CHALLENGES OF PARTICULARITY AND GENERALITY IN DEPICTING AND DISCUSSING TEACHING

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This essay focuses on representing and discussing the ephemeral, time-bound activity of teaching which, for the sake of this essay, we define as the interaction that happens in classrooms among teachers, students, and the subject matter that is taught [1, 2] How might one depict the activity of teaching so that the teaching can be discussed among practitioners across differences of context (*e g.*, school, students, curricula, . .)? To reflect on the deep challenges of such depiction and discussion, we consider non-fictional videotapes of actual classroom interaction and fictional animations depicting scenes from classrooms. Such representations of practice raise questions about the degree to which practitioners in different circumstances can discuss their circumstances by projecting them onto a common text.

To understand discussions of teaching mediated through these two sorts of depictions of classroom interaction, we look to mathematics where much effort is devoted to representing abstract aspects of human experience [3] in the service of making general claims. We examine representational systems mathematics has developed for representing the particular and the general when discussing the objects of Euclidean geometry and of algebra. These representational systems were once cutting edge technologies in mathematics that are now taken for granted Over time, norms for talking about these representational systems have been developed. And, since they are taught in school, mathematics educators have had the chance to see how novices to these representational systems have difficulty learning these norms Reference to these representational systems will aid us in contemplating discussion of teaching around depictions of classroom interaction.

Videotape of real classroom interaction to depict teaching and support discussions about teaching

Video and film of actual classrooms have long been used to represent teaching (consider, for example, the films that Robert Davis produced in the 60's; Davis, 1964). Much current professional development is based on videotapes of classroom interaction (see Seago, Mumme & Branca, 2004). Video records of practice are arguably a means to bring novice teachers in contact with practice (Lampert & Ball, 1998) and can stimulate reflection by experienced practitioners as well Van Es and Sherin (2008) note:

> Video has been used for decades in teacher learning and it appears to show promise in supporting teachers in learning to notice Video is able to capture much of the richness of classroom interactions, and it can be used in contexts that allow teachers time to reflect on these interactions. (p. 244)

Video has been used to showcase and transmit desirable techniques and strategies (*e g.*, Burns, 1996, and the related *Mathematics with Manipulatives* videotapes [4]) as well as to record and research intact instruction (*e.g.*, the videos collected by TIMSS; Stigler & Hiebert, 1997) Video has also been used to disseminate the findings of comparative studies of instruction (*e g.*, Stigler & Hiebert, 1997, and the related TIMSS videotapes [5])

Video is an appealing technology with which to represent teaching, possibly because it combines some of the immediacy of the experience with the opportunity to relive the experience multiple times The capacity to identify specific moments and replay them makes it possible to share the same visual and auditory experience repeatedly, though the meaning that different constituencies may make of it can vary (see Jacobs & Morita, 2002). As opposed to narrative cases, video allows one to point to an example rather than have to rely on language for describing it (see Chazan *et al*, 1998, for a related point). To the extent that our language for describing teaching is still under-developed, one cannot underestimate these advantages of video technologies: if our goal is to achieve some deep understanding of teaching through the consideration of the particulars of a teaching instance, making use of a video record of that instance is a reasonable choice. Still, while widely used and extremely useful, video records have their limitations.

Video as the diagram in a two-column proof

To flesh out the limitations of video, we draw on an analogy with the diagrams that accompany a proof in the two-column proof format (see Herbst, 2002, for the development of this practice over time), as well as challenges students sometimes experience when interacting with two-column proofs Netz (1999) argues that Greek mathematicians developed norms for using letters to allow diagrams to communicate the way in which geometric arguments were constructed But, this practice has limitations. Regardless of whether geometric objects are sketched or constructed, the resulting sketches or diagrams are always particular; they have particular angle measurements, particular lengths, and particular areas; any diagram has specific properties (Laborde, 2005, refers to them as spatio-graphical properties) that go beyond the properties necessitated by the figure or geometric object that the diagram purports to represent. Nineteenth century French textbook writers became concerned about the intrusion of particularity into a discourse of generality and therefore banned geometric diagrams from Euclidean geometry texts (Haggarty & Pepin, 2002).

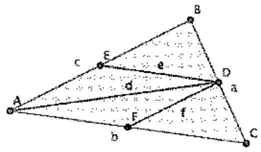
In US mathematics education, as Herbst (2002) outlines, developments in the ways of representing geometric objects took place around the turn of the 20th century, as mathematics educators faced the challenge of helping a larger percentage of the high school age cohort to learn to produce mathematical proofs in geometry classrooms. Cognizant of both the strengths of diagrams and their limitations, these educators developed a format for the communication of deductions regarding general if-then statements in Euclidean geometry. Following in the Greek, rather than French, tradition, in the two-column proof format, the "givens" and "to prove" are accompanied by a lettered diagram. As a particular representative of the class of figures being discussed (the geometrical object), the diagram creates a register that allows for the discussion of the general through the particular (requiring of the reader what Mason, 1989, calls a delicate shift of attention). The two-column proof states the "givens" and "to prove" (the characteristics that all figures in this class would be shown to have) in relationship to a particular diagram. Subsequent numbered statements about the general class of figures described by the "givens" are articulated with reference to the lettering in that diagram. Each of these numbered statements is supported by a previously proved theorem, an axiom, a definition, or the givens. These "reasons" justify that, given what is already known about this class of figures, the content of the statement must also be true.

Thus, if given three non-collinear points in the plane and the construction outlined by the givens and diagram, the argument in Figure 1 is an attempt to prove that in *all* such arrangements the original median divides the triangle into two triangles of equal area (in the more specific, diagrammatic register, that the areas of triangles ADB and ADC in the diagram in Figure 1 are equal).

Yet, research on students' understandings of this format as it appears in their instruction indicates that the goals of instruction are often unclear to students (*e.g.*, Fischbein, 1982); for example, for some students the two column proof format does not convey the intended generality. Chazan (1993) demonstrates that because there are no explicit markers of generality in a two-column proof, some students in his study felt that they had proven a result for the particular triangle pictured in the diagram accompanying the proof, the triangle with these angles and these lengths. In order to establish this result for a new triangle with its medians drawn in, particularly one that was visually quite different, they would need to repeat the steps of the proof. Thus, the norms developed by mathematicians and mathematics educators are not transparent to students. Students cannot automatically see through the particular to the general. Rather, in their geometry classrooms, they are being enculturated into a way of gaining the affordances of the particular to discuss the general, while minimizing the limitations that the particular has for discussing the general (see Chazan, 1990, for the work of a group of teachers on helping students with issues of particularity and generality exemplified with the construction implicit in Figure 1)

The median of a triangle divides the triangle into two triangles of equal area

Given: In triangle ABC, AD is the median from A, E is the midpoint of AB, F is the midpoint of AC.
To Prove: The area of ΔADB = the area of ΔADC



Statements	Reasons
1 m $\overline{AE}$ = m $\overline{EB}$ m $\overline{AF}$ = m $\overline{FB}$ m $\overline{BD}$ = m $\overline{CD}$	1 Given
2. $\overline{DE}$ is parallel to $\overline{AC}$; $\overline{DF}$ is parallel to $\overline{AB}$. m $\overline{DE}$ = ½*$\overline{AC}$; m $\overline{DF}$ = ½*$\overline{AB}$	2. Mid-segment Theorem
3 m ∠BDE = m ∠DCA; m ∠FDC = m ∠ABC	3. Corresponding angles
4. ΔCDF ≅ ΔDBE.	4. Angle-Side-Angle.
5. The area ΔCDF = the area ΔDBE.	5. Definition of area.
6 ΔEDA ≅ ΔFDA	6. Side-Side-Side.
7. The area ΔEDA = the area ΔFDA.	7. Definition of area.
8. The area ΔADB = the area ΔADC.	8. Adding equals to equals.

Figure 1 A two-column proof.

Learning from the analogy

Videotapes of actual classroom teaching practice, while much more complex than the diagrams in a two-column proof, are analogous to the diagrams of the two-column proof format in geometry in the way they can enable access to the general through a particular. A video record from a particular class can play the same role in a discussion about teaching that a particular diagram plays in a discussion about a class of figures. For whomever uses the video to make a point, a video record may stand as a sign for a general class of events in the same way that the diagram may stand as a representation of theoretical properties for the geometer.

But the particularity of the video can be a source of challenges, in addition to being a scaffold between a discourse of generality about teaching and the particular instances captures in a clip; viewers of the video, like the geometry students interviewed by Chazan (1993), may instead see it as a depiction of particular events. Thus, a video record of a classroom event may be meant as an illustration of a teaching strategy; yet the viewer may have difficulty teasing apart the strategy from other, inessential details of the classroom events in which the strategy was embedded. A similar challenge exists when using records to communicate how teaching is done in a particular culture: the viewer can have difficulty separating what is characteristic of the culture and what is incidental to the episodes recorded. These challenges also exist for teacher educators who broker conversations about teaching: to explore the boundaries between the typical and the atypical in teaching, facilitators may be challenged in distinguishing between what appears in an instance of practice and what the instance is meant to be a case of.

The analogy we are pursuing breaks down when it comes to the register for expressing generality; educators who use video do not have a register of generality analogous to the one available to geometers when they articulate a theorem without recourse to a diagram; nor do they have a mechanism to translate this generality into a specification of the givens and the to prove, which in geometry set the granularity with which one is to look at the diagram. Such coordination of the particular and the general in discussing teaching does exist, but is made possible through the artistry of facilitation rather than through shared conventional practices. In the absence of such conventions, some conversations about teaching are hard to sustain and benefit from: they may lose their focus on target facets of teaching and instead jump from one of the multitude of incidents and peculiarities shown in the video record to another, giving a sense of excessive richness in the video

In terms of particularity and generality, sometimes video records are perceived as too particular and that prevents discussion: viewers may want to discuss the teaching of lower track secondary mathematics and feel that a video that shows the teaching of fifth grade mathematics has nothing to say to them, even if the facilitator meant it as illustrating a general issue about teaching. On the other hand, sometimes video records do not seem to include enough particular information; they do not have what people seem to need in order to engage in a conversation about teaching: for example, they may lack information about what happened before and after the video clip.

In general, it seems that some of the same characteristics that recommend video as a useful tool for supporting conversations about teaching are also the sources of challenge. The richness and particularity of a video are what allow people to feel that they are in the presence of a lesson, yet the particularity of video can be such that it does not allow viewers to project their circumstances onto the provided representation of teaching and instead to focus too much on the circumstances of a different teacher in some other place.

Depicting with the unreal rather than the real

To return for the moment to the issue of representation of particularity and generality in geometry, the advent of computer technology brought a new wrinkle, the capturing of geometrical procedures by the computer and the capacity quickly to create multiple diagrams from a single procedure. Thus, in a software application that predated the computer mouse, one could do a construction on three particular starting points and then repeat this construction starting with other points (Schwartz & Yerushalmy, 1987). Or, in current software applications (as described by Goldenberg & Cuoco, 1998), one can create a diagram according to a construction and then drag base points to create new instances. The speed of the recreation of the images creates the sensation that one has a diagram with stretchy segments that can be dragged about, rather than a collection of diagrams.

While the static images that one sees on the computer screen are still particular, the ease with which they are changed supports the asking of all sorts of "what if" questions. What if we drag the points and make all three points collinear, what happens? Building on the earlier analogy between diagrams and video, the question is: what sorts of representations of teaching might be particular, but easily changed; what sorts of representations of teaching might support the asking of "what if" questions. What would be the equivalent of stretchy geometrical lengths in a representation of teaching?

In the Thought Experiments in Mathematics Teaching (ThEMaT) Project, we have explored two-dimensional, animated depictions of classroom interactions [6]. While, as will be explicated later, these animations are influenced by an idea of creating sketches of classroom interaction, in making these animations, we chose consciously to represent classrooms with characters that are patently unreal. An analogy with the use of literal symbols in algebra will help provide a rationale for this choice and will bring out other aspects of these depictions of classroom interaction.

An alternative mathematical strategy: generalizing with x

In mathematics, abstractions tend to be built out of other abstractions (using a process that Sfard and others call "reification", *e.g.*, Sfard and Linchevski, 1994). For example, people have abstracted counting numbers as a characteristic of collections of objects (Frege, 1884/1980). Once this aspect of experience was identified it was represented in tallies, using alphabets, and with the Hindu-Arabic numerals.

But, numbers, as characteristics of collections, were not just represented; with arithmetic, they were also acted upon and used to calculate. In what some (*e.g.*, Cajori, 1919) call rhetorical algebra, unknown quantities are represented with words and diagrams. But, such representations had their limits. Starting in the 17th century, the x's and y's that we are familiar with from school algebra were used to represent particular unknown numbers, representatives of a class of numbers, or the totality of the class all at once. While the ambiguity of this notational system is one of the key challenges in learning school algebra (as Yerushalmy & Chazan, 2002, argue), the power of this notation was soon evident. One can write equations involving numbers, but with a letter representing one of the numbers, and then reason about the letter as if it is a number. Viète (1983) noted that this strategy allows a mathematician to work analytically (as opposed to synthetically, in the Greek oppositional sense of these terms): one can assume that an equation has an answer, name that answer x, reason assuming that x represents an as-of-yet unknown number, and then in the end unmask the numerical identity of x. If one's assumption that there was an answer was incorrect, then one's reasoning will lead to a contradiction

A key aspect of this notational advance is to use one kind of thing to stand for another, to use a letter to stand for a number (in this use, the letter is sometimes referred to as a literal number). Precisely, because a letter does not usually denote a number, it has few of the characteristics associated with numbers. One cannot look at it and determine if it is even or odd. It does not telegraph whether or not it is prime. Alternatively, because the letter is not a number, it can be conceived of as representing any possible number, or even all numbers at once. While mathematics educators have developed clever ways to have young students see through the specificity of numbers to the role that they play in arithmetical equations (*e.g.*, Fujii, 2003), the affordance of literals (x's and y's), as opposed to numbers or geometrical diagrams, is that they aid in seeing generality without having to ignore some particularity.

Of course, use of this technology also requires enculturation. Algebra students are known to wonder what the x's and y's are all about (see Usiskin, 1995, for a response). As we have suggested earlier, some of this wondering is due to the different uses to which these symbols are put (see, for example, Usiskin, 1988). And, then there are important notational ambiguities to address: for example, to learn to use this technology students must learn that 2n does not imply a number in the twenties, but rather a number multiplied by 2; that n is not representing a digit, but rather a number, however many digits it may take to write that number.

Learning from this metaphor

As with the earlier mention of geometrical constructions that can be "dragged" with a mouse, the technology of literal symbols in algebra raises questions about depiction of teaching. For the purpose of depicting teaching, is there an analogue to the literal? Is there some way of depicting interaction among teachers and students that would suggest either an unknown particular or a more general class? Would there be discussions about teaching for which such a depiction of teaching would be useful?

ThEMaT has created animations peopled by cartoon characters. In an analogue to the literal number, inspired by McCloud (1994), our animations are built around nondescript cartoon characters. These characters are clearly not people, though the interactions between them model some aspects to be found in classrooms. In the same way that mathematical norms ask us to treat literals by the rules governing numbers, we ask participants in our study groups to discuss these characters as if they were teachers and students. While our teacher characters do not have all the characteristics that a particular teacher has (*e.g.*, they do not have hair, knees, etc.), by virtue of the role assigned to the cartoon characters in the talk depicted in the animation, they have some key characteristics, and a kind of indeterminacy, that potentially allow a wide range of teachers to identify with them as teachers.

One important side benefit of such indeterminacy is that one is no longer watching the classroom of some particular, real teacher. Instead, one is watching the classroom of a fictional teacher. And, alternative scripts can be represented by the same set of characters without one alternative having a privileged status over others. In the same way that x can be an even number in one problem and an odd number in another, a teacher character can be a "traditional" teacher in one alternative enactment of a story and a "reform-based" teacher in another. Finally, given the difficulties often noted around critiquing instances of actual practice represented in video, one can criticize the actions of this teacher character, and implicitly of the person who created the animation, without criticizing the teaching of any particular individual.

Thus, in the animations we have produced, one thing is substituting for another; a cartoon character is representing a real person. At the same time, they have the feeling of a sketch. Some essence is being conveyed, without much specificity about other aspects.

It is important to note, however, that in creating this sort of indeterminacy, as designers of the representation, we are not claiming that the aspects of teaching we represent specifically in the animation are the important ones, and therefore are to be represented and to be discussed, and that other aspects of the interaction are not important, and therefore are either left vague or not represented and should not be discussed. Instead, we are interested in identifying the characteristics that hold together the class of events that we seek to discuss and make these characteristics specific, while leaving other characteristics of the interaction indeterminate. To return to the analogy with diagrams in a two-column proof, we seek to make the givens salient, without providing too much additional specificity (in the same way that the diagram may also convey some information that goes beyond what is in the "givens"). This sketchiness allows participants in a conversation to project their own circumstances onto the parts of the interaction that we have left vague or have omitted.

Unlike the conventions around the discussion of geometric figures, conversations around an animation might thus productively focus on aspects of the classroom interaction that are left vague in the animation, or even omitted, not only those that are represented specifically. In addition, participants in the conversation might discuss whether something we have represented is a part of the "givens," or is a corollary of the "givens," or whether it is an artifactual, particular aspect of the interaction that could have been changed, or even left unrepresented.

Conclusion

In recent years, video recordings of classroom interaction have been a mainstay of our repertoire for supporting conversations about teaching. At the heart of how videotape supports conversations about teaching is the particularity of this representation and the multitude of characteristics it presents the viewer for interpretation. Video seems especially well suited to conversations in which teachers or prospective teachers are being taught to use evidence of the sort that videotapes can capture to make arguments about student thinking, for example.

In this essay, we have suggested that the very strengths of video for supporting some conversations about teaching may be drawbacks in allowing other conversations. Because of its specificity, teachers may feel that a video does not allow them to project their own circumstances onto an interaction and thus rules out possibilities that might be important for conversation. Stimulated by analogies with diagrams and literal symbols as tools for representing and discussing general claims in mathematics, we have argued for projective representations of teaching, representations which viewers can use to share their own perspectives.

Notes

1. The research reported in this article is supported by NSF grant ESI-0353285, to the authors. Opinions expressed here are the sole responsibility of the authors and do not reflect the views of the Foundation.

2. Of course, by doing so, we do not intend to suggest that classroom interaction represents all of the work of teaching.

3. Though, of course, there are important disagreements among philosophers of mathematics about the relationship between mathematics and human experience.

4. http://www.mathsolutions.com

5. http://nces.ed.gov/timss/video.asp

6. For a short excerpt see http://www.youtube.com/watch?v=lyINP3IXwqk, more of these animations can be seen in ThEMaT Online, http://grip.umich.edu/themat.

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From a constructivist perspective, concepts are not inherent in things but have to be individually built up by reflective abstraction; and reflective abstraction is not a matter of looking closely but of operating mentally in a way that happens to be compatible with the perceptual material at hand. Hence physical materials are indeed useful, but must be seen as opportunities to reflect and abstract, not as evident manifestations of desired concepts. Cuisenaire rods, for instance, are not embodiments of numbers, but their physical properties are such that they invite the construction of units and attentional iteration

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