Book Review

John Pottage, *Geometrical investigations: illustrating the art of discovery in the mathematical field* Addison-Wesley, 1983

At the recent International Congress on Mathematical Education in Adelaide I was struck by the ubiquitous influence of Imre Lakatos' remarkable *Proofs and refutations*. His work exercised an evident and powerful sway on the minds, attitudes and even the actions of many of the participants. Speakers and questioners alike frequently called on Lakatos' insights for support. In the sections on history, on problem-solving, on the psychology of learning, on philosophy and on the teaching of tertiary level mathematics, his influence was pervasive, a pervasiveness that stemmed as much from the depth as from the breadth of his work. For like good poetry, *Proofs and refutations* can be appreciated at different levels and in profoundly different ways. So it is with the book under review.

Although written for a different purpose and aimed at a different audience, John Pottage's *Geometrical investigations* evokes similar spirits to Lakatos' *Proofs and refutations*. Clearly the result of years of scholarship it ranges widely over the fields of heuristics, philosophy of science and mathematics, the psychology of invention and learning and the history of mathematics. Overall the text is stimulating, the notes rich in detail, the references surprising, the quotes charming, the problems challenging and the solutions rewarding.

Written as a dialogue between the Galilean characters Salviati, Sagredo and Simplicio, the author maintains the intellectual temperaments given them by Galileo. This literary device considerably aids readability but caused a "strange feeling (to be) induced in the writer ..... these characters did not so much need to be brought to us as we to them—that they were alive in another abode, more real, more lasting, more serene than our own, and that they had been able to keep in touch with much going on in our world, especially with matters connected with their own interests three and a half centuries ago"

Like Lakatos, Pottage's dialogue unfolds through attempts to resolve a conjecture. Are the circle and its circumscribing square the only plane figures for which the ratio of their areas equals the ratio of their perimeters? Totally in character Sagredo doubts his own conjecture: "... it occurs to me that perhaps an ellipse and its circumscribing rectangle might also be as intimately related". Through a wealth of geometrical insights, constructs, examples, counterexamples and theorems the characters explore the conjecture and its ramifications. *Inter alia* they adjudicate on competing formulae for the perimeter of an ellipse and investigate curves of constant width. Solid geometry, maxima and minima, a variety of inequalities and some elementary calculus present stumbling blocks to duly challenge the protangonists.

Following a mathematical dialogue is not easy; demands are made on the reader that require constant rethinking and recalculation. With care the author has ensured that the demands are not excessive, advanced high school mathematics is all that is needed for comprehension.

The thematic spirit of the book underlying the dialogue is "an illustrat(ion of) the art of discovery in the mathematical field." Pottage succeeds remarkably well in laying bare some of the heuristics and problem-solving stratagems used by mathematicians. Leaving aside (as if one can) lack of innate ability, the greatest hindrance to success in *doing* mathematics is a lack of heuristic, of problem-solving know-how. But how is the student supposed to acquire this art of discovering mathematics? Certainly not through any serious attempt by us to reveal our own heuristics. As teachers we rarely expose the *processes* that lead to the creation of mathematics and the solution of problems. The finished product yes, the process never. In his high school and undergraduate courses the student is expected to divine the heuristic art with minimal help for its priestly protectors. Even the bible of heuristic, George Polya's *How to solve it* is denied them.*

Those students who do acquire some problem-solving techniques are by every definition the successful ones. In their graduate years these students are finally given inklings of the priestly secrets. Graduate students are brought to appreciate that good heuristic is crucial to successful research. In their seminars and courses they see mathematicians explore and defend intuitions and strategems

* With one exception I have yet to see an undergraduate mathematics book list refer to any of Polya's works. References do occur in programming courses in computer science where the crucial need for learning heuristic is recognised. One cannot write formal, logically structured programmes until the problem at hand has been thought out at a higher level. Computer scientists do try to teach this art of "discovery in the computing field". Perhaps we should learn from them

With the burgeoning of artificial intelligence programs understanding heuristic has become a critical activity. For instance most automatic theorem-provers are of an expert-system type in which experts' heuristics have been taught to the machine

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This schizoid behaviour has two sources First, a belief that undergraduate mathematics is so technically easy, so transparent, there is no need to reveal any of the heuristic. More, as the mathematics is logically complete, heuristic is seen as a messy perversion, an affront to our aesthetic sense. Euclid's text, with its cold logical beauty and finely honed arguments and constructions has, for two millenia, been upheld as the exemplar of mathematical expression. Descartes' accusation that the Greeks wrote this way in order to exaggerate their obvious cleverness, is libellous in spite of the nugget of truth it contains. Notwithstanding this, there *are* valid intellectual reasons for writing and expressing mathematics in a tight, highly polished style, de-emphasising the process at the expense of glorifying the finished product. The very act of polishing and refining our proofs and techniques considerably sharpens what would otherwise remain dull mathematical tools. With each successive re-working deeper insights are gained Justice Brandeis' aphorism, "There is no such thing as good writing, only good re-writing," is equally true of mathematics † By implication the question Pottage asks is: is the fact that we *write* this way sufficient reason for *teaching* this way? That we do so is undeniable and again there is *some* justification for it.

Much of the joy of understanding mathematics comes from unravelling the clutter of symbols and discovering for oneself the strategies and intuitions through which the formal arguments were constructed. We understand a piece of mathematics only when it has been internalized, when we have played with it, explored ramifications, mentally filtered it, discovered and developed our own unique insights and intuitions. The more formal, rigorous and polished the mathematics, the greater the joy in finally understanding its essence By revealing some of the concrete mechanisms involved in this internalising process, Pottage's book could bring this joy within the compass of greater numbers of students. This may be its legacy.

The second source of schizophrenia is the prevalent view that it is impossible to teach others how to discover mathematics. But there are notable instances of just this type of teaching. Nineteenth century Cambridge was almost overpopulated with mathematical coaches who trained students to solve the extremely challenging problems of the Mathematical Tripos. As these problems were varied and of a high degree of difficulty I conjecture that the successful coaches must have formally imparted problem-solving techniques to their charges. In a similar vein it is now accepted that US high school students can be trained to solve the supposedly unconventional but elementary mathematical problems on the College Entrance Board examinations. Again it is difficult to imagine improvement on this score without the learning of heuristics, without learning strategies to attack problems and to discover solutions. But the most convincing example of the successful teaching of problem-solving technique occurs in the arena of the Mathematical Olympiads. Before joining battle, participants in these "games" are taken on training camps with resident experts teaching the young mathletes the arts of solving difficult and innovative problems. All seen to agree that problem solving skills increase dramatically as a result of this training. The moral is clear: heuristic can be taught. One final example comes to mind. R.L Moore, a famous American mathematician, ran his classes as a dialogue in which students were expected to formulate and criticize conjectures, construct examples, compose proofs and explore consequences. In short, the Moore-method taught heuristic. Moore and his students had no doubt as to its success. Indeed, an abnormally high percentage of his students became active research mathematicians.

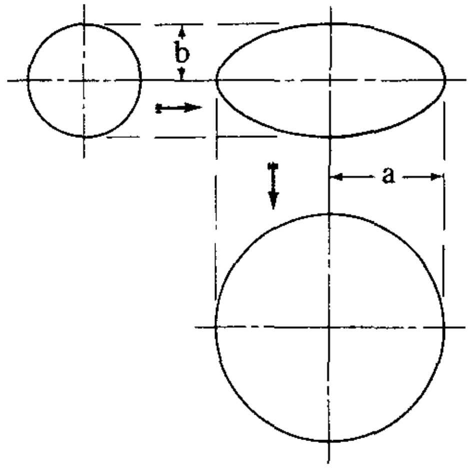
In traditionally structured mathematics courses it *is* difficult to reveal, let alone teach, the art of discovery. Particularly the weaker students, the ones who need it most, become dismayed and apprehensive by the unstructured nature of the tasks involved. When problems like (a) guess a formula for

$$\sum_{k=0}^{n} \frac{k}{(k+1)!}$$

and prove it by induction (from a first-year finite mathematics course); (b) find an unbounded sequence with a convergent subsequence (from a third year analysis course); and (c) find a property of $\mathbb{R}$ shared by neither $\mathbb{N}$ nor $\mathbb{Z}$ (from a third year logic course) cause insurmountable difficulties, we have surely failed to communicate a critical aspect of our discipline—how to attack a problem. So deeply have we failed that even the most primitive of all the arts of discovery, the art of testing formulae, theorems or conjectures on numerical examples is absent from most students' repertoires In the primordial days of slide-rules and log-tables laziness could be used as an excuse; but now? My attempts to gently inculcate this art-form have so far met with a resistance that often takes the form, "But this isn't mathematics." Pottage's book lays particular stress on the effective use of examples and counterexamples and on the Lakatosian dynamic between them and conjectures Simplicio, in an increasingly desperate search for a formula for the perimeter p of an ellipse, argues that p should be the geometric mean of the circumferences of the inscribed and circumscribed circles, $p = 2\pi\sqrt{ab}$ Through tests *in extremis* ($b = 0$) Sagredo convinces him that his formula "cannot in general provide correct measures of elliptical perimeters." Yet, like most learners, Simplicio is not convinced by emendation. Disquiet is a fair reflection of his mental state, a disquiet emanating from his heuristic not having been contradicted. "...(I) can (not) understand

† The analogy with writing is worth pursuing Students in courses on creative writing presumably see nought but the oft re-written, highly edited product Should they be exposed to the heuristic of writing? Should they see draft upon draft complete with editorial criticisms? Is there a book that is to creative writing what Pottage's book is to discovering mathematics?

why the perimeter is not multiplied by the same factor for each of the transformations



. . . what I am now appealing for is some *insight* to counter what still seems intuitively compelling to me . ." Not until Salviati exorcises his heuristic through similar transformations on rectangles does Simplicio's irritation dissolve

This highlights a fascinating learning dichotomy, one deserving of serious attention. In the context of logic, examples don't prove general assertions whereas counterexamples do disprove. Exactly the opposite often occurs in the pedagogical context. Students of mathematics frequently use examples to prove and are convinced by them. Dually they tend to be suspicious of the power of counterexamples to disprove, and often remian unconvinced by them At a behavioural level rarely do counterexamples lead to serious re-examination of hypotheses or assumptions *Geometrical investigations* could be used to instruct students in the intelligent use of examples From this book the serious student may learn to use examples as tools for exploration and illumination rather than equating them with proof. On the other hand the book shows how we the teachers must realize that to counter a conclusion with a clever example may not convince the student of his error and almost certainly will not change his behaviour. We should not equate silence with acquiescence but rather with helplessness. Not until we uncover and criticize the underlying heuristic will disquiet be abated. To teach in this way requires time, individual attention and a teaching framework sympathetic to mistakes and divergent learning styles. Pottage's work offers us models with which to construct such a framework.

Parts of the book will doubtless find their way into history of mathematics courses, at the very least as a source of excellent but elementary problems. All teachers of mathematics at all levels should have the book within easy reach so that they can periodically immerse themselves in it, from whence to return to their teaching more enriched and enlivened.

Any criticisms I have would be of the rather churlish kind, "If I had written this book I would have . . ." But I didn't, John Pottage did, and he wrote it well

Finally, the scholarly community should congratulate the publishers not only for their first-rate production but for having the courage to publish it at all, for by crossing so many disciplinary boundaries, by defying compartmentalization, the book surely has no large, readily identifiable market. I understand the publisher's courage may soon be rewarded by commercial gain.

JACK GRAY
University of New South Wales

This review also appears in the Newsletter of the Australian Association for the History and Philosophy of Science

Teachers are often urged to show enthusiasm for their subjects. Did you ever have to listen to a really enthusiastic specialist holding forth on something that you did not know and did not want to know anything about; say the bronze coinage of Poldavia in the twelfth century or "the doctrine of the enclitic *De*"? Well, then.

Ralph Boas