Form in Algebra: Reflecting, with Peacock, on Upper Secondary School Teaching

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After the first two years of upper secondary school the aim in learning algebra is not to be able to "do" algebraic calculations of ever increasing difficulty, but to be able to apply with certainty what has been learnt so far. As part of educational research, then, we must ask ourselves what kind of "control" of algebraic operations is acquired by the students: checking by substitution, recognizing known "forms", or mastering and recognising the rules of syntax

The author prefers the students to get into the habit of using the last two methods of control, i.e. she thinks that it is better, at a certain level, to underline explicitly the "leap" from arithmetical algebra to symbolic algebra, giving a major attention to "form". She takes a step back into the past, to the time when the basis for modern algebra was being laid when George Peacock underlined the two separate aspects of algebra, distinguishing between independent science on the one hand and "instrumental" science for discovery and investigation on the other.

Amongst the proposals recently set out in Italy for the last three years of secondary school, one of the *learning objectives* is:

> "the ability to use mathematical symbolism while recognising the rules of syntax for the transformation of formulae"

This statement is a "rare pearl" in programmes which, at least as far as algebra is concerned, do not make many new proposals nor do they provide specific indications for methods of application (for example, nowhere is the suggestion made to link the study of the graphs of functions with the solution of equations).

It is interesting to note that the objective mentioned at the beginning of this section is relative to the last three years of secondary school, whilst algebraic manipulation is taught in the first two years

At this second stage mathematical ability manifests itself on the one hand in operating with formulae and executing algorithms (syntax), and on the other in mathematization and in the interpretation of formulae (semantics). When discussing the algebra taught during this three-year period, not as much importance is to be given to learning some new rules (relating to radicals, logarithms, etc), as to the acquisition of a good command of algebraic (symbolic) calculations. Essentially, then, the aim is not to be able to "do" algebraic calculations of ever increasing difficulty, but to be able to apply with certainty (in analysis, in analytical geometry, etc) what has been learnt so far.

As part of our educational research, then, we must ask ourselves what kinds of "control" of algebraic operations are being acquired by the students: trials with numbers? the recognition of known "forms"? mastery of the rules of syntax?

Some aspects of this problem can be found reflected in a specific period of history. It is therefore useful to take a step back into the past, to the time when the basis for modern algebra was being laid.

1. Formal properties

1.1 George Peacock and symbolic algebra

In 1833 the English mathematician George Peacock (1791-1858) presented the Cambridge mathematical community with a comprehensive dissertation [Peacock, 1833]. Although the title ("Report on the Recent Progress and Present State of Certain Branches of Analysis") seemed to address itself to mathematical analysis, the dissertation was particularly dedicated to an area which was only just beginning to develop: symbolic algebra

> "The science of algebra may be considered under two points of view, the one having reference to its principles, and the other to its applications: the first regards its completeness as an independent science; the second its usefulness and power as an instrument of investigation and discovery, whether as respects the merely symbolical results which are deducible from the systematic development of its principles, or the application of those results, by interpretation, to the physical sciences" [ibid p 185]

The paper sets itself the task of developing the ideas introduced in the opening paragraph quoted above: to underline the two separate aspects of algebra, to distinguish clearly between independent science on the one hand and "instrumental" science for discovery and investigation on the other, this "instrument" being useful not only for experimental science but also for mathematics

According to Peacock the fundamental principles of algebra, as opposed to its applications, had not up to that date been given sufficient attention. In fact during the first half of the nineteenth century mathematicians used real and complex numbers without laying down any clear definitions. Letters were used implicitly assuming the properties of rational numbers; after all, no contradictions presented themselves if the usual methods and algorithms were adopted when other types of numbers were substituted for rational ones. Indeed algebra and analysis seemed to progress without difficulty.

Peacock was effectively one of the first people to tackle the problem of "justifying" operations on literal expressions or symbols (one could even say that the problem was more difficult to pose than to solve), and he maintained the need to devote his paper to symbolic algebra.

Even when making appropriate distinctions, I do not consider it unreasonable to draw a comparison between the prevailing situation at that time and that in which our secondary students now find themselves In fact they have learnt to replace numbers by letters, they can carry out practical exercises using them, but little by little they are distancing themselves from the numerical meanings in which the letters were born Textbooks and teachers tend very readily to take for granted a certain mastery of formalism, and they adopt a language which for many students is too obscure Let us look at an "educational" observation of Peacock's which seems to help illustrate this point.

> "[...] in the first place, the proper assumption and establishment of those principles involve metaphysical difficulties of a very serious kind, which present themselves to a learner at a period of his studies when his mind has not been subjected to such a system of mathematical discipline as may enable it to cope with them; in the second place, we are commonly taught to approach those difficulties under the cover of a much more simple and much less general science, by steps which are studiously smoothed down, in order to render the transition from one science to the other as gentle and as little startling as possible " [*ibid* p. 185-86]

Thus the student is not yet capable of a "mathematical discipline"; he seeks to refer to known entities, but, at the same time finds himself battling with abstract rules. The general opinion then, which is probably still common today, was that arithmetic provided a sufficient base for symbolic algebra

> "In reply however to such opinions it ought to be remarked that arithmetic and algebra, under no view of their relation to each other, can be considered as one science, whatever may be the nature of their connection with each other; that there is nothing in the nature of the symbols of algebra which can essentially confine or limit their signification or value; that it is an abuse of the term generalization to apply it to designate the process of mind by which we pass from the meaning of $a - b$, when a is greater than b, to its meaning when a is less than b " [*ibid.* p 194]

It must be observed that Peacock is here always referring to the transition from arithmetic (understood as a science using natural numbers) to formal algebra. In our secondary schools we generally deal with the transition from rational numbers (and occasionally real numbers) to algebraic calculation. The situation is not essentially any different: the problem is one of passing from numbers to letters.

Now one is also faced with an educational problem: is it right to pass over the properties (the axioms) of algebra as an independent science and present the formulae as an extension (generalization) of what is already known about numbers? Peacock does not seem to share this view. The first step taken by him along the road of the axiomatization of algebra was to introduce the *principle of permanence of the form*. Peacock distinguishes, with this, between arithmetic algebra and symbolic algebra.

This principle maintains that

> "Whatever equivalent form is discoverable in arithmetical algebra considered as the science of suggestion, when the symbols are general in their form, though specific in their value, will continue to be an equivalent form when the symbols are general in their nature as well as in their form " [*ibid* p 199]

Subsequently Peacock [1842-45] reaffirms his principle, also introducing a formal science of algebra. He underlines the fact that algebra, like geometry, is a deductive science It is therefore necessary to state fully the laws which regulate the operations used in the process (the axioms). The meaning of the symbols is only that given by the laws themselves.

1.2 Further historical developments

The theory of algebra as a science of symbols was supported and further developed by Duncan F. Gregory, Charles Babbage, Augustus de Morgan, and George Boole. Gregory [1840] underlines the fact that even algebra possesses theorems:

> "But the step which is taken from arithmetical algebra to symbolical algebra is that, leaving out of view the nature of the operations which the symbols we use represent, we suppose the existence of classes of unknown operations subject to the same laws; we are thus able to prove certain relations between the different classes of operations, which, when expressed between the symbols, are called algebraic theorems."

De Morgan goes further, maintaining that algebra is a collection of symbols devoid of any meaning and of operations performed on these symbols; the fundamental laws are chosen arbitrarily. It is also worth noting that De Morgan was interested in the difficulties associated with the teaching of mathematics: for example, he was opposed to the mechanical use of rules and proofs [De Morgan, 1831].

The same principle of the permanence of form was then linked to the axioms of algebra which, already by the middle of the nineteenth century, were very similar to those commonly accepted nowadays (even if only implicitly at a scholastic level) as regards algebraic calculation (commutative, distributive properties, etc.). This approach opened the way to a more abstract method of reasoning in algebra.

The principle of permanence of form was re-proposed in 1867 by Hermann Hankel, and only then was it fully accepted as a heuristic principle. Hankel also demonstrated that there are no extensions of the set of complex numbers to which the laws of classical arithmetic apply (in modern terminology, we would say that there are no algebraic extensions of the field of complex numbers) One year before the publication of Hilbert's "Foundations", A. Whitehead [1898] confirmed the independence of algebra from arithmetic with a statement altogether similar to that of Gregory quoted above

In texts of elementary mathematics at the beginning of the twentieth century we find a different statement: the sets of numbers are presented separately, and sequentially, with their properties, making a general reference to the principle of permanence, but not with the same intention as Peacock. For example, according to M. Cipolla [1929] the so-called *principle of permanence or of the conservation of formal properties* (now ascribed to Peacock-Hankel) is equivalent to the fact that, in the extension of a numerical structure, one *seeks* to define the operations in such a way that their formal properties are preserved (that is, those properties which do not depend on the numbers on which they operate, but instead on the way in which the operation is presented; that is, the *properties of the form*). The principle of permanence *can lead* to definitions, in the amplified set, of sum and product.

Also, even though this approach is currently followed in schools at an elementary level (the teaching of natural numbers coming before that of whole numbers, etc.) it is not usually expressed or confirmed at the same time as the introduction of algebraic calculation.

The principle of the permanence of form, however, should not be seen solely from a historical viewpoint. As a simple principle of generalization of operational properties it could perhaps be the questionable: it may have helped to define the sets of numbers to which it was applied, but it did not constitute a solid basis for algebra. However, even from an epistemological point of view, the principle enabled algebraic calculation to be released from the confines of a strictly numerical meaning, accepting an algebra made up of symbols, and thus of axioms for algebra. Peacock's statements regarding the meaning of algebraic symbols and the role of axioms are not so different from those of Hilbert on the meaning of geometrical entities. Therefore it is a matter of a crucial situation, even from an educational viewpoint: it is the moment at which concepts begin to reflect relations and not objects.

2. "Interpretation" in symbolic algebra

2.1 The syntax-semantics relationship

Returning to Peacock we can relive that short period of history while considering links with teaching.

> "In the absence, therefore, of such definitions of the meaning of the operations which these signs or forms of notation indicate, they become *assumptions*, which are independent of each other, and which serve to define, or rather to *interpret* the operations, when the specific nature of the symbols is known." [*ibid.* p. 197]

Thus Peacock maintains that a distinction between the two levels of algebraic understanding is vital, and that the link between them occurs through "interpretation".

In mathematics, at least in much mathematics at school (and university) level, an external reference point is missing. In branches *outside mathematics*, such as physics and biology, the reference to known measurements or relations constitutes a control which translates into an aid. A typical example, which is very simple though by no means trivial in its applications, is the control exerted by dimension.

Generally, I would explain *control* as the possibility of *interpreting* a considered algebraic expression or manipulation within a structure, a model. If algebra is seen as an instrumental science, interpretation occurs in a concrete model. But if algebra is seen as an independent science, the interpretative structure is not necessarily a numerical one. The *relations* that have been established between the objects of an (abstract) theory may provide a sort of mental image which is stronger than the objects themselves.

In developing a mathematical curriculum we find various examples of "concrete" control. A first example is given by so-called "real world problems". Many people put the application of algebraic methods to represent and solve a variety of "real world problems" amongst the aims of teaching algebra at secondary school level. But it is necessary to see what it is that makes an algebra problem real or concrete in the eyes of a student. The classic "tap" problem (to fill a bath with two taps of different delivery capacity) is definitely not real simply because it refers to concrete objects. And it is surely no more concrete than the one which asks you to prove that the product of two odd numbers is an odd number. Also computing has specific operational references (which, however, are linked by means of an *analogy of form*), as in the case of the two concepts of *variable* and *parameter*. We are, in fact, dealing here with two different levels, one higher than the other: the parameter is generally given on the computer keyboard and is varied manually by the user, while the variable changes according to a "cycle" carried out by the machine (the two levels are linked by quantifiers). Computing seems particularly to present both parameter and variable as *functional* images. Whereas, in algebraic calculation the idea of function should almost always be implicit.

It is often possible to grasp the semantics at an internal level of mathematics (in connection with graphs, geometry, properties of integers, etc.); the question is whether a semantics-type grasp also exists for the transformation of algebraic formulae, linked therefore to the "form" of the algebraic expression or to some "perceptual" type of ability.

2.2 Recognition of forms

Let us consider once more the aim of the three-year programme quoted at the start: the students must learn to operate with mathematical symbolism while recognising the rules of syntax for the transformation of formulae.

This statement seems to refer to the procedural method commonly used in logic. In fact there is a profound analogy between the manipulation of algebraic expressions and logical deduction. In both cases we are dealing with operations carried out directly on the symbols: in algebra the transformation does not depend on the value of the letters, just as with logical deduction no reference is made to the content of the statements. So the emphasis lies on the syntactical rather than the semantic aspect. Both proof and calculation proceed through a transformation of formulae, performed while observing certain rules.

How is such an "ability" developed, in the case of algebra? The question is the same as the one posed by Peacock: can you proceed with generalization while keeping in mind the idea of a number? Can we then rely on verifications carried out by making substitutions of the numbers in a formulae? My own opinion is that mastery of numbers is not enough, and that something more is needed; not exactly an axiomatics of algebra, but nonetheless an attention to "forms".

I shall clarify what I mean here by form, quoting some examples from a test set in Rome to third year secondary school students (16 years old) [Cannizzaro and Menghini, 1992]:

- Find the value of the following expression:

$$\frac{z^4}{c^2}(b-a) + \frac{z^4}{c^2}(a-b) = \ldots$$

For this exercise most of the students carried out very detailed calculations

- For which values of x is the following inequality true?

$$x^2 + x < x^2 + x + 1$$

In the best answers students carried out a comparison of the two parabolas represented by the two parts of the inequality.

- Express each of a, b and c in terms of the other two values

$$a-4c^2 = \frac{1}{b}$$

Only 50% of the students gave three correct answers.

- If m, n, p, q are natural numbers, the following equalities are true (always, never, sometimes):

$(2n + 1)(2m + 3) = 2p$
$(2n + 5)(2m + a) = 2p + 3$
$(2n + 1)(2m + 1) = 4pq$

This last exercise was set for university students (in the Mathematics Department, of course!) and very few answered correctly, most of them saying things like: "The first equality is true if $2p$ is an odd number. "

What seems to be missing from the answers to exercises such as those listed above is the ability to recognise more general laws and forms, or to be able to achieve an abstract understanding independent of the specific numerical context In these cases numerical examples are not as much help as the recognition of formal properties. Thus, the third exercise requires an understanding of inverses, whilst the first can be solved by "seeing" its distributivity. The second exercise asks for a little more: it is useless to think of "the square of an unknown number added to the number added to one"—one needs to see that a comparison is being made between something and the same thing with one added to it. In the fourth case one must recognise the formal ways of writing even and odd numbers.

There are various researches concerned with difficulties connected with algebraic manipulation. The problem of "unconscious manipulation" emerges from nearly all of the investigations on the teaching of algebra in secondary schools (cf. in particular [Harvey *et al.*, 1992], [Sfard, 1992], and Lee and Wheeler, cited in [Bell, 1987]), a problem which is linked to the inability to translate into symbols and interpret formulae. Many students seem to prefer long, monotonous, obviously repetitive processes, which, as they become automatic, require very little concentration or reasoning (but are "guaranteed") instead of brief, concise processes with few calculations which, however, require the active distinguishing of similarities and differences, and understanding of the rules, and the ability to synthesize. The classic algebraic mistakes, linked to the problems of *generalization,* are also "unconscious" [Mason *et al.*, 1985]. Not thinking it necessary to understand what one is doing is surely also linked to an image of classroom culture. But there is obviously much more to it than that.

Linked to the recognition of forms is also the ability to carry out *substitutions.* The substitution of numbers for letters and vice-versa is the reason for algebra: it is the same thing whether you substitute and then carry out the calculations or carry out the calculations and then substitute. More complex is the substitution of letter for letter. Difficulties arise in analysis when $x + \Delta x$ is substituted for x, or in the proof of the non-associativity of the arithmetic mean when it is required to find the arithmetical mean of $(a + b)/2$ and c. In fact, a certain level of formalism is required to enable substitution to work.

Freudenthal [Freudenthal, 1983] sees one possible solution in "going back", fastening again on the content. A different position is that of V. Byers and S. Erlwanger [1984], who claim that many teachers do not want to accept the idea that mathematical form requires a special kind of understanding, different to the understanding of context: "Understanding of mathematical form seems to lag behind the understanding of content".

Similarly, E. Gallo [1992] maintains that the key element to focus on is the transformation of the letter—seen as a symbol in place of a number— to its own structural use.

According to two mathematician-psychoanalysts [Canestri and Oliva, 1992]:

> "When learning mathematics the mastery of abstraction or conceptualisation is not sufficient: one also needs a higher level of proficiency in formalization "

In conclusion, recognition of forms derives from a mastery of syntactic rules, but is perhaps more than just that: one must be able to assign a meaning to an abstract structure in which those rules can be interpreted.

2.3 Structures

With Peacock, then, the step is made from a simple generalization to a real detachment from origins. But is it right for the students to follow an analogous road, thus detaching themselves from numerical meanings through algebraic calculations? It is not my intention to defend the idea of formalism (the primary objective still remains that of applying algebra to other branches), but I do not consider it foolish to follow a teaching programme which ties up with algebraic structures. If in algebraic calculations it proves useful to underline formal properties then it is worth not limiting such operations to exclusively numerical properties. Let us give the pupils more experience in the activity of *interpretation* which Peacock speaks about, studying sets with different properties. Such an interpretation is, after all, one of the key points in mathematics. In the Italian three-year plan algebraic structures are introduced in the fourth year of the classical or scientific "lyceum"; it mentions groups and rings, with an implicit reference to geometrical transformations and sets of numbers. In this sense, then, the structures seem more linked to some meaning other than algebraic calculation itself. But there are also other algebras to discuss, such as Boole's. Attempts at introducing them at school level have proved fruitful. Because Boole's algebras have simpler axioms? Because it is easy to check them, to interpret them in the context of sets? Or simply because a particular attention is devoted to the context in a first moment, and in the next it does not come up any more? If we had the means with which to answer these questions, we would also perhaps have further information regarding the question of teaching algebra. From experiences in schools we know that a knowledge of some formal-structural properties, such as the existence of inverses and opposites, does help to solve equations, more so than a continual referral to context.

But, of course, we must be cautious; our purpose is not to present a course of abstract algebra. Peacock himself seems to warn the teacher:

> "As long as we confine our attention to the principles of arithmetical algebra, we have to deal with a science all whose objects are distinctly defined and clearly understood, and all whose processes may be justified by demonstrative evidence. If we pass, however, beyond the limits which the principles of arithmetical algebra impose, both upon the representation of the symbols, and upon the extent of the operations to which they are subject, we are obliged to abandon the aid which is afforded by an immediate reference to the sensible objects of our reasoning. [...] But the necessity which is thus imposed upon us of dealing with abstractions of a nature so complete and comprehensive, renders it extremely difficult to give to the principles of this science such a form as may bring them perfectly within the reach of a student of ordinary powers, and which have not hitherto been invigorated by the severe discipline of a course of mathematical study." [*ibid.* p. 283]

Conclusion

Traditionally, algebra in schools has been dealt with at a syntactical level; the students have no "meta-control"; they know that they are allowed to do some things and not others, and obviously they sometimes makes mistakes.

Freudenthal confirms (in the paragraph "Formalising as means and aim") that for the majority of students who have got into contact with mathematics, formalising is mastering (or, in fact, not being able to master) formal rules. "What to do? Desisting from teaching mastery? After all, when students learn division they are learning a technique, not its meaning." [Freudenthal, 1983]

Indeed techniques work in a reasonable way, and for many they constitute a necessary base from which to carry out operations. Prodi and Villani [1982], who certainly do not defend "series" of exercises based on standard solution techniques, admit that "even in calculation, with appropriate training, it is necessary to create conditioned reflexes, so that when carrying out a mathematical process, the mind may be only partly absorbed by the functioning of the algebraic mechanism".

After all, this is one of the aims of algebra. Whitehead and Russell justify the symbolic writing in their works precisely as the unburdening of the mind, the precision achieved, and the "neatness" of the presentation. [Knobloch, 1981]

Of course it is always necessary to distinguish between mechanical learning and the automatic techniques that one acquires after having interiorized the learned matter

Solutions? Well, to improve the situation one can call to mind an algebra which is always linked to a context; not necessarily to the (often unreal) "real world problems", but to the properties of numbers, or to the manipulation of functions, in all cases where is is necessary to interpret the result. A. Bell claims that the students must become competent in the performance of these complete processes, not only of symbolic manipulation [Bell, 1992].

But there is something to suggest the need to underline that same symbolic aspect, presenting a sort of axiomatic of symbolic rules. This may eventually happen during the last three years of school, when one thinks that the students have acquired a certain "mathematical discipline". In fact, at a certain point the "leap" from arithmetical algebra to symbolic algebra must be *explicitly* underlined, not being content with the use of a language which has become ever more formal and detached from its initial justification

In this paper it is not my intention to make the case for the rediscovery of Peacock as a didactician, but I do maintain that the moment in history when a specific crucial point, an "epistemological obstacle", was explicitly analysed, is also worthwhile looking at from an educational point of view.

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If we think of education, in the broadest sense, in terms of the metaphor or concept of *transmission*, looking only from one mature generation to the next, we are looking at the process from a point of view analogous to that of the classical geneticist who takes the phenotypic character as the expression of what has been transmitted from a previous generation. When however we look at the *process* of intellectual development we see that the "transmission" must be replaced by a more complex conception of the interaction between the apparatus of thought and action which infants and children bring to the education situation and that which is brought to it by adults who purposefully or otherwise, play the part of teachers. What children assimilate and add to their repertoires of thought and action is not related in any simple or automatic way to the intent of the teacher. From this point of view human culture is not so much transmitted as, in each generation, reconstructed Just as variety in the biological phenotype is not in general simply the expression of genotypic variety, so the knowledge and habitude of one generation is not simply the expression of what the previous generation "transmitted" to it.

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