THE POST-PIAGETIAN CHILD: EARLY MATHEMATICAL DEVELOPMENTS AND A ROLE FOR STRUCTURED MATERIALS

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The first time I remember meeting the term 'mathematical development' was in a UK government publication prescribing a curriculum for state-funded education of the under-fives (DfEE/QCA, 2000). It struck me as odd then, and still does, that anything so ill-defined, varied and plural as what is called 'mathematics' might be thought to have a singular 'development', but there it was, officially sanctioned and prescribed for teachers in a series of stages called 'Stepping Stones'

That official document of 2000 reflected widespread agreement within the community of experts informing the UK government of the time that young children's 'development' of calculating is founded upon initial mastery of counting in 'everyday' situations, and that addition and subtraction are thereafter constructed by children through the invention of a predictable and necessary sequence of counting-in-ones strategies (see, *e.g.*, Fuson, 1992; Gray, 1991; Thompson, 1997). Interestingly, the view that counting-in-ones is the singular sufficient basis of children's invention of adding and subtracting has often been reported in the literature as part of a much deeper paradigm shift away from the domination of a Piagetian perspective during the 1960s, within which the kind of logical understanding that might lie behind children's 'pre-conservation' counting activities had notably been questioned. This general shift of thinking has also often been described more positively as involving a shift towards increasing recognition of the importance of Vygotsky's developmental theories - sometimes characterised rather globally as a shift from a Piagetian 'individualistic' view of logico-mathematical development toward the embracing of a thoroughgoing 'social' interpretation of thinking and its origins

Notable within this shifting interpretation to today's 'mathematical development' has been the work of Walkerdine, through her application of Foucault's ideas concerning 'discourse' and the historically situated nature of knowing. Her (1984, 1988) deconstruction of 'the Piagetian child' of the 1960s showed how the then prevailing discourse, exemplified in the English *Nuffield Mathematics Project*, 'regulated' classroom practices of the time, 'producing its own truths' and a 'Piagetian child subject' as if 'natural'. Her work has itself also inevitably constituted a key contribution to the formation of a more recent 'post-Piagetian' discourse. Since the 1980s many other influences have contributed to the regulating discourse of today's early schooling (most notably the influences of Vygotsky and of social anthropology), and the discourse of today is also 'producing its own (historically situated) truths' in constructing a *socially* natural 'post-Piagetian child subject'. The regulating of classroom practices within the UK by today's discourse has been effected particularly powerfully through the centralised prescriptions of 'informed' government, as well as by the discursive practices of those contributing experts by whom that government has considered itself informed. (The implementation of a government backed National Numeracy Strategy (NNS) in England from 1998 onwards has been optimistically characterised by one of the key civil servants responsible, Michael Barber, as a period of 'informed prescription' - see Fullan, 2003, p. 4.)

The complexities and regulating effects of today's prevailing discourse cannot be fully explored until history advances in ways that will permit a yet newer retrospective deconstruction (perhaps), from a yet to be constructed and contrasting perspective. My concern here and at this point in history is simply to point to some instances where the discourse of today is producing its own truths in an unhelpfully narrowing way, and to venture upon beginning an examination of the production of what might reasonably be called the 'Vygotskian student subject'. The 'mathematical development' of a 'Vygotskian student' is what is implicated most clearly within those expert views informing recent and current prescriptions of UK government. In particular, I question the view that children 'naturally' develop (or re-invent) addition and subtraction strategies by passing through a necessary sequence of counting-in-ones procedures arising within everyday contexts, and that numerical understanding may *only* develop with the use of physical objects and imagery somehow meaningfully 'situated' within 'everyday' pre-school contexts

The role of counting and the 'everyday' in today's mathematical development

The Vygotskian student moves through three key developmental stages, each characterised in terms of social institutions: pre-school, school, and work (adolescence). These Vygotskian social divisions have proved durable over time, maintaining a set of constructed oppositions in today's discourse between an 'everyday' world and a world of 'schooling', and between 'schooling' and an opposed world of 'work'. Developing and strengthening these discursive divisions, contributions from social anthropology have

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characterised practices of a number of children in Brazilian streets, of dairy workers, and of Liberian tailors as likewise opposed to those of traditional (or formal) 'schooling', and concluded that a 'situated' cognition explains how children's likewise constructed 'transitions' between such social worlds are inevitably problematic. Within this discourse, a preferred 'mathematical development' is produced that necessarily begins within 'everyday' social (and situatedly 'meaningful') practices of homes (or anywhere else 'not-school'), and is most effectively built upon by *enlightened* practices within schools that progress to the abstract objects of mathematics directly from 'everyday' situations and materials found in 'real life', thus obviating the constructed and problematic 'transitions' (see Aubrey, 1997). These constructed and opposing social worlds, and the difficulties of transitions between them, are evidenced within the discourse by opposing descriptions of selected examples and social interactions offered as representative of the 'everyday', of 'schooling', and of 'work' generally (see Lave, 1988 and Nunes & Bryant, 1996 for influential examples). Such discursive objectification (Sfard, 2008, chap. 2) of something called 'the everyday', although in some ways inevitable in general discussion, is nevertheless not real, and easily obscures important features of *actual* ('lived') everyday (and school, and work) situations in which both learning and teaching occur.

During one sequence of a UK NNS (2000) training video called *Every Child Counts*, a young child is seen 'making his own arithmetic discoveries' within the 'everyday' situation of cooking with his mother, at home. The voice-over explains how, "The everyday environment provides opportunities for all children to form mathematical concepts for themselves, in their own way" suggesting that the insights apparently arising on this occasion are sufficiently accounted for by the 'everydayness' of the 'opportunities' and the child's own mental activity in discussion with his mother within the meaningful 'situation'. What is not thought worthy of mention however is the fact that this particular 'everyday' environment placed before the child a 4×3 structured array of spaces in the form of a baking tray, and that the child uses this structure (by repeatedly pointing as he speaks and works) to explain his latest whole number calculating within '12' to his mother. A reasonable interpretation is that it is the *structure* he finds within the 'everyday' situation that is supporting his new insights, not the 'everyday' nature of the baking tray and the 'situated' cooking activity. Similarly, it is the *structure* implicated within the 'work practices' of dairy workers in another example that is exploited in their calculating (see Scribner, 1997).

To objectify 'the everyday' within a Vygotskian (or any other) discourse effectively conceals an infinite variety called up by the question '*Whose* 'everyday'?' (and similarly, '*Whose* 'schooling'?', '*Whose* 'work'?'), and can thus also lead to key assumptions about what is to count as 'everyday' in research studies. The whole corpus of studies purporting to show that young children's invention of calculating 'naturally develops' through a predictable sequence of counting-in-ones strategies (Fuson, 1992; Thompson, 1997) involves young children responding to word problems in situations and circumstances restricted to 'everyday' material, that is, word problems framed within 'everyday' contexts, and only discrete objects and fingers available to support children's problem solving. Thus, the very children who are claimed to be demonstrating a 'natural development' towards the invention of addition and subtraction through counting-in-ones strategies have actually been placed within a history and conditions that rule out other possibilities. The discursive 'truth' that children's learning in school is built upon pre-school understandings that are 'situated' within an 'everyday' world has determined the situated conditions of the studies themselves. In Walkerdine's (1988) terms, these 'empirical' research contributions allow the discourse to

> [...] produce the possibility of certain behaviours and then read them back as 'true', creating a normalizing vision of the 'natural' child. (p. 5)

Children use counting-in-ones strategies to solve addition and subtraction problems in circumstances where other possibilities have not arisen. Such conditions do not disclose a 'natural development', no matter how many times the research is replicated, merely one possible path to progress within which representational possibilities have been arbitrarily or consciously restricted to 'the everyday'.

Vygotsky's writing has not been the only influence contributing to objectification of the 'everyday' (anywhere 'not-school' in children's lives), which is why I prefer to call the constructed subject of today's prevailing discourse more broadly the 'post-Piagetian child' [1] to signify both manifold contributions and a place in history [2]. Hans Freudenthal (1973) argued that his preferred approach to

> Elementary arithmetic instruction is integrated [...] with the various *life activities* of the child. (pp. 139–40, emphasis added)

Freudenthal's influence is particularly significant because developers of the *Realistic Mathematics Education* (RME) approach he inspired frequently oppose a constructed 'everyday' ('realistic') world to an objectified 'traditional schooling' not only in the sense of noting *difference* (as Vygotsky had done) but in a further *evaluative* sense as well. Within RME the favoured 'realistic' approaches are commonly contrasted to a pejoratively termed 'transmission teaching' apparently characteristic of 'traditional schooling' – which is of course also to be seen by opposition, and also pejoratively, as 'unrealistic'.

Evaluative discursive objectification of the 'everyday', the 'realistic' (*whose* 'realistic'?), and of 'traditional schooling' enables what Goldin (2008) has called 'dismissive theorizing'. Once it is discursively established that 'natural development' builds upon early 'everyday' (or 'realistic') experiences, then organised 'schooling' must be adapted to build upon those very 'everyday' experiences if it is to be successful. Materials designed specifically for an educative function (such as Cuisenaire rods and Dienes blocks) become anathema through being confined to a world of 'schooling' – in this they stand opposed to 'everyday' objects and 'realistic' scenarios, and become therefore (again pejoratively) *artificial*, being outside everyday and realistic experience, and thereby effectively meaningless to children. More 'empirical' evidence can be accumulated showing that such artificial devices 'don't work' [3], and theorizing can be called upon to explain why this is so. Gravemeijer *et al.* (2000) for instance, claim that the 'realistic' and 'situated' approach of RME may be

> [...] contrasted with the traditional transmission view of instruction in which mathematical symbols are treated as referring unambiguously to fixed, given referents. The teacher's role in this traditional scheme is typically cast as that of explaining what symbols mean and how they are to be used by linking them to referents. Frequently this involves the use of concrete materials or visual models designed to ensure that students learn mathematics meaningfully. Implicit in this approach is the assumption that such models embody the mathematical concepts and relationships to be learned. However, from our perspective, the explanatory power of such didactic models can be seen "only in the eye of the beholder" (p 226)

And in Gravemeijer, Lehrer, van Oers, and Verschaffel (2002),

> As expert adults, we are able to see these [number] relationships in the material because we have already constructed these relationships, but for the students who have not already constructed these relationships, the Dienes blocks are just pieces of wood. *This does not leave the teacher many options, other than to spell out the correspondences between the blocks and the algorithm in detail.* The consequences of that policy, however, will sooner be rote algorithmization [sic] than understanding (pp 11-12, emphasis added)

I call such arguments 'dismissive theorizing' because a carefully limited and flawed theoretical argument is constructed and positioned as the alternative to the writers' own preferred rationale. In Gravemeijer *et al.* (2002, pp. 11-14), for instance, it is argued that the use of 'manipulatives' such as Dienes blocks involves presuming that the materials somehow 'embody' number ideas, and that this entails a commitment to questionably dualist 'correspondence' theories of truth. It is further suggested that in practice it is only possible to interpret Dienes blocks as 'embodying' number ideas once one has already developed those same number ideas by some other [socio-cultural] means. Dienes blocks (in this case) are dismissed as the concomitant of a flawed teaching theory, which theory itself may be used to explain the 'empirical confirmation' of the inevitably poor teaching that results from using them. Thus these artificial (purely *pedagogic*) materials may be safely dismissed from our consideration on both theoretical and empirical grounds and a more natural 'development' preferred solely from 'everyday' and 'realistic' situations (Note that a failed (embodiment) theory is put forward to explain why such pedagogic materials must fail *in use*; RME itself also features purely pedagogic materials (such as bead strings and arithmetic racks) but their use is explained with the support of an entirely different 'learning trajectory' theory.)

Consilience

As an alternative to such 'dismissive theorizing' Goldin (2008, p 196) in his discussion of representation suggests that a spirit of consilience may be more helpful to us all, within which 'a coherence and compatibility of knowledge in different domains' can contribute to a 'theoretical framework for mathematics education that can unify useful and valid ideas'. In a similar spirit, Bruner (2006) invited us all to 'celebrate the divergence' of Piaget and Vygotsky, noting, "Just as depth perception requires a disparity between two views of a scene, so in the human sciences the same may be true: depth demands disparity" (p 195). Within such an openness of spirit, I suggest that many different 'mathematical developments' are both possible and successful, that as Dienes himself proposed (see Dienes & Holt, 1973) children may build mathematical understanding with both 'everyday' and with pedagogic materials, and that key choices to be made between representational possibilities (and hence 'developments') when teaching mathematics will be influenced significantly by underlying views (not always either conscious or expressed) upon what is to be called 'mathematical'. Important further considerations in practice concern communication and economy in teaching

With regard to what is called 'mathematical', I begin by noting that in my experience the word is always used in connection with actions, objects, and/or situations at least involving *structure* of some kind or kinds (apart from trivial nominal uses such as, "What homework have you got today?" "Oh, mathematics "). Structure may be being sought, found, formed, recognised, described, developed or utilised - it doesn't matter which. Secondly, although there may be many who still regard 'mathematics' as a body of knowledge I note that all experiences that are called 'mathematical' also involve action of some kind(s) at some level(s); people *do* 'mathematics' Structure and action may thus be seen as together necessary to what is deemed distinctively 'mathematical' Also, for Gattegno,

> [...] the essence of the mental attitude which characterises the mathematician is the substitution of the virtual for the actual, the performing of virtual actions which are envisaged as being real or realisable, once there is no reason to suppose that they might not become actual. (Cuisenaire & Gattegno, 1957, p 57)

While for Freudenthal,

> [ ] whatever you think about it, the exclamation "so it goes on" *is* mathematics, it is the first mathematics mankind produced and individuals are producing. It is great and important mathematics, it is first and last mathematics; it is the loftiest and the most profound mathematics. (1973, p 173)

It is difficult to imagine how any 'virtual action' in Gattegno's sense, or any observation 'so it goes on' would be possible without some form(s) of structure having been discerned. Already, with these views, there arises the possibility of distinguishing within any situation *when* action may be deemed 'mathematical'. This is not a trivial observation since it raises significant questions about whether actions involving 'the use of numbers as labels' (part of the UK government prescribed 'mathematical development' - DfES/QCA, 2003) can reasonably be called 'mathematical' at all [4], and invites further analysis of many 'situated' social practices to determine precisely *which* actions among all those occurring in the situation are to be picked out as 'mathematical'. Indeed, in some situations the 'situatedness' of the understanding involved might be the very quality that precludes it from qualifying as 'mathematical'

Note I am not claiming that Gattegno and Freudenthal have somehow together authoritatively captured what some ideal 'thing' called 'mathematics' is; I am observing that the distinguishing of any child development, or any instance of a 'practice within a community' as 'mathematical', is contingent upon an implicit or explicit view of what is to be called 'mathematical', and that in their view 'mathematical' activity essentially involves some form or forms of virtual action. In a spirit of consilience I am suggesting that all experiences deemed 'mathematical' at least involve both structure and action, and I invite readers further to consider an additional operational criterion that it could be *virtual* action in particular that is distinctively 'mathematical' Much has been said and written about 'the objects of mathematics' over millennia that resonates with movement between what is 'real' and what is 'virtual' (see Sfard, 2008, chap 6); when the use of words such as 'two', 'three' etc. switches from adjectival to nominal in children's development (and from 'real' action to 'virtual' object) we are inclined to judge that they have (re-) invented the virtual mathematical objects we call 'numbers'. This last observation reminds us that essentially involved within 'mathematical' activity is what we may call 'mathematical' communication – certainly self-communication and possibly also interpersonal; in this I follow Sfard's (2008) operational definition of thinking as 'individualized interpersonal communication' (p 81).

If children are to 'develop mathematically' then, it seems to me there must be communicating, structure and action involved. With Freudenthal and Gattegno, one could also expect there to be *virtual* action, indeed given the distinctive ways in which mathematical objects are formed one would also expect movement between real and virtual action. Key features of situations within which children will 'develop mathematically' then concern which objects, which actions, and which structures are involved, and importantly *when* and *how* these elements introduce themselves or are introduced Crucial decisions for teachers (as Freudenthal's notion of 'mathematizing' suggests) are when, how, and by whom structure is to be introduced; importantly, *how much* structure is to be 'offered', and how much (and which) structur*ing* is to be expected of children? Pedagogic economy is to be judged in this as well.

Various kinds of physically structured pedagogic materials have been invented to support children's number development, and are well documented at least since the time of Froebel (1782–1852). To these may be added further structured materials such as notches on sticks, lines drawn in sand, and various forms of abacus that were perhaps never designed with a pedagogic purpose in mind but nevertheless may be most useful in teaching. All of these physical materials may afford representational possibilities with the potential to support children's active encounters with structure; different materials may thus be seen with different 'representational affordances' I suggest that such affordances may be seen more or less to emphasise either number *notation* conventions (such as bead strings, abacus and Dienes blocks) or number *relations* (such as Cuisenaire and Numicon); unstructured materials (such as counters and cubes) may be actively structured *in use* to emphasise either. Emphasis upon number notation in teaching (for instance 'place value') may reflect a primary concern with 'mathematics' as *symbolic system* (*cf.* Vygotsky?), and emphasis upon number relations may reflect a primary concern with 'mathematics' as *study of structure* (*cf.* Piaget?). Calculating of course involves both, but not always simultaneously; compare multiplying by '10' with a relating of 'factors'

Unifying 'useful and valid ideas'?

Vygotsky used his pre-school 'everyday', 'school', and 'work' distinctions to mark out and characterise broad differences that he claimed lay between approaches to 'concept formation'. In his view, development is not simply built upon 'everyday concepts' but is the product of two-way *links between* 'the concrete and empirical' and 'conscious awareness and volition'; his link is the zone of proximal and actual development (see Hedegaard, 2007). His characterisation of 'proximal development' brings to the fore one key manifestation of the importance of social interaction in development, and of course the key role for him of language in the development of thinking. It is important to note however that nowhere in this vision does he suggest that the only *physical objects* children can learn with before 'school' are 'everyday' *objects*; his pre-school world is simply characterised by a distinctive and social approach to 'concept formation' – a manner of learning that only happens *contingently* to involve 'everyday' materials. And Gravemeijer (2002) is correct in suggesting that when children meet Dienes blocks (or Cuisenaire rods) for the first time they are 'just pieces of wood'. I suggest that Piaget's attention to the logico-mathematical developmental significance of *action* and *structure* may be aligned with Vygotsky's attention to the importance of *social interaction* and *language* to outline perfectly feasible 'mathematical developments' that are restricted neither to 'everyday' objects nor to 'realistic' scenarios. I further suggest that there are some potential economies and advantages to be gained through the affordances of particular structured materials that emphasise number *relations* within such developments.

Contrary to views expressed by critics of Dienes blocks, it is possible to outline productive 'mathematical developments' involving pedagogic structured physical materials in either 'dualist' or 'non-dualist' terms; it is not necessary to theorise about the existence of 'meanings' of words and symbols and pieces of wood for such perceptually accessible material to prove effective in use. Also pragmatically, for a teacher it is not necessary to choose between ontological and/or epistemological positions unless to do so results in improvements to practice. I refer to a particular non-dualist account next because it offers the important possibility of novel empirical investigation with a view thence to pedagogical improvement

In Sfard (2008), there is offered a carefully non-dualist account of mathematical development that involves no objectified 'everyday' or 'realistic' worlds, and consequently no dismissal of particular physical materials as 'artificial'.

Instead there is a carefully developed account of how action with *any* physical object(s) may become implicated within a Vygotskian (used in a broad sense here; Sfard, importantly for my argument, writes of thinking as 'individualized interpersonal communication' rather than as 'internalised speech or language') discursive 'mathematical development'. On such a view structured pedagogic materials such as Cuisenaire rods and Dienes blocks may become equivalent 'realizations' of number signifiers alongside any other forms - concrete, iconic, or symbolic - linked within what Sfard calls 'realization trees'. Differing affordances of different physical objects thus offer a range of 'Vygotskian' developmental possibilities, some perhaps more economical and effective with some children than others.

For brief and illustrative detail, I address here two crucial aspects of number development: the reification of counting processes into mathematical objects signified with words such as 'two', three', *etc.*, and children's perception/construction of relations between such objects. It is the relating of number objects *to each other* and between *the names we give them* that characterises effective calculating and its developments; in this relating we can note a crucial role played by both structure and structuring. In prevailing accounts of development (see Fuson, 1992; Sfard, 2008) we find transitions described from counting-in-ones procedures to calculating (relating) with 'whole' number objects, with *perceptual structuring* apparently playing no mediating role. It is worth exploring the potential economies and effectiveness involved in enactive and visual structuring however as children 'develop' their thinking *beyond* counting-in-ones toward 'numbers', for it is noted often enough that many children remain limited to counting-in-ones procedures instead of calculating by relating whole numbers (see, *e.g.*, Gray, 1991).

Reifying a counting procedure into a virtual object (a number) is both a mysterious process and not inevitable; some children appear to achieve this only partially. There is a kind of 'compression' involved, and as Sfard observes (2008, p. 171), there is '[...] replacement of talk about processes with talk about objects'. Mysteriously though, the virtual number object somehow just 'arises' during this time. In the same way that Sfard acknowledges crucial roles for visual mediation in interpersonal communication and for realizations of mathematical signifiers in different media, I suggest that there is opportunity for a helpful *visually structuring* role within the developmental process of reification itself - a structuring in two senses: the real objects involved may themselves be structured (*e.g.*, fingers), but discrete real objects may also be structured *in a systematic way* so as to invite a relating structure between the various virtual objects (*i.e.*, numbers) that are produced. Dienes blocks, arithmetic racks, and bead strings involve structurings that reflect notation conventions; Numicon pieces involve structurings that reflect initially number relations and subsequently notation. Cuisenaire rods invite a development that will proceed in parallel with reification of counting-in-ones instead of building upon it, with discourse developing around ordinal relations between ungraduated lengths usually before discourse concerning cardinal values is subsumed. In short, the various and different structur*ing* actions with, and the physical structures of, the real objects implicated within the talk in a reifying process may support that process significantly; the *lack* of structure in random arrangements of discrete physical objects offers only minimal support to children whose *mathematical* development depends upon them both reifying separated counting-in-ones procedures *and* seeing connections between the virtual objects that result. Gestalt psychology offers many examples, and thoughtful discussion, of human perceptual structuring that are relevant here (Wertheimer, 1945).

Some concluding remarks

Just as there is no reason to assume that children may only 'develop mathematically' with 'everyday' objects, there is no reason to assume that children may only (or even principally) form 'numbers' by reifying counting-in-ones procedures. Cardinality may be determined by *grouping* discrete objects into systematic visual patterns, as well as by counting-in-ones, and such active structuring may form a rich perceptual basis for exploring of both number notation and number relations [5]. Nor should we forget that Gattegno proposed a perfectly feasible mathematical development based upon relations of magnitude (both ordinal and algebraic) before cardinality is studied at all.

Dismissive theorizing acts to support ideological positions that risk precluding significant possibilities until ideology has, eventually, to be abandoned (Goldin, 2008). Freudenthal (1983), in his passionate advocacy of number lines dismissed Cuisenaire rods as *opposed to* work with such imagery (p. 10). In his false implication of a necessity for *choice between* number lines and Cuisenaire rods he ignored the rich possibilities of involving *both* within 'mathematical developments'. The 'part-whole' relationships so fundamental to effective calculating and rational number can be explored most effectively with such visual realizations and in their 'universality' of possibility the rods present no necessary obstacle to experience of continuity with a number line. Consilience, I suggest, offers a richer future; following Bruner, we limit the depth of our understanding in advance by asking ourselves to choose between Piaget and Vygotsky, similarly in giving ourselves many other false choices.

Vygotsky's characterisation of developmental stages in terms of social institutions involved the generalising of social practices of his time and place. Discursively objectifying 'the everyday' and 'schooling' and 'work' subsequently as if these opposing constructions were as real and present to the senses as material substances risks discourse that produces only arbitrarily narrow truths, and may blind us to details of significance in *actual* lived experience. There is no magical property inhering in 'everyday' or any other perceptually accessible objects; *any* physical object may become a 'communication mediator' within the practising of a human discourse.

Working with operational definitions of what is called 'mathematical', of 'mathematical communication', and of 'thinking' as individualised interpersonal communication opens up exploration of discursive 'mathematical developments' that both recognises Vygotsky's insights regarding the social dimension of thinking and makes possible new avenues of empirical enquiry. Adding Gestalt observations concerning perceptual structuring (with others) to such dialogic discourse widens enquiry still further.

There is no space within a single article such as this to lay out the detail of varying 'mathematical developments'; my aim in this paper is simply to point to their possibility. Much work remains to be done investigating perceptual activity within discursive developments, and in particular the possibilities of visual structuring – how much, when, how, and by whom? And within this, investigation of not just notation structures, but number relations as well Similarly, empirical investigation of 'routines', 'rituals', 'deeds' and 'explorations' (Sfard, 2008, chaps 7 & 8) within developments involving variously structured (and structuring) visual 'communication mediators' could reveal much to us that remains closed when we restrict ourselves unnecessarily to just 'everyday' objects and 'realistic' scenarios Children are more inventive than that and as Gattegno often pointed out, the evidence we have daily from children's learning of their mother tongues shows us just what an incredible eye (and ear, and feeling) they have for pattern and structure

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Endnotes

[1] A distinction between Piagetian 'child' development and Vygotskian 'student' development is important, and to discuss the constructed subject of today's prevailing post-Piagetian discourse as either 'child' or 'student' risks over simplifying a complex and shifting discursive situation Since I later argue that Piagetian child development is not simply to be replaced by a Vygotskian perspective, and because I am here concerned to discuss the mathematical development of 'students' who are also 'children', I have preferred to use the term 'the post-Piagetian child subject'. Readers should be aware however that from this point I am referring to a more technically correct, but stylistically more awkward, constructed 'post-Piagetian child/student' subject

[2] Even Piaget himself had observed, "It is a great mistake to suppose that a child acquires the notion of number and other mathematical concepts just from teaching On the contrary, to a remarkable degree he develops them himself, independently and spontaneously" (Piaget, 1953, p 2)

[3] Walkerdine (1988) for example, forensically savages an episode of poor teaching with analysis of several pages of transcript (pp 159-182) in order to demonstrate that the struggling teacher's 'reading' of her classroom use of Dienes's materials was wildly different from the children's own experiences (see also Hart, 1989)

[4] The use of *numerals* to designate bus routes, or telephone addresses, is distinguished within the French language (for instance) by the use of 'numero' as opposed to the cardinal 'nombre'; the English language currently involves no such lexical distinction, both uses being covered by the use of the same word 'number' Describing 'the use of numbers as labels' as 'mathematical' may be one instance in which a socially anthropological view of 'mathematics as 'social practices within communities' is implicated without acknowledgement or explanation

[5] I am grateful to Wacek Zawadowski for pointing out the significance of Pythagorean reasoning here; since linguistic deductive logic had not been invented in their time, the demonstrating of numerical relationships by early Pythagoreans frequently involved the dynamic imagery of 'showing' I am loath to believe that humans have somehow lost this capacity since the subsequent invention of formal deductive reasoning

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