## Communications

## Images, fractions and their intricate links: true, false or somewhere in-between

DAVE HEWITT, DAVID PIMM

Dave Hewitt: I recently observed a mathematics lesson being taught by a trainee teacher. This lesson had been planned by another teacher and was used by several teachers within the school. It seemed the school was proud of this lesson, as it was one based on a 'Mastery' approach, heavily influenced by teaching approaches from Shanghai. There were many things which I considered positive about it, but one part troubled me. The teacher was using images of shapes divided up into parts potentially to represent fractions. Figure 1 shows one particular group of images.

During the lesson, there was a strong emphasis on the need for dividing shapes into equal regions. The students were asked to say whether each of the drawings in Figure 1 represented a quarter of the rectangle area. After a while, the teacher announced (purported) answers to each of these and said that the latter two drawings did not represent a quarter (but the first two did) and moved on to other questions. One student in the class asked about the fourth drawing, saying that he thought a quarter was shaded. The teacher replied, "Are there equal parts?" The student replied, "No", and the teacher said, "Well, there you are then", and moved on to the next bit of the lesson.

It seemed to me that the emphasis had been so much on "Are there equal parts?" in the drawings that this dominated other mathematical considerations. At the end of the lesson, I sat in on a debrief between the regular class teacher, who had been observing, and the trainee. To my surprise, nothing was mentioned of this incident. So, I redrew the fourth picture and asked the trainee teacher whether there was a quarter of the rectangle shaded. They felt it was not, as there were not equal parts. It took me a while, by comparing the


Figure 1. Four different divisions of a rectangle
first and last drawings in Figure 1, and pointing out that the shading had not changed between the two, to convince not only the student teacher, but also the class's regular teacher, that there was nevertheless one quarter shaded. I was troubled by the unquestioning acceptance of this received lesson plan, for it seemed to come with the apparent authority of being based upon a Shanghai approach.

David Pimm: In Figure 1, for me the first one is $1 / 4$; the second is $2 / 8$ rather than $1 / 4$ (but does not actually match the seemingly required image of it being divided into four (or eight) equal parts); the third one is $1 / 8$ (but also does not match ...) and the fourth one is $1 / 4$ (but also does not ...). But that is me focusing on relative areas and dynamically moving the shaded area around the 'whole' and simply counting, while ignoring many of the internal lines. The images that are termed 'false' are static for me. It may be the presence of the additional lines that is (mathematically) distracting, yet not, by themselves, falsifying.

When you told me about this, and that you had seen other instances on-line, I found a link [1] to an NCETM video of a visiting teacher from Shanghai, Ms Dai, model-teaching in England with a Year 5 (c. 9 years old) class and a broad audience of teachers present, a common situation in China. She was just starting off by recalling depictions of fractions from the previous lesson, which had been more centrally on diagrams having equal parts (a video which unfortunately was not available on-line) and displayed an image on the screen for the class to consider (see Figure 2a) along with the question: "True or False (Can these fractions show the colored parts?)".
Her spoken justification for 'it' being false was, "Because the whole rectangle is not divided into equal parts" (and then she put up the second, revised image, Figure 2b). This seemed to me to comprise a similar flaw, in that the relevant shaded area certainly is $3 / 5$ in both instances, not least because the coloured area and the whole rectangle have not altered and so the fraction does 'show the coloured parts'. So, just as in your fourth example in Figure 1, changing the division of one line inside the image but outside of the designated region does not, for me, affect the fractional relationship of the part to the whole. Later, in discussing with the adult audience after the lesson, Ms Dai talked briefly about the importance of "dividing a whole into equal parts".

It is not apparent in the video (as the numerical fractions


Figure 2. (a) An initial and then (b) a revised screen image from Ms Dai's lesson.
were already created beforehand），but the Chinese conven－ tion for describing fractions is to name the denominator first and then declare the numerator as a cardinal．The Chinese structure for written fractions is：Denominator＋分（part（s））之（of）＋Numerator．Thus，五分之三 is，in character sequence， ＇five（equal）parts of（which）three＇，which in English would be＇three fifths＇．The writing of the fractional notation is commonly done first by writing the denominator，then by drawing in the horizontal line and finally by adding the numerator，an action which matches the temporal order of what is being said［2］．Many teachers in China do teach and require the symbolic construction in this order［3］．

In regard to shaded rectangles rather than fractions，I also recall Lakatos＇account of a local－but－not－global counterex－ ample（one that violated a purported proof，but not the actual statement of the claimed conjecture）：

Teacher：let me introduce the following terminology．I shall call a＇local counterexample＇an example which refutes a lemma（without necessarily refuting the main conjecture），and I shall call a＇global counterexample＇ an example which refutes the main conjecture itself． Thus［addressing Gamma］your counterexample is local but not global．A local，but not global，counterex－ ample is a criticism of the proof，but not of the conjecture．（1976，pp．10－11）
I wonder to what extent this is related to our discussion．It is not exactly apt，not least because there is no specific coun－ terexample to a general theorem．But if we imagine the ＇theorem＇states that the fraction of the shaded area to the whole in the video is three fifths（which it unquestionably is！），then what the teacher perhaps seemed to be indicating is that the image does not justify that．But that is，at most，a local problem，not a global one．And as Lakatos instantiates， a global counterexample can also throw into dispute defini－ tions of pertinent objects．

A tale about three stages in a baseball umpire＇s career came to mind．The novice says，＂I call it the way it is＂；the medium－experienced one says，＂I call it the way I see it＂；the expert says，＂Until I call it，it isn＇t anything＂．So，if the image is to be called＇false＇in regard to it not being $3 / 5$ ，what does that say about the image？That it isn＇t anything？

So，is this a language issue or is it a visual representation issue（where the question and answer are supposedly present in the same image，like a theorem being both conjecture and proof）？Or is it actually a question of the co－ordination（or otherwise）of language and image？What is representing what and what does the representation（including the nota－ tion）represent？And what aspects do the numerator and denominator reflect of the image？

Dave Hewitt：Your argument about this being akin to a local－ but－not－global counterexample is an interesting one．As you point out，the issue is really the relationship between the visual representation and the verbal／notational statement（in the instance of Figure 1，of a quarter）．If the emphasis is on the denominator，that being the first thing which is written in China and certain other Asian countries（including Japan， Myanmar and Korea）－as well as the first thing spoken－ then this appears to be a key emphasis of the lesson I
observed and also，presumably，of the lesson taught before the one which appears on the NCETM website．

This is supported by the teacher＇s emphasis on whether or not there are equal parts in the visual representation．My own reaction to considering each of the drawings in Figure 1 was primarily on the area（s）which are shaded．I make assumptions about where the shading starts and finishes；I assume the shading in the fourth drawing ends half－way down the vertical side of the overall rectangle and also half－ way along the top horizontal side of the rectangle．

With these assumptions，I say with confidence that the area shaded is a quarter of the whole．I am treating the frac－ tional notational representation $1 / 4$ as a single object，as a representation of＇a quarter＇．This is different from treating $1 / 4$ as meaning＇one out of four＇，where the numerator and denominator are considered separately．If I were viewing the drawing on being asked＂Does this represent one out of four？＂，I might respond，＂Yes，but not those four＂．Within the act of counting，one of the important awarenesses is that the choice I make in the order of the count does not affect the result［4］．Likewise，if I want one out of four，then I would like the result not to be affected by which one I choose．One out of four should still be OK whichever one I choose．In the fourth drawing in Figure 1，it does matter which one of the four areas I choose（and consequently shade in），whereas with the first drawing，it does not．

This makes me think about where attention is being placed．In deciding about the area shaded in the fourth dia－ gram in Figure 1，and also in the video of the Shanghai teacher＇s lesson，I pay attention to the boundary of the area in relation to the whole．Given the assumptions I make about where the shading stops，I decide on the fraction shaded of the whole without any consideration of other lines which may or may not be drawn．Thus，I would happily say that a quarter has been shaded in all of the diagrams in Figure 3.

My attention is not with the lines，but with the shading and the overall shape．If I shift my attention to the lines， rather than the shading，then none of the ways the lines have been drawn in Figure 3 represent the whole area being divided up into quarters．Mason（1989）talks about a delicate shift of attention，and here this shift in attention from a sin－ gle area（compared with the larger rectangle）to the positioning of the lines dividing up the rectangle changes my awareness from＂Yes，it is one quarter（of the whole）＂to ＂No，they are not quarters（dividing the whole）＂．

Notice that my language changes from singular to plural （although，in Chinese，I gather there is no marking of singular


Figure 3．A quarter is a quarter，no matter where other lines are drawn．
or plural，either on nouns or verbs）．In the first case，I am considering only one area and its relationship to the whole， whereas with the second I am considering all the parts simultaneously．The fact that，in China，the denominator is both written and said first might indicate that attention is with the whole in all of its parts，which must be equal to be so named，and then the numerator is just a matter of how many of those parts have been shaded．

If this is the way that attention is placed，then the dia－ grams we have focused on fail at the first hurdle；they fail through consideration of the denominator alone．If my atten－ tion is with the singular，the single shaded area，then it is，of course，a quarter．In fact，English has particular words for a few special fractions which do not explicitly state the numerator or denominator，of which＇quarter＇is one．Neither ＇one＇nor＇four＇make an appearance in this word，which they do if it was said＇one fourth＇．It appears there is not a general linguistic equivalence in Chinese of such a singular word， although there is a specific character in relation to time：刻 （quarter of an hour）．

The last point I will make is one which concerns the direc－ tion of the relationship between the visual representation and what is written or said．If it is a one－way relationship，then which way is the relationship arrow going？There might be a difference between considering whether $1 / 4$ represents the diagram，or whether the diagram represents $1 / 4$ ．Or do they both represent the concept of quarter？

David Pimm：What you have raised made me wonder how that relates to or compares with the English notion of＇part＇ and＇whole＇？Must there be equal parts in place before＇a part＇can even be discussed（see Figure 2）？Is the shaded area ＇a part＇or＇three parts＇？

I was also taken with your distinction between singular and plural．This has made me realise that there is significant difference between＇three fifths＇and＇three－fifths＇．In Eng－ lish，the first one is like＇three chairs＇or＇three diamonds＇， namely a（whole number）numerical quantifier（larger than one）followed by a plural noun．But the second one is a sin－ gle entity．And the mathematical notation（as in so many other instances）does not distinguish between them．So，per－ haps，in the sense of Figure 2a，it is not three fifths．But， nonetheless，it is three－fifths（the singular relationship of the part to the whole）．And the same could be true of the differ－ ence between＇one quarter＇and＇one－quarter＇．Going back to Lakatos，it might mean that while Figure 2a could be seen as a（linguistic？）local－but－not－global counterexample if $3 / 5$ ＇means＇three－fifths，while if $3 / 5$＇means＇three fifths，it could be seen（in Chinese，at least）as a global counterexample（at the diagrammatic level）．

One more thing．Consider $3 / 5>4 / 7$ and then say it aloud in English．The difference between two standard possibilities is reflected in the corresponding singular or plural verb．Do we say，＂three fifths are more than four sevenths＂or do we say， ＂three－fifths is more than four－sevenths＂？So，when you transcribe someone talking about fractions in English，the inclusion of the hyphen or not should match whether the verb spoken is singular or plural［5］．

Dave Hewitt：Your remark about the difference between ＇three fifths＇and＇three－fifths＇is very helpful for me．It has reminded me of a lesson I have often given to eleven－year－ olds，where I tell them I am amazingly good at division．I claim that they can give me any division to do and I will write the answer on the board within three seconds．I invite someone to give me a difficult division－say，three hundred and forty－eight divided by seventeen－and I pretend to think for a second and then write $348 / 17$ ，saying＂three hundred and forty－eight seventeenths＂［6］．The purpose of this little inter－ action is for these students to become aware that the notation they tend to view as a fraction，as a single number，is also representing a division．The fraction notation is both an answer and a question，an object and a process，a singular and a plural．

## Acknowledgements

We are grateful for assistance from our reviewers and also from Qiang Lin and Pauline Tiong，particularly with regard to semantic and syntactic aspects of Chinese characters．

## Notes

［1］https：／／www．ncetm．org．uk／classroom－resources／lv－year－5－shanghai－ showcase－lesson／
［2］The Chinese character 分 is basically equivalent to＂divide／separate （verb or noun）＂．In general，the items divided do not have to be equal（or equivalent）parts．Where does the tacit or explicit sense of sameness come from，when dealing with fractions using this character？The Chinese（non－ compound）characters for＇denominator＇are 分母 and for＇numerator＇分子． Intriguingly，母 means＂mother＂and 子 means＂son or child＂．
［3］For more on this，see Bartolini Bussi et al．，2014，and my response， Pimm，2014，not least about the connection between unit fractions and ordi－ nals in English．See also Pimm and Sinclair， 2015.
［4］＂In order to count，you have to know what counts．＂
［5］This is because mathematical notation is not part of English，or any other natural language，but when people are speaking mathematics，it is all within natural languages：for more on this，see Pimm（2021）．
［6］See also the topologist William Thurston＇s（1990，p．847）reminiscence about his fractional childhood．

## References

Bartolini Bussi，M．，Baccaglini－Frank，A．E．\＆Ramploud，A．（2014）Inter－ cultural dialogue and the geography and history of thought．For the Learning of Mathematics 34（1），31－33．
Lakatos，I．（1976）Proofs and Refutations：The Logic of Mathematical Dis－ covery．Cambridge，UK：Cambridge University Press．
Mason，J．（1989）Mathematical abstraction as the result of a delicate shift of attention．For the Learning of Mathematics 9（2），2－8．
Pimm，D．（2014）Unthought knowns．For the Learning of Mathematics 34（3），15－16．
Pimm，D．（2021）Language，paralinguistic phenomena and the（same－old） mathematics register．In Planas，N．，Morgan，C．\＆Schütte，M．（Eds．） Classroom Research on Mathematics and Language：Seeing Learners and Teachers Differently，pp．22－40．London：Routledge．
Pimm，D．\＆Sinclair，N．（2015）The ordinal and the fractional：Some remarks on a trans－linguistic study．In Sun，X．，Kaur，B．\＆Novotna，J． （Eds．）Proceedings of the 23rd ICMI Study 23 ＇primary mathematics study on whole numbers＇，pp．354－361．Macau：University of Macau．
Thurston，W．（1990）Mathematical education．Notices of the American Mathematical Society 37（7），844－850．

# Preparing students to reason about (existential) risk: lessons from the pandemic 

NENAD RADAKOVIC

As we search for the meaning in the current global pandemic of COVID-19, there are different ways to make sense of it. One way is to see this pandemic through the lens of risk [1]. The question for mathematics education in particular then becomes how we teach about risk and how we foster sound decision making about risk. It is helpful to introduce the concept of existential risk which can be defined as any risk that can endanger humanity as a whole and its existence. Such risks include global pandemics, climate change, nuclear war, etc. In short, we can think of existential risk as any risk that could "lead to human extinction or civilizational collapse" [2]. In this communication, I outline some of the lessons regarding risk and mathematics education arising from the ongoing COVID-19 pandemic.

## There is a tension between risk education and risk communication

Understanding the structure and dynamics of existential risk events involves substantial mathematical content (see e.g., Radakovic \& Chernoff, 2020). The mathematical content and practices necessary to mathematize risk are already present in the mathematics education literature. Such content includes exponential growth, quantity and number sense, representation and analysis of data. Beyond content, critical skills and dispositions are also necessary.

This content would be sufficient for risk education-students would have the tools needed to analyze data and draw their own conclusions from it. However, when faced with existential risk, it is not enough to analyze the data. We must be concerned with the wellbeing of our students. Communicating about risk should include information about preparation for, and mitigation of, foreseeable risks. Mathematization of risk must be accompanied by learning about resources and safety. For example, projecting exponential growth of COVID-19 cases in Florida during an upcoming school break might not be sufficient to convince students to change their travel plans. In this case, it is better to concentrate on risk communication (Spiegelhalter \& Gage, 2015). Furthermore, we should aim for transparent risk communication which requires data representations that are "easy to understand and [present] the facts objectively" (Bodemer \& Gaissmaier, 2012, p. 623).

## Risk involves coordination between knowledge, feelings, and beliefs

More than a year into the pandemic, we understand that individuals' behavior and decision making about COVID-19 risk varies and is influenced by their prior experiences, feelings, and beliefs (Levinson et al., 2012). For example, because of my prior traumatic experiences, I tend to be very risk averse. This shows the importance of affect in making
decisions about risk. Slovic et al. (2010), influenced by dual process theory (Kahneman \& Frederick, 2002), suggest that human reasoning about risk consists of two cognitive systems: one is the experiential, intuitive system that helps us make quick assessments about the safety of a situation (a gut feeling), the other is the analytic system that helps us evaluate our thinking. Slovic et al. do not want to fall into a trap of deficit theorists by favouring the analytic system over the experiential. According to research, affect that stems from the experiential system helps us to make decisions quickly in an uncertain and dangerous world:

We now recognize that the experiential mode of thinking and the analytic mode of thinking are continually active, interacting in what we have characterized as the "dance of affect and reason" (Finucane et al., 2003). While we may be able to "do the right thing" without analysis (e.g., dodge a falling object), it is unlikely that we can employ analytic thinking rationally without guidance from affect somewhere along the line. Affect is essential to rational action. (p. 24)
An example of an affective heuristic is the feeling of dread (Fischhoff et al., 1978), which has been shown to be a major predictor of public perception of risk for a wide range of phenomena (Slovic et al., 2010). In other words, as much as I was guided by the information about the pandemic and the mathematics behind it, my aversion of risk and need to be in control played a big role in my decision to stay home and shop online.

How does this translate to teaching and learning? One possibility is taking a self-based narrative approach (Breen, 2004; Chapman, 2020) in order to help individuals to make sense of their own sense-making. In order for learning of risk to make sense, students should start from self-study and reflection on their own ways of coping with risk. This is necessary before we start introducing students to analytical and mathematical tools to deal with risk.

## Problem solving about risk is epistemologically different from mathematical problem solving

Risk theorists have identified one crucial way in which risk differs from scientific and mathematical research. When dealing with risk, Type II errors (false negatives) are more serious than Type I errors (false positives). In traditional science, the opposite is the case; Type I errors are to be avoided. Most of Western science and mathematics has been focused on avoiding Type I errors (something that is not true but is classified as true). An historical example of Type I error is the one committed by Sciaparreli and later by Lowell in asserting that there are canals on Mars, while the canals were nothing more than figments of their imaginations. In the study of risk, however, we are more concerned with avoiding Type II errors (something that is true, but is classified as not true). In other words, we hope to be wrong about our assertions but prepared to deal with a wide range of possible catastrophes. For example, in mid-May 2020, I was hoping that the mathematically reasonable estimate of 100,000 US coronavirus deaths by the end of May was wrong and that the number would be much lower-unfortunately, mathematical models [3] predicted the number
correctly. Even if the models had overestimated the risk, it would have been better to take steps to limit the fatalities than to ignore the predictions.

## Problem solving about risk is affectively and aesthetically different from mathematical problem solving

The first time I came up with a way to generate an arbitrary number of primitive Pythagorean triples, I was ecstatic. I then shared the proof with my friends who challenged my thinking by checking whether the generated triples are indeed Pythagorean. The satisfaction that comes from solving mathematical problems and tasks is well documented. But when predicting risks, as we are focused on avoiding Type II errors, we do not want to be right; being right about the severity of a catastrophic event is not satisfying in the same way as solving a mathematical task can be.

The sense of being right and being pleased about a discovery is difficult in case of predicting a catastrophic event. I cannot deny that I tried really hard not to think "I told you so" when cases of COVID in South Carolina (where I live) started to grow exponentially (again) after safety measures were eased. The affective and aesthetic properties of riskbased decision making is something that we should take into consideration when teaching about risk and also something that needs further research.

## In case of life altering/catastrophic events, the focus is on well-being and justice and not on mathematics education

Learning about a crisis should include learning about the structural injustices revealed by the crisis, even at expense of mathematical analysis. Shah (2019) poses the question of whether we should teach less mathematics if that means that students have more opportunities to learn about climate policy, health care reform, and public ethics. In other words, mathematics education should "prioritize a goal of justice for minoritized groups and do so with urgency, even if it means there should be less mathematics education" (p. 31). We should teach about how the pandemic reveals structural inequities and how to mobilize against them. It is also ethically problematic to see the crisis and tragedy as an opportunity to learn exciting mathematics or to concentrate on teacher online effectiveness, when people are sick and dying. Instead, the focus should be on the wellbeing of students.

## Conclusion

At the risk of sounding alarmist, it is important to note that the pandemic we are experiencing is not the last existential threat we will face. In order to make sense of such risks, we have to look into ourselves first. John Mason has described mathematical research as learning about ourselves in relation to others (Mason, 2010). This is particularly relevant for the research on understanding of existential risk. This is why studying ourselves and our relationship to risk is an appropriate research and pedagogical methodology. I invite readers to reflect on the lessons I present here and to identify their own lessons. This will allow educators to reach for greater under-
standing through sharing our feelings, experiences, and observations. As we teach and learn about the pandemic during the pandemic, it is helpful to examine ourselves in relation to the world and each other. We can then lay a foundation of how to guide our students to be self-reflective and more capable of making the meaning of and decisions about risk.

## Notes

[1] There are many definitions of risk. The definitions include risk as a probability of an unwanted event and in terms of expected utility (this definition involves the coordination of probability of an event with its impact). For discussion of other definitions of risk, see Hannson (2012).
[2] As described by the Centre for Study of Existential Risk at https://www.cser.ac.uk/news/covid-19-update/
[3] Such as those at Worldometer, https://www.worldometers.info/coronavirus/country/us/

## References

Bodemer, N. \& Gaissmaier, W. (2012) Risk communication in health. In Roeser, S., Hillerbrand, R., Sandin, P. \& Peterson, M. (Eds.) Handbook of Risk Theory: Epistemology, Decision Theory, Ethics, and Social Implications of Risk, pp. 623-680. Berlin: Springer.
Breen, C. (2004) Opening the space of possibility-for myself (and others?) For the Learning of Mathematics 25(1), 24-27.
Chapman, O. (2020) Mathematics teacher educators' use of narrative in research, learning, and teaching. For the Learning of Mathematics 40(0), 21-27.
Fischhoff, B., Slovic P., Lichtenstein S., Read, S. \& Combs B. (1978) How safe is safe enough? A psychometric study of attitudes towards technological risks and benefits. Policy Sciences 9, 127-152.
Finucane, M., Slovic, P. Mertz, C.K., Flynn, J. \& Satterfield, T. (2010) Gender, race and perceived risk: the 'white-male' effect. In Slovic, P. (Ed.) The Feeling of Risk: New Perspectives on Risk Perception, pp. 21-36. London: Earthscan.
Hannson, S.O. (2012) A panorama of the philosophy of risk. In Roeser, S., Hillerbrand, R., Sandin, P. \& Peterson, M. (Eds.) Handbook of Risk Theory: Epistemology, Decision Theory, Ethics, and Social Implications of Risk, pp. 28-54. Berlin: Springer.
Kahneman, D. \& Frederick, S. (2002) Representativeness revisited: attribute substitution in intuitive judgment. In Gilovich, T., Griffin, D. \& Kahneman, D. (Eds.) Heuristics and Biases: The Psychology of Intuitive Judgment, pp. 49-81. Cambridge: Cambridge University Press.
Levinson, R., Kent, P., Pratt, D., Kapadia, R. \& Yogui, C. (2012) Riskbased decision making in a scientific issue: a study of teachers discussing a dilemma through a microworld. Science Education 96(2), 212-233.
Mason, J. (2010) Mathematics education: theory, practice \& memories over 50 years. International Journal for the Learning of Mathematics 30(3), 3-9.
Radakovic, N. \& Chernoff, E.J. (2018) Risk education. In Lerman, S. (Ed.) Encyclopedia of Mathematics Education. Cham, Switzerland: Springer.
Shah, N. (2019). Should there be less mathematics education? In Otten, S., Candela, A.G., de Araujo, Z., Haines, C. \& Munter, C. (Eds.) Proceedings of the Forty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, pp. 29-32. St Louis, MO: University of Missouri.
Slovic, P., Finucane, M.L., Peters, E. \& MacGregor, D.G. (2010) Risk as analysis and risk as feelings: some thoughts about affect, reason, risk, and rationality. In Slovic, P. (Ed.) The Feeling of Risk: New Perspectives on Risk Perception, pp. 21-36. New York: Routledge.
Spiegelhalter, D. \& Gage, J. (2015) What can education learn from real-life communication of risk and uncertainty? The Mathematics Enthusiast 12(1-3), 4-10.

# Disability, mathematics, and the Goldilocks conundrum: implications for mathematics education 

## DANIEL LEE REINHOLZ

Mathematics and disability have a long and complicated relationship. On one hand, students who struggle with mathematics have been pathologized and labeled as learning disabled. On the other hand, many famous mathematicians are portrayed as insane and troubled geniuses. This is a Goldilocks conundrum; a person needs a 'just right' relationship with mathematics to be seen as normal. This essay aims moves beyond tropes of disability as deficit or superpower. I draw on my experiences as a disabled mathematics educator and argue for a reconceptualization of disability and mathematics.

In mathematics education research, disabled students are typically seen through deficit lenses, framed as a problem to be solved (Lambert \& Tan, 2017). Although disability designations ostensibly aim to help students by garnering additional support, they also stigmatize students, which often leads to further marginalization. Such labels are also wielded against racially minoritized students as a form of discipline and control, which reifies structural racism and ableism in schools. Similar tactics were used in the eugenics movement, when IQ tests labeled racially minoritized populations as 'feeble-minded'. Clearly, mathematics is a powerful tool for signifying normality and oppressing the non-normative.

People who are 'too good' at mathematics are also stigmatized. For example, both Newton and Einstein, notable historical figures stereotyped as odd geniuses, are suspected to have been Autistic (James, 2003). Yet, historical accounts do not reflect what is now understood about non-neurotypical behavior. Other mathematicians have had mental illness or depression, like John Nash or Georg Cantor. Narratives about these mathematicians follow tropes about mad scientists, or 'supercrips' who overcome their disabilities to succeed. These mathematicians were so talented that they were seen as abnormal. Similarly, narratives like 'Asians are good at math' position an entire race as too good at mathematics, in the process dehumanizing them (Shah, 2019).

In reality, disability is a complex embodied experience. Different bodies have different experiences. Some Blind mathematicians have had exceptional visualization skills; Euler, for example, made many discoveries after he became Blind. Similarly, neurodiverse mathematicians think differently, and a logical, Autistic mind can be a huge asset in making mathematical discoveries. It is clear that disability is not purely a deficit or an asset, but a normal variation in human experiences. The disability justice movement provides tools to reframe disability and humanize mathematics education.

## Disability culture and justice

Disability justice as a movement recognizes the embodiment of human experience and the intersections between disabil-
ity, race, gender, and other identities (Sins Invalid, 2019). It centers the experiences of those who are multiply marginalized, particularly disabled people of color and queer/ nonbinary disabled people. Activists in this movement problematize the socially constructed notion of a 'normal' person, instead highlighting the natural variation in human experience.

Much as ethnic studies provides an avenue of resistance for racially minoritized people, disability justice helps disabled people of all races reframe their experiences. Typically, disabled people learn little that is positive about their own communities in school (Mueller, 2021). In contrast, recent efforts show that institutions can celebrate and embrace disability culture as a way to push back against damaging deficit narratives (see, e.g., Chiang, 2020). Mathematics classrooms are also well-positioned to disrupt these narratives, because of their unique relationship to disability.
Here I offer three possible implications for mathematics education building on insights from disability justice, related to embodiment, access and culture.

## Centering embodiment

In contrast to commonsense notions that mathematics is entirely logical or disembodied, researchers are increasingly recognizing that mathematics is an embodied experience. When students can explore mathematics with their bodies, it opens up new learning opportunities (Goldin-Meadow, Wagner Cook, \& Mitchell, 2009). Centering embodiment is also a step towards a broader framing of mathematics as a historical, cultural, and evolving set of practices. This reframing is needed, to problematize a singular, correct way of knowing or doing mathematics.

Mathematics makes use of written symbols in very precise ways. A small change in the shape, size or position of a symbol can change its meaning considerably. For example, $3 x^{2}$ has a very different interpretation compared to $3 \times 2$. This is a barrier for many disabled students. Dyslexic students struggle with the specific notation and terminology used in mathematics (Perkin \& Croft, 2007). The use of colors or different visual representations may help dyslexic students to overcome some of these issues (Nieminen \& Reinholz, under review), and such practices can also benefit nondisabled students. In this way, a disability justice lens pushes us to revisit these singular and taken-for-granted ways of knowing or doing mathematics, and to embrace more varied forms of engagement. Conventions that have been established over hundreds of years may need to be changed. In general, disability justice requires us to take a careful look at what is considered normal and normative and more deeply consider alternatives.

## Using access talk

Disability justice also draws attention to the importance of providing access. Access needs are often conceptualized in terms of sign language interpretation, image captions, or gender-neutral bathrooms. But access needs also relate to the types of language or visual representations used, or the way students interact throughout the learning process. Mathematics education scholarship prizes verbal and social engagement, but this is often done without attention to the
barriers such learning environments create. For example, students experience racial and gender microaggressions in collaborative settings (see, e.g., Ryan, 2019). Similarly, Autistic people may be framed as mathematics geniuses, and yet be marginalized in an active environment without appropriate support (Gin, Guerrero, Cooper \& Brownell, 2020). Rather than assuming we know what students need to learn, we can provide more opportunities for self-advocacy through access talk. We can ask students: What do you need to access a space? What would help you do your best work? Access talk reframes the teaching and learning process. Rather than thinking of students as the recipients of education from an expert, teachers and students are co-participants.

Access considerations also challenge assessment practices. Mathematics classrooms are typically organized around 'objective' assessment through high stakes exams. This positions disabled people as defective, in need of accommodations. Yet, the accommodation process itself is power-laden and can further marginalize disabled students (Nieminen, 2020). Moreover, high-stakes exams do little to reflect the real-world experiences of mathematically intensive professions. Drawing upon principles from Universal Design for Learning (Rose \& Meyer, 2002), mathematics educators can radically reimagine what is considered mathematical competence and how to measure it. Assessment can be a tool for learning through practices like peer/self-assessment, portfolios, and mastery learning.

## Embracing disability culture

Although disability is often framed as an individual impairment, in fact, disabled people have a variety of strengths and their own diverse cultures. Mathematics educators can benefit from understanding these cultures and incorporating them into their classes. Just as a culturally-relevant pedagogy is needed to meaningfully engage racially and lingusitically minoritized students, such a pedagogy is also needed for disabled students. Especially for nondisabled mathematics educators, cultural competence is needed to better relate to the experiences of disabled students.

At the most basic level, educators need to develop better language practices (e.g., understanding identity-first vs. per-son-first language), and avoiding non-ableist terms. We can also teach the experiences of disabled mathematicians in a way that humanizes disability. Disability culture recognizes 'disability gain' (Fox, Krings \& Vierke, 2019) rather than only looking at drawbacks of disability. From this perspective, some Autistic and Deaf people may not even identify as having a disability. Disability culture also values interdependence over independence; this challenges the notion of mathematics as individual and competitive. In this way, embracing disability culture can build community and solidarity for disabled students, who are often stigmatized and isolated. Beyond these specific examples, awareness of disability culture can generally frame the way that educators interact with disabled (and nondisabled) students. Given the
interconnections between ableism, racism, sexism, and so forth, this is also an important site for challenging intersectional oppression. Beyond supporting disabled students, practices from disability justice (like a focus on access, wholeness, intersectionality, and working sustainably) are ostensibly good for all students, and their teachers.

## Discussion

Mathematics and disability have a complex relationship. The interconnections between mathematics and normality (through ableism) allow mathematics to be used as a tool of oppression. Simultaneously, mathematics classrooms are a powerful site for reconceptualizing disability, and the disability justice movement offers concrete tools that educators can leverage for this reconceptualization. For example, disability justice pushes us to re-evaluate taken-for-granted practices around embodiment, representation, access, interaction, assessment, and disability culture. These practices have the potential to greatly benefit disabled and nondisabled students alike, moving towards a more humanizing and empowering mathematics education.

## References

Chiang, E.S. (2020) Disability cultural centers: how colleges can move beyond access to inclusion. Disability \& Society 35(7), 1183-1188.
Fox, A.M., Krings, M. \& Vierke, U. (2019) 'Disability gain' and the limits of representing alternative beauty. In Liebelt, C., Böllinger, S. \& Vierke, U. (Eds.) Beauty and the Norm: Debating Standardization in Bodily Appearance, pp. 105-125. Cham, Switzerland: Springer.
Gin, L.E., Guerrero, F.A., Cooper, K.M. \& Brownell, S.E. (2020) Is active learning accessible? exploring the process of providing accommodations to students with disabilities. CBE-Life Sciences Education 19(4), es. 12.
Goldin-Meadow, S., Cook, S.W. \& Mitchell, Z.A. (2009) Gesturing gives children new ideas about math. Psychological Science 20(3), 267-272.
James, I. (2003) Singular scientists. Journal of the Royal Society of Medicine 96(1), 36-39.
Lambert, R. \& Tan, P. (2017) Conceptualizations of students with and without disabilities as mathematical problem solvers in educational research: a critical review. Education Sciences 7(2), art. 51.
Mueller, C.O. (2021) "I didn't know people with disabilities could grow up to be adults": disability history, curriculum, and identity in special education. Teacher Education and Special Education, Advance online publication. https://doi.org/10.1177/0888406421996069
Nieminen, J.H. (2020) Student conceptions of assessment accommodations in university mathematics: an analysis of power. Nordic Studies in Mathematics Education 25(3-4), 27-49.
Nieminen, J.H. \& Reinholz, D.L. (under review) Ableism in postsecondary mathematics education: toward a structural/systemic conceptualisation.
Perkin, G. \& Croft, T. (2007) The dyslexic student and mathematics in higher education. Dyslexia 13(3), 193-210.
Rose, D.H. \& Meyer, A. (2002) Teaching Every Student in the Digital Age: Universal Design for Learning. Alexandria, VA: Association for Supervision and Curriculum Development.
Ryan, U. (2019) Mathematical preciseness and epistemological sanctions. For the Learning of Mathematics 39(2), 25-29.
Shah, N. (2019) "Asians are good at math" is not a compliment: STEM success as a threat to personhood. Harvard Educational Review 89(4), 661-686.
Sins Invalid (2019) Skin, Tooth, and Bone-the Basis of Movement is Our People: a Disability Justice Primer (2nd ed.). Berkeley, CA: Sins Invalid.

