.,* add negative 5) or divide by 5 (*i.e.,* multiply by 1/5)? (2) What is the relationship between the number in the original equation and the operation you performed on both sides?; (3) After the first step, what number is ultimately being added to x (*i.e.,* 0, as in $x + 0$) or multiplied by x (*i.e.,* 1, as in $1 \cdot x$) on the left side?; and (4) What is special about those numbers (*i.e.,* 0 and 1) in addition and multiplication? Mrs. Billups hopes that slowing down and asking students to think about the underlying mathematics behind isolating a variable (foundationally, the mathematical property that operation with an inverse element produces the identity element for a group) will help lead students to avoid cancelling in situations that do not make sense (*e.g.,* accidentally turning $5x + (x + 5)/5 = 4$ into $6x = 4$ instead of $5.2x + 1 = 4$) and to approach solving equations more critically than procedurally.

Mrs. Billups used her knowledge of abstract algebra to inform her instructional planning, specifically to incorporate student *reasoning* about solving equations into her lesson. While she did not teach students the four group axioms, her knowledge of group theory gave her a framework to select questions that would lead students to reflect on important mathematical ideas, such as the additive and multiplicative identities and an element's inverse under a given operation. Her goal in highlighting these fundamental ideas within the equation solving process was to turn mathematics into a reasoning and sense-making activity for students, helping them to justify more carefully and precisely the process of cancellation or simplification, as well as to prepare them for future uses of inverse and cancellation.

Conclusion

In this communication, we have discussed prevailing conceptions of the mathematical horizon and looked at some current examples of its impact on teaching practices. Previous literature has documented an impact on the in-action work of teaching, primarily teachers' responses to comments and queries from students. As an extension of this work, we have described opportunities where knowledge of the mathematical horizon could be informative and transformative to the work of planning in mathematics education. The two vignettes presented in this communication illustrate ways that horizon knowledge may impact teachers' planned classroom practices. In each of these vignettes, knowledge of the mathematical horizon was the impetus for additions or alterations to the teachers' instructional plans. The teachers' decisions were rooted in various pedagogical purposes, such as engagement and student reasoning. Horizon content knowledge is an important aspect of teachers' subject matter knowledge: understanding how it impacts teachers' work in the classroom, both in in-action and in planned practices, informs our conception of and work regarding this domain of subject matter knowledge.

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From the archives

Editor's note: *The following exchange is extracted from a dialogue on ethnomathematics between Marcia Ascher and Ubi D'Ambrosio (1994), published in FLM14(2). Marcia Ascher died earlier this year.*

Marcia: Most mathematicians believe that what gives mathematics its power is the manipulation of symbols standing for anything and having no context. In a problem, x is x and nothing more. However, I believe mathematics could be even more powerful by retaining some recognition of what the symbols stand for and gearing the approaches used to that. If, for example, you are dealing with x's referring to numbers of human beings, you should only be seeking integer solutions and selecting solution methods accordingly.

Ubi: … something that has a past, and seems more natural. The mind of a child seems more receptive to the idea of number as attached to something else. This is true too in many cultures. This has been characterized as a lack of capability for abstract thinking, which is an obvious historical perversion. Simply said, we use numbers according to what we want to design or classify with the number. The fact that we deal with numbers by insisting on their abstract meaning may reveal a distortion in our civilization, the distortion of decontextualization, essentially, which is a form of reductionism.

This criticism leads to what some have called a holistic view.

Marcia: I also think that, especially in the education of children, values are being taught through our emphasis on contextless numbers. Very often school examples are phrased in terms of money. In the examples, numerical equivalents are stated for labor, for objects, for health care, for food, *etc.* Then the students are told to focus on and manipulate only these numerical equivalents. As such, it is more than just teaching the mathematics for itself; it is teaching the students to view the quantities and their manipulations as contextless and divorced from meaning. I think this is a very important issue for people in education to consider.

Ubi: The absolute suppression of context, and the quantification of values for comparison, in order to value something more than another, is what I consider a sign of the philosophical damage done to modern thought; it leads to a world deprived of human values, even of human feelings. Everything valuable has to be quantified.

Marcia: One of my favourite examples is from linear programming in which you are minimizing or maximizing some function under linear constraints. One early application was the creation of a diet for pigs. The constraining equations were their daily nutrient requirements and then, using current prices, the most economical diet was found. But the pigs wouldn't eat it! There are a number of similar examples where the mathematics was absolutely correct but the solutions couldn't be used because the audiences and their tastes and values were ignored during the mathematical formulation.

Ubi: This has also much to do with the problems we face with the environment. The way we look at our behaviour, in general, trying to quantify it, without attaching any value—in the ethical sense—to the quantifiers, is probably the main reason why we have been so unwise in dealing with the environment.

Marcia: I also think that this is one of the causes of the dislike of mathematics among young people. There is a feeling, often articulated by those alienated by mathematics, that it is emotionless and lacks feeling. Even students who like mathematics seem to associate it with a certain inhumanity.

Ubi: We do mathematics systematically out of context. We put ourselves inside the discipline, look at a few parameters, and apply the solving techniques according to those parameters. But we know that the real situation is so complex. Has such a multiplicity of parameters, that in simplifying it we inevitably limit the overall view.

Marcia: I agree that this is a very important issue. Particularly in school settings, people try to make the ideas relevant to the students' world by creating word problems or story problems. I generally find that students can more easily grasp the more theoretical topics like number theory. It is at the interface between mathematics and unreal "real" problems that the greatest confusion and over-simplification occurs. It is exceptionally difficult to take abstract mathematical statements and apply them in a real world context. It would, perhaps, help if more time was devoted to teaching about the creation of mathematical models. Then students could learn to be more critical of them and could learn to distinguish between the validity of the mathematics and the sufficiency of the model. When mathematical models were based primarily on physical theories, the omitted variables represented smaller effects and so what resulted was a first approximation. As the applications are moved into social, economic, or global settings, the problems become more difficult. There are far more variables, far less clarity on the hierarchy of their effects, and more value judgements are involved in deciding which effects to consider or ignore.

Ubi: Putting all this in the context of ethnomathematics: this is the reason I call ethnomathematics a program in the history and philosophy of mathematics. It's a program with a holistic approach, much broader than current historiography and epistemology which have clearly selected only a few variables for analysis. This program has implications for pedagogy. I think we do not disagree on this.

Marcia: I very much agree. This is why I believe there are two distinct aspects of ethnomathematics which are definitely related but sometimes have to be more clearly separated. One aspect is seeking understanding of the relationship between mathematical ideas and culture. For this understanding, more investigations, study, and research, and more thinking about these historical/philosophical issues are needed. Once there is this deeper understanding, *then* can come the educational aspect which addresses the question of how to incorporate it; how should we or how do we modify education? The two aspects share in seeking a broader view of mathematics in which mathematical ideas are not restricted to any one group, profession, culture, or historical time. But they differ in that modification of education depends largely on the goals of the educator and the setting of the education. This is not a question of research but of clarifying one's goals specifically enough to develop methods or practices that move towards those goals.

[…]

Marcia: And there are the other aspects we were talking about earlier that have been greatly downplayed because of the extreme rationalistic approach. For example, in the non-Western math course that I teach, one of the topics is magic squares. I talk about all of the time and devotion that was put into constructing the squares by people in diverse cultures and that, when they did this, it was to order the universe or with some other religious motivation and feeling. Students are very surprised that such feelings are involved because most things in school are presented as rational—and the emphasis is that rational is better. In class, we often get into a discussion as to whether they believe in magic. They immediately say "No!" because in college you are not allowed to admit this, you are supposed to be scientific. But then there emerge examples of lucky pencils for exams. Special objects or words associated with certain sports events. And so on. All of this is a big problem *for* mathematics because so many people point to it as the ultimate in rationality.

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