DEVELOPING SYSTEMS OF NOTATION AS A TRACE OF REASONING

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Consider this scenario. Middle school students are solving *The Ribbon Problem*: "I have $\frac{3}{4}$ of a yard of ribbon. My friend needs $\frac{2}{5}$ of that amount. Draw a picture of how much my friend needs and determine how much of a yard she needs. Then, write mathematical notation to represent your reasoning." One student's solution is as follows:

> Since I have three-fourths of a yard, I marked the 1-yard rectangle into four equal parts and drew out three of them below (see Figure 1, top). Then I want two-fifths of that. So I can take two-fifths of each of the three parts. Two-fifths of one part is two-twentieths. So I need two-twentieths plus two-twentieths plus two-twentieths, or six-twentieths (see Figure 1, bottom).

The teacher praises the student's picture and explanation, and asks about notation. The student writes: "$\frac{2}{5}\times\frac{3}{4}=\frac{6}{20}$." Although this notation is correct, it elides the student's reasoning. More precisely, this notation stands for only the very beginning of the process (take $\frac{2}{5}$ of $\frac{3}{4}$) and the result ($\frac{6}{20}$), hiding all that went on in between. This phenomenon is dissatisfying because it means that the student is not developing notation that is a trace of the processes of her reasoning. In contrast, when notation is a trace of the processes of students' reasoning, it can make these processes visible (Kaput, Blanton & Moreno, 2008), which opens possibilities for students to become *aware* of these processes. We consider this type of awareness to be in the province of what Piaget (2001) called a reflected abstraction—a retroactive thematization of one's ways of operating and a significant mechanism for learning (see Simon, Tzur, Heinz & Kinzel, 2004). Thus, working with students to develop systems of notation that trace the processes of their reasoning has the potential to contribute to their learning.

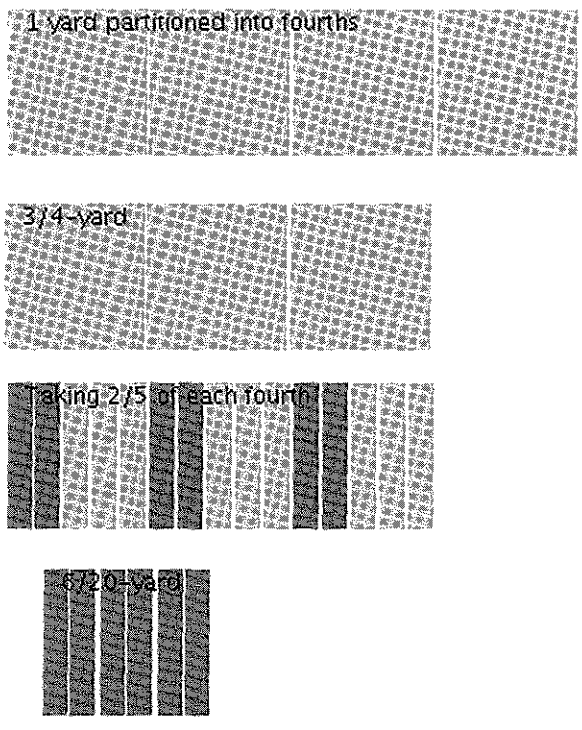

Figure 1. A length representation to illustrate distributive reasoning to take $\frac{2}{5}\times\frac{3}{4}=\frac{6}{20}$.

In this paper, we explore this issue by engaging in a thought experiment. We analyze the notation that students *might* produce, and how this notation could develop, based on models of how students take fractions of fractions, *i.e.*, how students compose fractions multiplicatively (Hackenberg & Tillema, 2009; Mack, 2001; Steffe, 2003; Steffe & Olive, 2010). Our goals are as follows: (1) to show how differences in students' reasoning could influence the notation they produce to trace the entire process of their reasoning; (2) to describe how teachers and students could develop systems of notation that are more sophisticated and yet still faithful to that reasoning; and (3) to discuss the kind of mathematical learning that could ensue from this way of developing systems of notation. To accomplish these goals, we first present our conceptual orientation to students' mathematical learning, notating activity, and reasoning with fractions.

Conceptual orientation to students' mathematical learning, notating activity, and reasoning with fractions

We view mathematical learning in the context of schemes, where a *scheme* is a goal-directed way of operating that consists of three parts: an assimilated situation, an activity, and a result (Piaget, 1970; von Glasersfeld, 1995). For example, a situation that students may assimilate using their partitive fraction scheme is a request to make $\frac{3}{5}$ of a yard, given the whole yard. The activity of the scheme is to partition the given yard into five equal parts, disembed one of the parts, and iterate the part three times (Figure 2). The result of the scheme is three out of five parts, made separately from the whole.

When a person *interiorizes* a scheme, the activity and results of the scheme become available prior to operating in a situation. For example, once a person has interiorized their partitive fraction scheme she could use the result of the

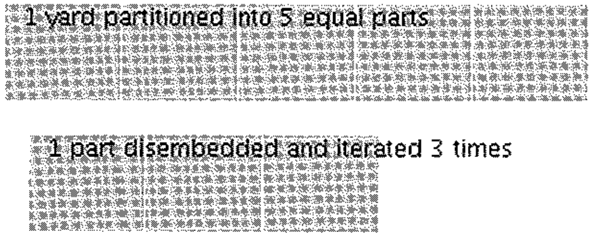

Figure 2 An illustration of the result of a partitive fraction scheme

scheme, $\frac{3}{5}$, to assimilate a new situation without having to carry out the activity of the scheme From this perspective, *concepts* are interiorized results of schemes, that is, results of schemes that are available to a person in their assimilation of a situation.

When a person makes a modification to a scheme, we consider her to have made an *accommodation*, which for us is an act of learning Accommodations are driven by reflective abstractions (Steffe, 1991; Thompson, 1994; von Glasersfeld, 1995). Piaget (1977/2001) identified four different types of reflective abstractions, two of which are reflecting abstractions and reflected abstractions. *Reflecting abstractions* involve a projection, in which schemes developed at one level are abstracted and applied to a higher level, as well as a cognitive reorganization of these schemes Whether someone who makes this type of *reflective abstraction* becomes aware of this process is uncertain In contrast, awareness is likely required for reflected abstraction, which involves thematization of the results of mental activity conducted in retrospect (von Glasersfeld, 1995). Reflected abstractions drive a person's conscious recognition of, and reflection on, her patterns of mathematical activity For Piaget, reflecting abstraction was the means by which people produce a higher state of knowledge from a lower one (*i e* , construct schemes and concepts), while reflected abstraction involved reflecting on reflecting abstractions (*i.e.*, organizing these constructions).

Most of the prior research on students' fractional knowledge from this perspective has investigated scheme construction: accommodations students make in their whole number schemes that yield what researchers have called fraction schemes (*e g*., Hackenberg, 2007, 2010; Hackenberg & Tillema, 2009; Norton, 2008; Steffe & Olive, 2010; Tzur, 2004). Much of this learning may not be within students' awareness, in the sense that they have not necessarily reflected on their constructions (even though they may indeed be aware that they have learned). One might think of this body of research as the study of students' *implicit* organization of their ways of operating with fractions. Less attention has been paid from this perspective, thus far, to students' thematization of their fraction schemes, that is, to learning that is in the province of reflected abstractions. We view students' notating activity as a site for making reflected abstractions, although certainly we do not rule out students making other types of reflective abstractions as they generate systems of notation. In other words, we view generating systems of notation to be a way to support students' *explicit* organization of their ways of operating. To explain this position, we discuss our orientation to students' notating activity

Students' notating activity

We consider a student to be engaged in *notating activity* when she produces any alphanumeric character during the functioning of her schemes (see Kaput *et al*., 2008; Lehrer, Schauble, Carpenter & Penner, 2000). This definition intentionally excludes other types of graphic items like a line segment that represents one yard, which are critical to students' symbolizing activity more broadly, but which we do not address in this paper (Tillema, 2010). We use the term *system of notation* when a student uses alphanumeric characters in a way that indicates that these characters are related to each other by a set of syntactical rules (Kaput *et al*., 2008; von Glasersfeld, 1974; see also Ernest, 2006; Hoffman, 2006).

In our view, teachers [1] are critical in helping students develop systems of notation In particular, *if teachers encourage students to develop systems of notation that are a trace of their reasoning,* then the notation that students produce can serve as a record of the functioning of their schemes (Tillema, 2010) Given this orientation, teachers and students will necessarily expand conventional systems of notation because such systems frequently reflect only the assimilatory part and the result of a scheme (see Ernest, 2005) For example, as we discussed in the opening of this article, the *Ribbon Problem* would conventionally be notated as $\frac{2}{5}\times\frac{3}{4}=\frac{6}{20}$ However, notating the situation in this way ignores the *activity* that students use to transform the fractions they use in assimilation of the situation, $\frac{2}{5}$ and $\frac{3}{4}$, to the one that they produce as a result, $\frac{6}{20}$ Thus, this orientation means a teacher may aim to help different students develop different systems of notation based on differences in students' schemes (Brizuela, 2004).

Furthermore, observing the notation that students develop as a trace of their reasoning can put the three parts of a scheme in a position to be reviewed and re-processed by the students Simon *et al.* (2004) have posited such review and reprocessing as a mechanism for making reflective abstractions, although they have not differentiated between reflecting and reflected abstractions (Piaget, 1977/2001; see also Hoffman, 2005). We posit that developing systems of notation that are a trace of the functioning of students' schemes can be a site for making *reflected* abstractions because this process requires slowing down the functioning of the schemes and produces a record for retrospective review

Students' reasoning with fractions

The study of reflected abstractions cannot precede the study of students' scheme construction because it is impossible to study how students' retroactively thematize their schemes without first making a model of these schemes Researchers who have made models of students' fraction schemes have approached fractions as quantities and specifically as measurable extents, or lengths. JavaBars, a computer microworld designed to facilitate students' construction of fractional knowledge (Biddlecomb & Olive, 2000), supports this approach. In JavaBars, students can draw bars (rectangles) of various sizes and physically operate on these bars. For example, they can mark a bar into some number of equal parts, pull out a part from a whole while leaving the whole intact, and repeat a part to make a larger fractional amount of a given bar. For students, these physical actions can be the basis for building key operations such as partitioning, disembedding and iterating (Biddlecomb, 1994). For researchers, JavaBars can indicate the operations a student uses to solve fraction problems and it has therefore been an integral part of making models of students' fraction schemes and accommodations in those schemes (*e.g.*, Hackenberg & Tillema, 2009; Steffe & Olive, 2010; Tzur, 2004).

In formulating models of students' schemes for composing fractions multiplicatively, we have also relied on making distinctions among students' multiplicative concepts. Following Steffe (1992, 1994), we conceive of students' multiplicative concepts as the coordinations of units that students have interiorized. For example, when a student has interiorized the coordination of two levels of units, she has the potential to treat a length quantity as a unit that contains some number of equally sized units, prior to implementing any activity in the situation. This student can treat a length that represents one yard as a unit that contains four equal units—a unit of units structure—without actually having to partition the length into four equal parts.

Students who have interiorized the coordination of two levels of units are able to coordinate three levels of units *in activity*. That is, as part of the activity that they carry out in a situation, they can insert more units into a unit of units structure. For example, a student who conceives of a yard as a unit containing four equally sized units may insert five units into each of the four units to create the yard as a unit that contains four units, each of which contain five units—a three-levels-of-units structure. However, this structure is somewhat ephemeral for the student in that she does not maintain it in further activity. So, even though the student produced a 20-part bar as a result of inserting units into units, in further activity the bar becomes solely that: a unit of 20 units, a two-levels-of-units structure (Figure 3).

In contrast, students who have interiorized the coordination of three levels of units have the potential to treat a length quantity as a unit of units of units structure prior to implementing any activity in a situation. In the example above, this student would maintain the view of the 20-part bar as a unit of four units each containing five units (Figure 4), and further, the student could create a view of the 20-part bar as a unit of five units each containing four units. In short, the student can not only structure a length quantity as a unit of units of units structure, she can use this structure in further activity, such as switching to a different three-levels-of-units view in the process of problem solving (Hackenberg, 2010).

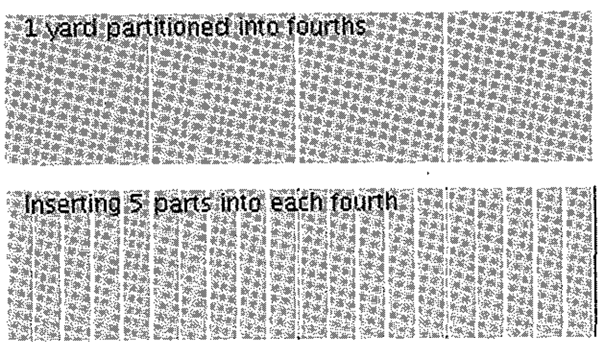

Figure 3 *A bar with four units (top) and the insertion of five units into each of the four units (bottom)*

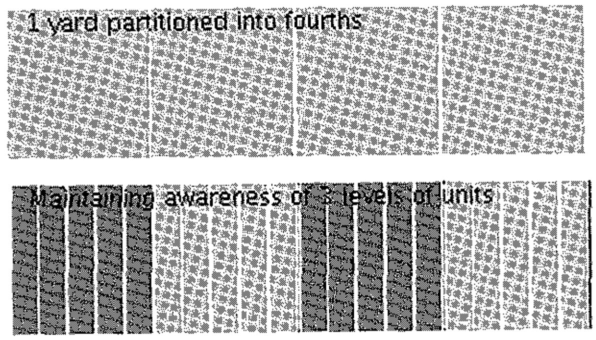

Figure 4. *The shading is used to indicate that the student maintains awareness of each of the four units after inserting five units into each of the four units*

In prior research, the interiorization of two versus three levels of units has been found to be a factor in students' construction of fraction composition schemes (Hackenberg & Tillema, 2009; Steffe & Olive 2010). Although students with either multiplicative concept can construct a scheme for taking a fraction of a fraction, their schemes have some differences. In this paper, we refer to these differences as two different levels of a *fraction composition scheme* (FCS). For example, students at each level of the scheme can take two-fifths of three-fourths of a yard of ribbon as in the *Ribbon Problem,* but they do so in different ways. We articulate these differences in the next two sections, when we explore the different kinds of notation that we conjecture students at each level may develop.

Generating notation to reflect two levels of an FCS: reasoning at the first level

To take a fraction of a fraction, students at the first level of an FCS typically insert more units into each unit fraction that comprises the given fraction. For example, a student who is taking two-fifths of three-fourths may partition each of the three parts that constitute three-fourths into five equal parts, since the student wants to take fifths (Figure 5). In doing so, the student creates a three-levels-of-units structure in activity because inserting five parts into each of the three parts involves treating the bar as a unit of three units, each of which contain five units. Because the student is constrained to producing a three-levels-of-units structure in activity, she can operate further on the 15-part bar, but in this process the 15-part bar loses its status as a three-levels-of-units structure—it becomes a unit of 15 units. Typically the student finishes her solution by reasoning that one-fifth of the 15-part bar would be three parts, and so two-fifths of the 15-part bar would be six parts. In doing so, the student creates a second three-levels-of-units structure in activity: she structures the 15-part bar as a unit of five units each containing three units, where two of these five units of three is two-fifths of the 15-part bar (Figure 5, bottom two bars).

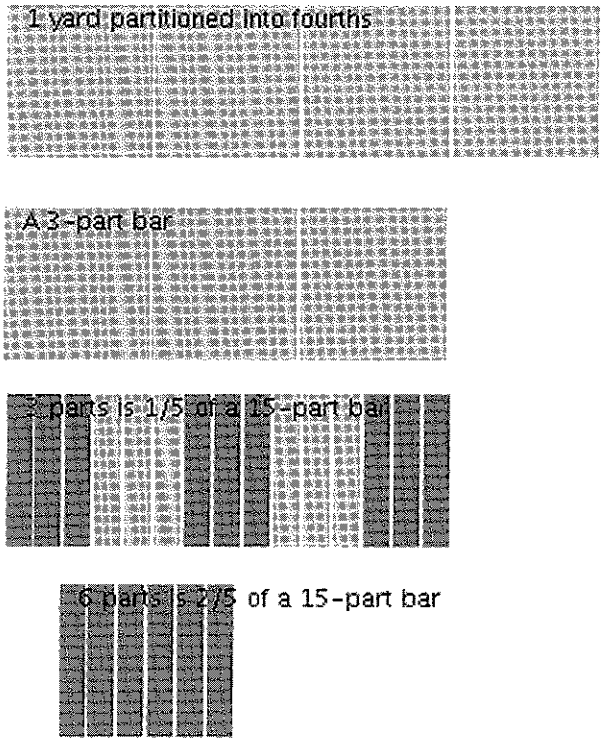

Figure 5 *A student at the first level of the FCS takes three of the four parts of the yard (second rectangle), inserts five units into each of the three units and then restructures that bar as a unit that contains five units, each containing three units (third rectangle).*

At this point, typically the student will call the 6-part bar six-fifteenths, as opposed to six-twentieths. The student takes the 15-part bar as "the whole" because it is a two-levels-of-units structure on which she has twice created three-levels-of-units structures. Since she has interiorized only two levels of units, she has yet to be able to track the 15-part bar as three fourths (each containing five parts) in relation to the 4/4-bar with which she began. However, we have found that when the student's attention is re-focused on the original whole either by another student or a teacher, she can then name the resulting fraction in relation to the original whole (Hackenberg & Tillema, 2009). So, in the example above, she can state the result of her activity to be six-twentieths

Notating at the first level of an FCS

In our experience, students who reason in the way described above often use primarily whole numbers when they begin notating their solution, as shown in Table 1. We note that students do not always include all of the statements in Table 1. For example, they may only write "$3 \times 5 = 15; \frac{6}{20}$" (see Olive, 1999, pp 307-08; Steffe & Ulrich, 2010, pp 257-58). [2] However, a teacher can encourage students to "slow down" their thinking and write notation for each mental action in their reasoning process. This slowing down can engender greater awareness of the various parts of students' reasoning by allowing students to reflect on the activity that links

$3 \times 5 = 15$	I started with 3 parts and inserted 5 parts into each, which yields 15 mini-parts.
$15 \div 5 = 3$	I need to find two-fifths of the 15 mini-parts. I can do that by first finding one-fifth of them. To do so I divide the 15 mini-parts by 5, which gives 3 mini-parts
$3 \times 2 = 6$	To find two-fifths of the 15 mini-parts, I can take the 3 mini-parts two times, which yields 6 mini-parts
$\frac{6}{15}$	Each mini-part is one part out of 15 mini-parts; since I have 6 mini-parts, I have six-fifteenths.
$\frac{6}{20}$	(Upon prompting to reflect on the relationship of the mini-parts to the whole unit bar): The whole unit bar (yard) contains another fourth, in addition to the three-fourths of the yard I was working with. This fourth would also be partitioned into 5 parts, and so there would be 20 mini-parts in the entire yard. Thus, since I have 6 mini-parts, I have six-twentieths.

Table 1 *Initial notation for a student reasoning at the first level of an FCS*

the fractions they used in assimilation to the result of their scheme (Simon *et al*, 2004). In this case, such reflection might bring about for students the idea that although this is a fraction problem, the students used whole number multiplication and division to notate their solution

A goal for students who are reasoning at this level is to integrate fractions into their notation *while preserving the notation as a trace of their reasoning*. A teacher can facilitate this transition by helping students to coordinate fraction notation with particular actions they have taken on the bars, while also recognizing that students are not coordinating all of the units in the situation. For example, once students have partitioned each part of the 3-part bar into five parts, a teacher can question students about a *fraction* name that they would give to the 15-part bar. Students are likely to call the bar fifteen-fifteenths rather than fifteen-twentieths because the bar is broken into 15 equal parts, and breaking the bar into 15 parts involved creating a three-levels-of-units structure as part of their activity. The teacher may introduce the need for further fraction notation by suggesting that it does not make sense to write "$3 \times 5 = \frac{15}{15}$" because 3 times 5 is 15, not $\frac{15}{15}$. The teacher might then ask students if they can replace 3 and 5 with fraction notation in the statement "$3 \times 5 = \frac{15}{15}$". [3] This type of dialogue between a teacher and student can yield notation like that shown in Table 2.

Developing notation in this way opens further opportunities for students to thematize how they are solving the problem. As before, producing the notation in Table 2 will involve students in slowing down their reasoning, which can allow students to reflect on the activity linking the fractions used in assimilation to the result. For example, students began with $\frac{3}{4}$ of a yard but they notated that amount as $\frac{3}{3}$. At this level of the scheme, this notation makes sense to stu-

$\frac{2}{5}\times\frac{3}{4}$	This is the fraction multiplication problem I am starting with.
$\frac{5}{5}\times\frac{3}{3}=\frac{15}{15}$	Since I partitioned each of the three parts of my bar into fifths, I made fifteenths. I can notate this action as making the 3/3-bar into a 15/15-bar by partitioning each of the thirds into fifths.
$\frac{15}{15}\times\frac{1}{5}=\frac{3}{15}$	One-fifth of my 15/15-bar is three-fifteenths. This is because one-fifteenth of 15 units is 3 units, but now I am thinking about the 15 parts as fractions
$2\times\frac{3}{15}=\frac{6}{15}$	To get two-fifths I can take two times three-fifteenths, producing six-fifteenths
$\frac{4}{4}\times\frac{5}{5}=\frac{20}{20}$ $\frac{6}{20}$	(Upon prompting to reflect on the relationship of the mini-parts to the whole unit bar): the parts are actually twentieths of a yard, not fifteenths. (See lowest right cell of Table 1 for reasoning.)

Table 2 Integration of fraction notation for a student reasoning at the first level of an FCS.

dents because they conceive of making fifteenths when they partition each part of the 3-part bar into fifths. Nevertheless, some students may notice that they began with $\frac{3}{4}$ and then used $\frac{3}{3}$ in the notation—and this may strike them as odd or interesting. A discussion about this issue may help students at this level see more clearly that a fraction can be named in relation to different wholes, which can prompt a greater awareness of the process they are using to make the fraction composition (*e.g.*, "I started out thinking of this as three-fourths, but I am subsequently going to notate it as three-thirds because that is how I think about it in the process of my problem solving activity.")

In addition, developing this notation can help students become aware that they can use fraction notation to symbolize actions that they initially think of as whole number multiplication and whole number division. For example, multiplying 3 parts by the 5 mini-parts inserted into each of these parts does yield 15 *mini*-parts in all. However, from a fractions perspective this aspect of the reasoning involves taking fifths of thirds and making fifteenths. Similarly, finding one-fifth of 15 parts can be accomplished by dividing 15 by 5, but from a fractions perspective this idea can be shown by taking one-fifth of fifteen-fifteenths. This kind of shift in perspective from whole number to fraction notation can allow students to appreciate and understand that fraction notation can be a trace for their reasoning—and we suggest that this shift is a result of students retroactively thematizing the actions they perform on quantities as *fraction* operations as opposed to whole number operations.

Generating notation to reflect two levels of an FCS: reasoning at the second level

In contrast with a student who is reasoning at the first level of an FCS, students reasoning at the second level of an FCS are able to coordinate multiple three-levels-of-units structures simultaneously (Hackenberg, 2010). Doing so means that they are able to develop more complex ways of operating and more complex systems of notation when solving fraction composition problems.

For example, students at the second level of an FCS have the potential to use a distributive operation to make fraction compositions (Hackenberg & Tillema, 2009). Students may use this operation in several different ways, one of which was shown at the opening of this paper. In that example, a student used her distributive operation by taking two-fifths of the first fourth, two fifths of the second fourth, and two fifths of the third fourth. So, to create two-fifths of three-fourths she took two-fifths of each one-fourth, which she determined to be two-twentieths, two-twentieths, and two-twentieths, yielding a result of six-twentieths. [4] In using a distributive operation the student reasoned through the entire solution using the two fractions that she used in assimilation (*i.e.*, two-fifths and three-fourths). That is, the student operated on each of the three one-fourths with two-fifths in order to produce the result. This reasoning opens possibilities for particular advances in notation.

Notating the distributive way of reasoning

In our experience, when a student develops her reasoning as discussed above, a teacher can encourage students to create notation like that shown in Table 3.

As with the notation shown in Table 2, a teacher can encourage students to generate this notation by asking questions that help students slow down their reasoning and link different actions on the bars with notation for expressing each action. For example, a teacher could ask students to write a fraction multiplication problem for the situation, and then ask them to express three-fourths as an addition problem, which would help students express their intention to take two-fifths of each one-fourth. The notation in Table 3 differs from Tables 1 and 2 because it demonstrates that students continue to operate on the fractions used in assimilation to produce the result. That is, the students begin with $\frac{3}{4}$ of a yard and continue to operate on $\frac{3}{4}$ (not on whole numbers or on $\frac{3}{3}$). Thus, in contrast with the notation in Tables 1 and 2, this notation displays a sequence of equivalent expressions.

Encouraging students to create notation like that shown in Table 3 can help students to thematize their reasoning in at least two ways. First, students may recognize that the syntax of the notation with fractions is similar to the syntax of the notation that is used to notate the distributive property with whole numbers. Thus, they may begin to see that distributive reasoning is a powerful type of multiplicative reasoning that extends beyond whole number contexts. If students do see this syntactical relationship, then they are

$\frac{2}{5}\times\frac{3}{4}=\frac{2}{5}\times(\frac{1}{4}+\frac{1}{4}+\frac{1}{4})=$ $(\frac{2}{5}\times\frac{1}{4}+\frac{2}{5}\times\frac{1}{4}+\frac{2}{5}\times\frac{1}{4})=$ $\frac{2}{20}+\frac{2}{20}+\frac{2}{20}=\frac{6}{20}$	I took two-fifths of three-fourths by taking two-fifths of each one-fourth, which is two-twentieths plus two-twentieths plus two-twentieths, or six-twentieths.

Table 3 Notation of a distributive operation at the second level of an FCS

primed for a discussion about the relationship between distributive reasoning with fractions and distributive reasoning with whole numbers.

Second, this notation may help students see that embedded in the process of composing a non-unit fraction with a non-unit fraction (*i.e.*, taking $\frac{2}{5}$ of $\frac{3}{4}$) there is a less complex process: taking a non-unit fraction of a unit fraction (*i.e.*, taking $\frac{2}{5}$ of $\frac{1}{4}$). It may help students to see this process because developing this notation creates an occasion to record this feature of students' solution of the problem. In turn, seeing that this less complex process is embedded in the more complex problem can help students begin to thematize their ideas about what is required to compose any two fractions multiplicatively—"Oh, this notation shows me that even for more complex problems it is important to be able to compose a non-unit fraction and a unit fraction." As students begin to make this type of thematization, they are likely to condense the reasoning they use to solve fraction composition problems.

Condensing the distributive notation

Table 4 shows condensed notation that a teacher can support students to generate as they condense their reasoning.

A teacher can help students to develop the notation in Table 4 by using the notation in Table 3 as a guide for questioning students. For example, a teacher could ask students to examine the notation in Table 3 to determine how many times they took $\frac{2}{5}$ of $\frac{1}{4}$, and then ask students if they can use multiplication to express this conceptualization, as opposed to using addition as shown in Table 3. This question can enable students to develop new notation from the old notation and, in this process, to condense their notation, while continuing to use it as a trace of their reasoning.

Creating the condensation of reasoning and notating in Table 4 contributes to students' thematization of their ways of operating in at least two ways, which are parallel to the ways discussed for Table 3. First, this condensation relies on associative reasoning. So, as students develop this notation, they may recognize that the syntax for the associative property is similar in whole number and fraction contexts. Thus, students may learn to view associative reasoning as a powerful type of reasoning that extends beyond whole number contexts.

Second, this notation may help students see that to compute any fraction multiplication problem involves determining a unit fraction of a unit fraction (*e.g.*, one-fourth of one-fifth); multiplying this product by the numerator of the fraction that is the operator; and then multiplying this result by the numerator of the other fraction. The multiplication by the two numerators has quantitative meaning: multiplying by the first numerator indicates the number of times larger the product of unit fractions is when the operator is a non-unit fraction (*e.g.*, two-fifths is two times one-fifth of one-fourth); multiplying by the second numerator indicates the number of times that this process needs to be repeated (*e.g.*, this process is repeated three times, once for each of the fourths). This thematization can lead to the development of an algorithm for fraction multiplication that is rooted in notating students' reasoning with fractions as quantities.

$\frac{2}{5}\times\frac{3}{4}=$ $(\frac{2}{5}\times\frac{1}{4})\times 3=(2\times(\frac{1}{5}\times\frac{1}{4}))\times 3=$ $(2\times\frac{1}{20})\times 3=$ $\frac{2}{20}\times 3=\frac{6}{20}$	To take two-fifths of three-fourths, I will take two-fifths of one-fourth three times. I know that two-fifths of one-fourth is really just twice one-fifth of one-fourth, which is one-twentieth. So two-fifths of one-fourth is two-twentieths. Then two-twentieths times three is six-twentieths.

Table 4. Notation that multiplicatively relates factors and products. [5]

Concluding remarks

These examples of how students could develop systems of notation are not meant to be exhaustive of all the systems of notation that students may produce to symbolize their fraction composition schemes. They are intended, instead, to give a sense of what we mean by developing notation that is a trace of reasoning at two different levels of a scheme, and to indicate the kind of learning that could result. Thus, we see our thought experiment as situating notating activity within a framework for learning in a way that specifies the type of learning we expect might to occur as teachers and students work together to develop systems of notation. As we have stated, how students and their teacher develop systems of notation and how this process contributes to learning has received less research attention thus far, and for good reason: researchers need to establish how students construct schemes in particular mathematical areas such as fractions prior to being able to explore how students retroactively thematize these schemes.

Yet we do not intend to suggest that students' experiences of scheme construction and thematization are non-intersecting processes: as students refine notation that is a trace of their reasoning, we fully acknowledge that the notation they develop will influence how they conceive of the problem that they are solving (Kaput *et al.*, 2008; Saenz-Ludlow, 2006; Sfard, 2000). In other words, we do not intend to portray scheme construction and thematization as independent processes. We admit that thematization of one's scheme could certainly lead to changes in the operations of that scheme: that is, to accommodations that could be considered in the province of scheme construction. However, we have proposed that developing notation with students that is a trace of their reasoning is more likely to contribute to thematization of their reasoning, because we see it as a place where students slow down their reasoning and, in doing so, reprocess and review it. Ultimately, for students, using notation that is consistently a trace of reasoning can foster the view that notation is itself useful to reason with—which is essential for learning algebra (Kaput *et al.*, 2008), as well as, more generally, for becoming a powerful mathematical thinker.

Notes

[1] We use the word *teacher* to include a researcher who is using teaching to investigate mathematics learning.

[2] These references are of data excerpts where students produce systems of

notation for making fraction compositions. These data excerpts support the hypothetical systems of notation we outline in this paper. However, the hypothetical systems of notation we outline here are more complex than the ones that have been reported. Therefore, they provide further possibilities for what a teacher could aim for when working with students.

[3] We have found that students at this level usually recognize why they would notate taking one-fifth of one-fourth as "1/4 x 1/5 = 1/20", and also why they would notate taking five-fifths of one-fourth as "1/4 x 5/5 = 5/20". Helping students develop notation for taking a fractional part of a unit fraction is an essential pre-cursor to helping students create notation for more complicated situations like the one we discuss here

[4] Students at this level can also use distributive reasoning by taking one-fifth of each one-fourth to get three-twentieths, and then add three-twentieths twice (Hackenberg & Tillema, 2009). The notation in Table 3 can be adjusted to reflect this conceptualization.

[5] In our experience, students often do not capture all of these steps in their notation; the important point is that they try to use notation to record how they are thinking about the situation

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