Dilemmas in Teaching and Learning Mathematics

BILL BYERS

A few years ago I taught a course in real analysis to a small class made up of final year undergraduate honours students plus a few qualifying year students Over the years the mathematics department has used this course to help it decide whether or not a particular student was suited for graduate work in mathematics. The fact that serious decisions might be made as a result of my evaluation of the students' performance led me to some careful reflection upon my pedagogical goals. What should I be looking for from these students? Clearly I could not settle for the usual memorization of pages of definitions, theorems and proofs. No, what I was seeking from them was some insight into the basic concepts around which the course was built. This insight or understanding would ideally be demonstrated through the student's ability to do something original with the material. For example, this might mean working through certain types of problems, isolating the essential element in the proof of the theorem, or transferring an idea from one domain to a slightly different one. Thus, I hoped that the students would develop some understanding which was more than superficial and that this understanding would be demonstrated in certain types of mathematical activity

Of course, every teacher is faced with the problem of teaching for "understanding" as opposed to teaching facts or techniques. Also we are all aware that there are different levels of understanding. However, while I was preparing to teach this course I was made aware of a dilemma which I had never before seen so clearly and which is shared, I believe, by many teachers. This dilemma involved certain conflicting goals which existed for me in this particular course and which I suspect may be present in many teaching situations To begin with I was committed to teaching for "understanding" as I have explained. However I was well aware that the subject matter of the course—uniform convergence, function spaces, Lebesgue integration, and the like—contained much that was subtle and difficult to grasp. I anticipated that mastery of these concepts would involve a fair amount of struggle and temporary discouragement on the part of my students. Thus my primary goal conflicted with my general desire for harmony in the classroom—my desire to be considered a "good" teacher by shielding my students from exactly this struggle and discouragement. In the past I had dealt with this problem by presenting my own understanding to the students in a very patient and detailed manner, hoping in this way to provide them with a path through the material which was relatively free of conceptual ambiguity. That is, I had attempted to present a pre-digested version of the subject. This style of teaching is certainly popular but I had noticed that it resulted in a certain superficiality. Certainly it had never led to the depth of understanding that I was hoping for in this course. It therefore became clear to me that there was no way in which I could save my students from their individual struggle with the material. As a corollary, I could not hope for some simple means through which to extricate myself from the dilemma with which I was confronted.

More generally, I have noticed that there are various sets of teaching goals which are at least potentially in conflict. For example there is usually the desire to keep ideas as simple and straightforward as possible within the framework of the course However there is also the opposed desire to explain as completely as possible the phenomena under consideration. Let us consider a simple example Quadratic and cubic equations which arise in the mathematics classroom often have solutions which are small integers. This is so often the case that if the students discover a root like $(-1+\sqrt{5})/2$, for example, they are prone to conclude that they have made some error. A complete discussion of the solutions of even a quadratic equation would of course involve a fairly deep investigation of the real and complex number systems. We are able to present a relatively simple formula for the solution of quadratic equations but we often are unable to deal effectievely with the kinds of numbers which may be generated by this formula.

Another basic conflict involves the goal of technical mastery versus that of theoretical understanding. We shall later discuss this problem in relation to the teaching of calculus but for the moment let us consider a more elementary example: the teaching of "carrying" in addition Here the question is whether to emphasize rote manupulations which will result in a certain technical competence; or to stress acquiring an understanding of the algorithm in question by working with some underlying mathematical principles (place-value notation, for example). The experience of recent years has taught us that an emphasis on delving into the abstract mathematical structure which formally underlies a given computational procedure need not in itself ensure computational mastery. On the other hand few would wish to settle for mindless computational skill. An attempt may be made to integrate both of these approaches but in practice this is often difficult to achieve. The conflict usually remains unresolved.

Certainly the reader can add to this list of conflicting or potentially conflicting goals from his own classroom experience. A key point of this article is the *inevitability* of such opposing tendencies in the teaching situation. I would maintain that such contradictions reside in the heart of the pedagogical process and cannot be completely resolved by any ingenious new curriculum or teaching technique. To come to grips with the actual teaching situation it is necessary to acknowledge that a given set of pedagogical goals may be inconsistent—certainly in practice but perhaps even in theory

It is possible that teachers of mathematics are especially inclined to be impatient with situations which may appear inconsistent or ambiguous. We are prone to model the teaching process on what we conceive to be the basic nature of the subject. Often what is found to be attractive about mathematics involves thes basic characteristics of precision, consistency and formality. However it is important to point out that these attributes describe only one dimension of the mathematical experience—that of the formal mathematical structure. The process of actually doing mathematics, either by the mathematical researcher or by the student in his classroom, would be described in very different terms. For example, the mathematician is faced with describing in the theorem certain regularities he observes in a body of mathematical phenomena. This theorem can be made to apply in a more general setting if the conclusions are correspondingly weakened. The eventual theorem is often a trade-off between these two factors There is no "best" theorem; a profound mathematical idea often finds applications in many domains. At the research level, mathematics is amorphous. The field of mathematical phenomena is vast, the world of formal mathematical structure is negligible by comparison.

In fact the mathematics of our century has been characterized by a "loss of certainty" [1]. Work by Gödel and others on the foundations of mathematics show that incompleteness and perhaps even inconsistency cannot be totally eliminated from the subject. In fact this conflict between completeness and consistency in the foundations of mathematics is very similar in form to the teaching dilemma of completeness versus simplicity which we have described above.

Of course the existence of conflicting goals is a dilemma and living with the dilemma inevitably generates stress. Our usual reaction to stress is to eliminate its causes if that is at all possible. In our situation this would mean suppressing one of the given pair of conflicting objectives. However we claim that the greatest potential in the teaching situation lies in learning to work with the inevitable dilemmas which arise in the classroom. Here tension in the classroom is not viewed as evidence of inadequacy on the part of student or teacher. Rather it points to the fact that real educational work is going on. The existence of this creative tension can then be viewed in its proper perspective—as a valuable opportunity for learning to take place. Unfortunately, this is not the usual reaction.

Teaching limits

Let us examine a specific teaching problem in some detail. The problem of teaching limits has bothered untold numbers of teachers of calculus. A fundamental difficulty is the following. Most teachers, influenced by the formalist vision of mathematics, feel that there is a "correct" or logical order through which to approach the idea of a derivative and that this order passes through the concepts of real number, absolute value, function, and limit.

Let us review the $\epsilon - \delta$ definition of limit which is usually presented in this formal sequence. We say that the function $f(x)$ tends to the limit L as the number x tends to a, and write $\lim_{x \to a} f(x) = L$, if for any given positive number ϵ there exists a corresponding positive number δ (whose value depends upon the value of ϵ) such that if a real number x (different from a) lies within an interval of radius δ around a then the corresponding value of $f(x)$ must be within an interval of radius ϵ around L. Symbolically this becomes

$$\forall\ \epsilon > 0,\ \exists\ \delta > 0 \quad \text{such that}$$
$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

This is perhaps the most complicated logical statement that the student has ever encountered. He may be told that this definition means that "$f(x)$ gets close to L as x gets close to a". This is a more geometrically obvious statement which the student may find easier to accept. However it is not precise enough for formal mathematics. In fact we may consider the second formulation as the intuitive content of the first To prove theorems involving limits in the currently acceptable way, one must come to grips with the logical complexities of the former definition. These difficulties are easily demonstrated by asking people who have already been exposed to this $\epsilon - \delta$ definition to show that a given function has no limit; for example, $f(x) = \sin 1/x$ as x approaches 0. Here a careful sketch will show that the function oscillates between $+1$ and -1 infinitely often in any interval around 0. Thus $f(x)$ does not approach any specific value as x approaches 0 However the formal proof would involve showing that *no* value L can be the limit of $\sin 1/x$. The latter involves negating the convoluted logical statement involved in the $\epsilon - \delta$ definition. The purely logical aspects of this task are beyond the ability of many students.

Thus working with the formal $\epsilon - \delta$ of limit is very difficult. Any "honest" attempt to deal with this definition causes so much tension in the classroom that the instructor is tempted to avoid it completely This temptation is strengthened by the realization that once the safe terrain of differentiating is reached the whole subject of calculus becomes straightforward. The idea of a limit quickly vanishes behind a mass of computational techniques

However most instructors refrain from taking the path of least resistance and suppressing all mention of the limit concept. This is due to the uncomfortable feeling that to do so would be intellectualy dishonest, a misrepresentation of the true nature of the subject, and therefore demeaning to their view of themselves as conscientious teachers.

What then are the conventional responses to this dilemma? My feeling is that teaching strategies often evolve which have as their aim (consciously or unconsciously) the reduction of the tension which emanates from this dilemma. Amongst university lecturers in mathematics

I have been able to distinguish two basic types of tension-reduction techniques.

The first of these belongs to the formalist, definition-theorem-proof school of lecturing which has been for some time now the dominant style of teaching mathematics at the post-secondary level. The subject is seen as a fixed body of knowledge which needs to be consumed by the student. The lecturer perceives his primary task to be a logical, coherent presentation of the material.

This teaching strategy minimizes stress on the teacher while maximizing that on the student. Pressure on the teacher is reduced by means of the illusion of objectivity which is created. This illusion is based on a confusion between naming or defining a concept on the one hand and understanding it on the other. This is realy a confusion between the formal structure of mathematics and the process dimension of the subject, since for most people understanding arises out of experience with specific examples or by relating a new situation to a familiar one. All too often a purely formal approach will present a few concrete examples to the student and even these will be illustrations which follow the presentation of the rigorous definition as opposed to motivating it. If we think of the limit definition as an attempt to capture the essence of the limiting behaviour which we observe in a great many mathematical situations, then it is probable that no one definition will be able to definitively pin down the concept of limit. In this sense *all* definitions are tentative and one's understanding can always be deepened.

It is the feeling of objectivity which allows the teacher to keep the student and his problems at arm's length. If the latter does not grasp the material it must be due to his lack of ability—the teacher need not feel any sense of responsibility. Of course, in our view, the greater the success of this particular attempt at tension-elimination the less actual teaching is going on. The tension produced in the students often produces a large drop-out rate. Paradoxically, a high drop-out or failure rate is often attributed by such a teacher to his high standards and so reinforces his view of the appropriateness of his teaching strategy. Such a tough approach appeals to the instructor precisely because it avoids the ambiguity of the dilemma.

The teaching dilemma may also be avoided by trivializing or ignoring the difficult concepts. This may be accomplished by letting the students know in a direct or indirect way that they are not responsible for understanding a certain idea. More subtly, dealing directly with the difficulty of the concept may be replaced by the mastering of a few technical skills. In the case of limits this might involve learning how to compute the values of a few standard limits. This type of strategy greatly reduces the tension felt by the students. If they master the technical skills which the teacher has substituted for the conceptual difficulties, the latter is often evaluated as having done a good job of teaching. Of course this type of instruction has its place, but mathematics cannot be reduced to what is gained through this type of instruction. Damage is done when students are given the impression that their level of understanding is deeper than it in fact is. A type of student-teacher *folie-à-deux* ensues in which each side has a vested interest in concluding that the student understands. The fact that the student has learned very little is often left to his next teacher to discover. The end result of these subterfuges gaining widespread acceptance (sometimes as a result of well-intentioned attempts to improve teaching through the use of teaching evaluation) are the educational disasters of which so much has been written in recent years. The point which is relevant for us here is that the attempt to avoid the tension which arises naturally in a valid teaching-learning environment often short-circuits the educational process as a whole. There is no painless road to learning. The true educational task consists of managing, not eliminating, tension.

The student's position

If the teacher is faced with the dilemma of working with conflicting goals and expectations, how much more difficult is the student's situation. Interestingly enough, the most anxious groups of students with which I have had to deal have been groups of high school and college teachers of mathematics. These professionals seem to experience all of the difficulties which younger students have to face—the resultant stress not mitigated at all by the fact they they had been in front of their own classes just hours earlier. This leads me to conclude that the stress that students feel is intrinsic to the learning situation, it is not something that experience can eliminate. In fact it may be that these teacher-students felt greater stress precisely because they were more open to the dilemma of the situation

Let us spend a moment considering the educational dilemma viewed from the student's perspective. Of course to learn anything one must first venture into the unknown. One may have achieved a certain mastery of a subject; that is, one may have achieved a comfortable kind of equilibrium in which one feels at home and relatively secure. The teacher then introduces some new task for which one's present skills and knowledge prove to be inadequate. As a result the equilibrium threatens to break down.

Here we have a fundamental dilemma which underlies all learning situations. On the one hand a homeostasis (or equilibrium) has by definition an intrinsic stability and thus tends to preserve itself. As a result the demands of the new situation may be rejected or its importance denied. This may lead to a general resentment of the teacher or of the entire situation

However if the forces of homeostasis were the only ones present, then learning would never take place. They are opposed by a set of forces which demand change and development. These range from the desire to please teachers and parents to the desire for success and the need for self-esteem. They may also include a tendency towards cognitive development and growth in general.

The interplay between these two types of forces is involved in all learning. Here again we can observe a conflict between opposing tendencies. Many problems which teachers encounter in the attitudes of their students—from rebelliousness to passivity—may be understood as the students' reaction to the tension which arises out of this conflict

Similarities between the roles of student and teacher

It is important to stress that since there is only one teaching-learning process going on in the classroom, it is artificial to consider one of these processes independently of the other. Thus the teaching is only successful if learning is going on, but also the learning is affected by teaching. We have seen that both teacher and student are caught up in dilemmas arising from various sets of opposing demands which are intrinsic to their respective positions. In this sense there is a parallel between the two positions In fact the ultimate equality in the student-teacher relationship lies in both sides' openness to the dilemmas which they are forced to confront. We usually focus on the division between student and teacher becasue we are forced to deal with the conflicts which arise as each side attempts to manipulate the other in an effort to reduce the stress in their particular situation. However we may neglect to stress the sense in which the relationship is an equal one. This equality is to be found in the existence of a fundamental dilemma which is basic to each role. It has been maintained that work involves "facing and resolving the dilemma" [2]. In any classroom there is plenty of real work in the above sense for both student and teacher.

The teacher's role

Even though there exist deep parallels between the situations of student and teacher, there are, of course, many differences in terms both of role and or responsibility. What then is the task of the teacher which follows from our analysis of the teaching/learning environment?

Let us begin by stating the obvious, namely, that no teacher can learn for his students All he can do is to create an atmosphere within which learning may take place In order to do this he must first recognize the existence of the tension which we have maintained is an inevitable part of a functioning learning environment It follows that one of the teacher's essential jobs is to create the proper atmosphere of *controlled tension* and to help his students extract the true potential from this atmosphere. Thus he must endeavour to maintain that level of tension (or excitement) which is most beneficial to the learning process If the level is too high, one risks alienating the student or reducing him to utter passivity. If the teacher is overly protective and shields the student from any struggle, he is reallys stealing from the students the possibility of accomplishment and real learning. Clearly the search for the optimal level is a complex one which depends on the intuition and sensitivity of the teacher. Factors which are under his control include the rate of introduction of new material and the demanded level of abstraction. In fact referring back to the various pairs of conflicting goals which were mentioned earlier, the tension level produced will depend on the extent to which simplicity is favoured over completeness or technical manipulations over the more theoretical aspects of the subject.

It is possible to put certain Piagetian ideas into the above discussion. For example the duality of the technical versus the theoretical which plays a large role in the teaching of university level mathematics may be looked at in terms of the cognitive development of the student. Piaget's final two stages of cognitive development are called the "concrete operational" and the "formal operational". At the university today one meets many students who appear to operate pretty consistently at the concrete operational stage, at least in so far as mathematics is concerned. They enjoy great success in more elementary courses where manipulative skill is emphasized but get into trouble with an analysis course, for example. The equilibrium which these students may have achieved is often very stable and subtle. Thus they may disguise their lack of reasoning skills through rote manipulation of formal mathematical objects, for example the memorization of the proofs of abstract mathematical theorems. However if the teacher investigates carefully enough he will discover that these students lack any independent reasoning skills and any feeling for the reality of abstract mathematical objects. To these students advanced mathematics is like magic, everything appears arbitrary and anything is possible. For them the educational dilemma is created by their levels of cognitive development. The dilemma of their teacher on the other hand is compounded by having students with different levels of cognitive development in the same class.

Learning can be broken down into two stages. In the first a previously held equilibrium is shown to be inadequate and is broken down. In the second a new state of equilibrium is established. "New structures only emerge from awareness of conflict within the cognitive system". One role of the teacher is to provide "experiences that promote cognitive conflict—experiences which disequilibrate the system" [3] Without the sustained tension generated by these conflicts there is no hope for students to break out of the equilibrium within which they are trapped.

Finally the teacher can contribute by validating the student's attempts to work with the material in the way which we have described. Through his example and encouragement the student can develop his capacity to live in and work with this sustained tension.

Resolution of tension

The second stage of learning involves the establishment of a higher order equilibrium, thereby resolving the tension generated by a particular dilemma. For example, in problem solving the tension is released when the solution is perceived. It is fascinating to speculate on the mechanism of the mind which underlies the seemingly instantaneous insight which resolves tension and marks the success of the learning experience I shall restrict myself to a few general observations.

In the first place the processes of problem solving, of learning in general, and even of creativity seem to have certain basic similarities. If this is in fact the case then in training a student to learn we are also giving him lessons in creativity. It is not so much what is learned which is important as much as the learning process itself.

The mechanism through which understanding is achieved and a new equilibrium established does not involve primarily analytic thought but rather our intuitive or sythetic abilities. Education is often seen as a training of our logical or deductive faculties, yet we give little thought to the possibilities of training these crucial but often underdeveloped intuitive abilities. It is clear that one cannot reason one's way to a great work of art. Neither can one exclusively reason one's way to a deep understanding of some subtle mathematical concept.

Creative people often have a reluctance to analyse too deeply the mechanisms which underlie their creativity. However the emergence of creative insight often seems to be preceded by a period of sustained tension centred around the subject in question. The individual's ability to sustain this tension over an extended period of time is basic to his ability to make these kinds of creative leaps. Sir Isaac Newton is reported to have been capable of "sustained concentration on his problems for hours and days and weeks" "When asked how he made his discoveries, he said 'I keep the subject constantly before me and wait till the first dawnings open little by little into the full light'." [4] Sustained concentration is a form of sustained tension. Though we cannot hope to make our students into Isaac Newtons, we can attempt to strengthen their tolerance for tension. This will not only increase their capacity for learning but also help them in many aspects of their lives both in and out of school.

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References

[1] Kline, M *Mathematics: the loss of certainty* New York: Oxford University Press, 1980
[2] Low, A. *Zen and creative management.* New York: Anchor Press-Doubleday, 1976
[3] Do students think as we do? Progress with Piaget. *Improving College and University Teaching,* Vol 30, No. 4, 154-158
[4] Simmons, George F *Differential equations with applications and historical notes* New York: McGraw-Hill 1972

The teacher, as has been recognized at least since Plato's *Meno,* is not primarily someone who knows instructing someone who does not know. He is rather someone who attempts to re-create the subject in the student's mind, and his strategy in doing this is first of all to get the student to recognize what he already potentially knows, which includes breaking up the powers of repression in his mind that keep him from knowing what he knows. That is why it is the teacher, rather than the student, who asks most of the questions. The teaching element in my own books has caused some resentment among my readers, a resentment often motivated by loyalty to different teachers. This is connected with a feeling of deliberate elusiveness on my part, prompted mainly by the fact that I am not dispensing with the quality of irony that all teachers from Socrates on have found essential. Not all elusiveness, however, is merely that. Even the parables of Jesus were *ainoi,* fables with a riddling quality. In other areas such as Zen Buddhism, the teacher is often a man who shows his qualification to teach by refusing to answer questions, or by brushing them off with a paradox. To answer a question . . . is to consolidate the mental level on which the question is asked. Unless something is kept in reserve, suggesting the possibility of better and fuller questions, the student's mental advance is blocked.

Northrop Frye
