SECONDARY SCHOOL STUDENTS' MATHEMATICAL ARGUMENTATION AND INDIAN LOGIC (NYAYA)

BRUNO D'AMORE

Over a long period of time, I have been observing students involved in demonstrating theorems. [1] The demonstrations were intended to be individual activity, but were carried out within a classroom setting and were therefore able to be considered as social activity. I noticed that, although there were many typologies of more or less spontaneous behaviour presented to satisfy the teacher's request (*i.e.*, a pattern more or less bound to Aristotelian or Megarian-Stoic logic), some students' demonstrative modality could have been thought about in a different way from the one expected by the teacher. This modality brought clearly to mind quite a different kind of logic, highlighting certain factors such as the use of examples or the preliminary enunciation of the thesis.

The *nyaya* approach described in this article shows that other cultures have produced intellectual mechanisms of 'truth' generalization and predication different from Aristotle's logic (see, for example, Needham, 1959; D'Amore and Matteuzzi, 1976; and more recently, Sarma, 2005; Sarukkai, 2005). Both the ideas of collecting evidence of the implicit and unaware use of this logic, along with an analytic perspective, gave rise to this article. I initially introduce the basic elements of this logic, followed by the presentation of three illustrative cases. I finally present some concluding remarks and reflections.

I explicitly want to emphasise that my purpose here is not to substitute one logical model for another in school education. From the explorations I have conducted and relate here, it emerges that, in order to reason properly, students have a strong tendency to use particular cases in order to 'read' and 'see' the general in them. There is evidence of a dialectic between generalisation (or abstraction) and concretisation (or specialisation) that must necessarily be taken into account while teaching.

Even after this piece of research, I do not believe that these students think in exact accordance with *nyaya* logic. Nevertheless, having recourse to this logic for the analysis of students' mathematical reasoning highlights the fact that any didactic analysis presupposes, in one way or another, a frame of reference: it is possible to use several logical frames to account for the students' deductive behaviour. Each interpretation of deduction presupposes a logical frame of reference, and deviant behaviour is judged only relative to the particular frame of reference considered.

Some characteristics of the nyaya philosophical school

Although European syllabi seldom take into account the study of oriental philosophies, I believe it is fairly well known that in India a philosophical doctrine called *nyaya* (in Sanskrit) opposed the classic Buddhist school. (*Nyaya* literally means *logic*, so it is redundant to call the logic of that school *nyaya*.) *Nyaya* considered rational speculation to comprise a fundamental basis for a coherent doctrine of knowledge, coming close to what we would now call deductive logic, relinquished by Buddhist philosophy.

The basis of the *nyaya* school was empiricism: as we shall see, such logic was consequently quite different from the Greek Aristotelian model that prevailed throughout the Western world at the time. Among other things, Aristotelian logic also shaped, in a form that still persists today, the way of handling mathematical demonstration. However, while this is certainly true at an academic level, it is not so for young students at school. [2]

By denying a transcendent principle of the universe (typical of many Indian doctrines), *nyaya* built an atomistic physics, within a realistic mould, that supported the existence of nine primordial substances and a system of sixteen objective categories immanent within the real. Its gnosiology (philosophy of knowledge) was based on a unity between purely sensory knowledge (relative to the external world) and the conversational one (relative to no matter which communicative language). The *nyaya* doctrine thus acknowledged a form of existence where communicable concepts are also real entities.

The *nyaya* school claimed the hegemony of four means of gaining knowledge (*pramana*):

- witness/testimony
- analogy
- perception
- inference.

I will examine each of these concepts in more detail:

Witness (sabda) deals with what is reliable from handed-down communication, written or oral. This would include such things as God's revelations, handed-down history, prayers and sacred poems.

Analogy (*upamana*, also translated as *comparison* or *equivalence*) is the way of reasoning that defines an object in terms of resemblance to others. Note that *nyaya* analogy classifies objects according to categories or classes of analogues, distinguishing between two classes on the basis that they do not have analogous terms. Since an analogy between existing objects depends on considerations regarding the object itself (and therefore these are not abstract considerations but classificatory and experimental ones), this form of knowledge relates to some present-day concepts also found

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in mathematics. We could refer to geometry, with its closely related type and proximal genus and specific differences in definitions. Or we could mention more developed ones here, so-called analytic definitions that characterise a class of objects through the quotient construction: in other words, by means of an equivalence relation.

Perception (pratyaska) is the relation between the visible (that which we see with eyes) – or at least with the senses (a relation that is produced by the contact of the sensory organ with the object) – and the image we have of that object. I avoid considerations regarding the six senses that *nyaya* philosophers attributed to human beings, highlighting instead the importance ascribed to the sixth sense, the intellect (*manas*), because of the regulating and mediating function this 'organ' has, with respect to the other five. According to *nyaya* philosophy, communicable concepts acquire their own reality in contrast with Buddhism that assigns them the role of a mental image.

Finally, I consider *inference (anumana)* that in the *nyaya* school represents the sublime stage. The *nyaya syllogism* (so called because its form is apparently similar to the Aristotelian one) is not widely known. *Nyaya* distinguished five assertive elements in its syllogism (compared with three in an Aristotelian syllogism):

- the statement (*pratijna*) (not proved; the enunciation we want to prove)
- the reason (*hetu*)
- the general proposition or enunciation (*udaharana*), followed by an example
- the application (*upanay*), also called the second statement
- the conclusion (*nigamana*).

The following example is a classic *nyaya* (as well as Socrates's pseudo-syllogism for Aristotelian logic):

1. object A moves (statement);
2. because of a force applied to it (reason);
3. whenever we apply a force to an object, the object moves (general proposition); for example: if oxen pull a cart, the cart moves (example);
4. a force is applied to object A (application); therefore:
5. object A moves (conclusion).

It is relatively easy to put this reasoning into symbolic form. Before I proceed to do so, here is a convenient formalism:

Let A, B be given objects, and X a general object;
P(X): the open predicate statement, "X moves";
F(X): "a force is applied to X".

The open statement F(*X*) is always true if F(A) is experimentally verifiable whenever we replace the variable *X* with a constant A (we experience its truth through our senses, at least according to *nyaya*'s empiricist interpretation).

Nyaya's syllogism can be formally interpreted as follows:

Statement:	1.	$P(A)$	unproved statement
Reason:	2.	$F(A)$	cause for P(A)
Thesis:	3.	$(\forall X)\,[F(X) \rightarrow P(X)]$ $F(B) \rightarrow P(B)$	general proposition example
Application:	4.	$F(A)$	from the general case, we return to the case we are studying: a force exerts an action on A
Conclusion:	5.	$P(A)$	A moves

Classic Buddhist critics argue against the first and the second steps, because they do not belong to true reasoning but can nonetheless be included in the thesis. I would like to emphasise that this is frequently carried out in our common way of reasoning: for example, in didactic action. We point out the final object of the proof right from the beginning: otherwise, it would be impossible to organise *that* kind of reasoning exactly. I shall return to this issue later on.

Buddhists wrongly refused the fifth step, a sort of *modus ponens* extended to predicate calculus, a logically correct and essential operation for that type of syllogism. The relative formal linguistic expression is:

$$\{(\forall X)\,[(F(X) \rightarrow P(X)) \wedge F(X)] \rightarrow P(X)\} \rightarrow \{[(F(A) \rightarrow P(A)) \wedge F(A)] \rightarrow P(A)\}$$

The logical analysis of language, related to the close connection assigned to the dichotomous language-object of thought, leads to an exact language criticism similar to that of rhetoric in modern times.

According to *nyaya*, the enemies of correct deduction and speaking are:

- *ambiguity (chala)*, resulting from an improper use of a term (*i.e.*, a wrong use of analogy)
- *unfinished thinking (jati)*, looped speech without content
- *absurd arguments (nigrahastama)*, adopted by someone without logic: his fate is to be defeated dialectically by someone who operates with logic and rational arguments.

Nyaya philosophers studied the instances in which their syllogisms led to sophisms. Here are the principal cases of this harmful reduction:

- an incorrect correspondence between the syllogism's constituent parts, therefore there is no relationship between terms
- an intrinsic absurdity that appears in a term stating the opposite to that which it should affirm
- an explicit absurdity due to the contrast between two terms of the syllogism that exclude each other
- the lack of a proof or a test of one of the terms supporting the reasoning
- the falsity of the major term, the non-existence of the object [3] we are referring to or the attribution of false properties to it.

It is evident how *nyaya* differs from the Aristotelian logic, since its arguments are based upon empirical tests and on contact with the external world. [4] *Nyaya* regards not only objects and facts but also thoughts as real entities (I use the word 'real' and not 'existing' to avoid any comparison with Platonism).

The current distinction between propositional logic and predicate logic fails to acknowledge the actual historical development of the discipline. Propositional/enunciative logic was not as prominent in Aristotle's works as it is in those of contemporary logicians. It derives from the studies of Megarian and Stoic philosophers and, paradoxically, established itself later, whereas predicate logic is essential to the understanding of Aristotle's syllogisms from a modern point of view. In the classroom, in high school logic lessons, teachers mainly use propositional logic and they try to apply it, as an illustration, to geometrical proofs, although it is not always completely suitable. For example, such proofs often require quantifying over variables, an action that does not make sense in propositional logic.

An in-depth analysis of the manners of reasoning and their logical modelling both by experts (mathematicians, university teachers) and by university students (first year) can be found in the work of Durand-Guerrier and Arsac (2003). Among others things, the authors indicate different conceptions of the use of and need for quantifiers in proofs by both experts and students.

Classroom arguments and proofs

For various reasons, the three examples I propose do not always correspond perfectly to the *nyaya* logic, although they are very similar. What is particularly interesting is to study the logic that the students employ in the face of the demonstration, since this can often prove to be a logic quite different from that of Aristotle. Three examples of this follow.

The case of Filippo

I would like to highlight the role of reasoning in the case of Filippo, a fourteen-year-old student faced with the following exercise (written on a sheet of paper that was given to him). Filippo's responses were videotaped. In the following transcript, I add comments on what he did in square brackets.

> Prove that if a triangle has two congruent sides then it also has two congruent angles.

Filippo draws a scalene triangle and then erases it. After that, he draws an isosceles triangle using the side different in length from the other two as a base, *i.e.,* parallel to the short side of the page nearest to him (see Figure 1).

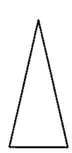

Figure 1: The initial isosceles triangle.

At this point, he puts letters on the vertices (see Figure 2):

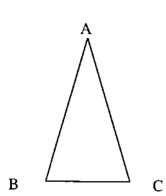

Figure 2: Isosceles triangle with letters on the vertices.

During this activity, he utters an unintelligible sound as if he were concentrating. Then he looks at the researcher and asks him:

F: Do I have to put in the angles?
R: What do you mean?
F: Do I have to write them?

It is clear what he is trying to say but I pretend not to understand.

R: Do what you consider correct.

Filippo then adds the names of the base angles to the drawing (see Figure 3). He looks at the researcher satisfied, looking for approval.

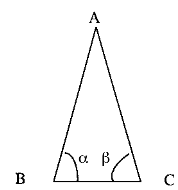

Figure 3: Angles added.

R: Go ahead.

Filippo carefully reads the piece of paper where the text of the assignment is written, looking at his drawing now and then. Then he exclaims:

F: Alpha is equal to beta. Yes, alpha [*he indicates it with the point of the pencil*] is equal to beta [*he indicates this second angle now with the point of the pencil*].

Then he looks at the researcher.

R: Is it what you want to prove or where you start from?

Filippo remains silent. He reads the text again, looks at the drawing, reads the text again and says:

F: No, no, I don't start from here; this is what you are asking me.
R: So?
F: I know that AB is equal to AC; here [*he points with the point of the pencil to the two sides on the drawing, sliding the pencil precisely over both sides*]. These two are equal.

The researcher keeps quiet.

F: If these two are equal [*he points to the sides, but only touching a single point on each of them with the pencil*] and also these must be necessarily equal [*similarly indicating a point inside each angle*], well then the angles must be equal.
R: Ah, yes?
F: How can it not be so? It must be like this, if AB is 3 and AC is 3, then alpha and beta will be, let's say, 60.
R: Why 60 degrees? Couldn't they be 40 degrees?
F: Yes, I believe. 60 degrees just came out, but I believe it could be anything [*he refers to the amplitude*].

Filippo looks at the researcher as if he has concluded the exercise.

R: So? Can we conclude the exercise? What can you affirm?
F: I believe that whenever the sides are equal, also these two angles here, the ones on the bottom [*he touches the two angles with the point of the pencil*] must be equal. Therefore, I believe it is like this, what I said is correct. The two sides here of the triangle [*and he touches inside the triangle*] are certainly equal and therefore also the angles, or not?

If we examine Filippo's argumentative/demonstrative behaviour, he follows almost exactly the steps outlined by *nyaya* philosophers:

1.	Alpha is equal to beta. Yes, alpha is equal to beta.	P(A)
2.	I know that AB is equal to AC; here, these two are equal.	F(A)
3.	If these two are equal, also these, well the angles must be equal.	$(\forall X)\ [F(X) \rightarrow P(X)]$
	If AB is 3 and AC is 3, then alpha and beta will be, let's say 60.	Example
4.	The two sides here of the triangle are certainly equal.	F(A)
5.	and therefore also the angles [are equal].	P(A)

Indeed, Filippo did *not* prove the proposed theorem, but he argued as if taking the implication (AB = AC implies angle B = angle C) for granted. I am not examining here the correctness of the assignment's development: I am examining Filippo's *spontaneous* behaviour in coping with the task. His main concern was not to carry out the Euclidean proof, but to convince himself (or the researcher) that the assignment's written text corresponded to reality.

In truth, Filippo's argument is only one instance taken from about ten interviews verifying the idea *that students' argumentative/demonstrative behaviour in spontaneous situations is sometimes empirically closer to nyaya rather than to Aristotelian or StoicMegarian logic*.

Among all the fifteen- and sixteen-year-old students who were interviewed, Filippo provided one of the clearest examples, because, in my opinion, he goes through all the *nyaya* steps. But many other students tend to behave in this way, even if the more academic (D'Amore, 1999) ones are less inclined to follow such spontaneous behaviour. They try, at least at the beginning, to, for example, extend sides AB and AC somehow or to draw segments, according to what they remember having seen or done in the past. But, in a more or less evident and recognisable way, many students follow the *nyaya* steps, looking for an example at step 3 or being satisfied merely by a drawing, without necessarily displaying measurements, as in Filippo's example.

Giada's 'double' example

Giada, fifteen-years old, is in ninth grade and considered to be 'gifted' in mathematics by her teacher. I offer the following proof (taken from her textbook exercises):

Given the quadrilateral ABCD, PQRS are the mid-points of its sides; join these points; prove that the quadrilateral you obtain is a parallelogram.

Giada makes a drawing (see Figure 4):

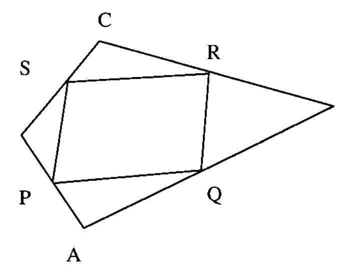

Figure 4: Giada's initial drawing.

G: Here it is.
R: Yes?
G: I did it badly.
R: No, no, it is very clear like this.

Giada reads the text again.

G: Then PQRS must be a parallelogram…

A moment of silence.

G: ...yes, because [*she writes on the page, speaking aloud at the same time*] PQ//RS and PS//QR. Yes. [*She looks at the interviewer.*]
R: Ah.
G: Yes, no, it's like this. When the two sides of a square [*she means a quadrilateral*] are two by two parallel, then the square [*but she means quadrilateral*] is a parallelogram. [*Giada looks at the interviewer then at her drawing.*]

Giada puts the pencil into her mouth, and then she draws (see Figure 5):

Figure 5: "Sides two by two parallel."

saying at the same time:

G: See, this one, for example, has the sides two by two parallel [*she touches two by two the opposite sides with the point of the pencil*].
R: Here it is.
G: In our case, it works because PS is parallel to QR and also PQ to PS, not to SR.

Silence.

G: Therefore it is true: PQRS is really a parallelogram [*doubting, she looks at the interviewer*].
R: OK, but how can you tell that PQ is parallel to RS?

Giada looks at the initial drawing.

G: Ah, oh, I have to show that PQ is parallel [*she starts with an affirmative tone, later it changes to* *an interrogative one*] ...to RS? Yes, parallel, PQ parallel to RS.

She reflects for a while.

G: Well, because I believe both are maybe parallel to AC [*she draws AC on the first figure, the one at the beginning (see Figure 6)*].
G: Yes. I remember. It is because there is the triangle ACD, and so S and P are those midpoints. You can see it.
R: Well. Therefore?
G: [*It sounds like she is quoting a set phrase*.] If two lines [*she means straight lines*] are both parallel to a same line then the two lines are parallel to each other. One can see it here also. [*with the tip of her pencil she draws over SP, RQ and CA again*.]

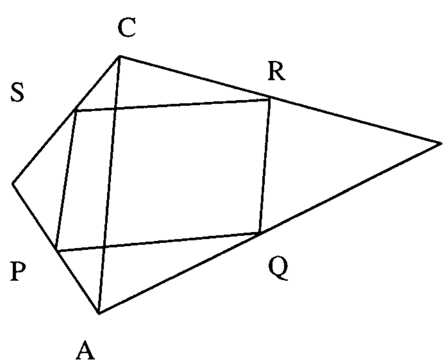

Figure 6: AC drawn on the initial diagram

R: Ah.
G: Yes [*She reflects*] Well, it is the same for the other two. [*she means that analogous reasoning applies to RS and PQ with regard to BD*.]
R: Which ones?

Giada silently goes over the three segments RS, PQ and BD.

G: Yes, they are all parallel. SP, RQ, AC and then SR, PQ, BD. But also… I have it! This was the thesis, or not?

I called Giada's example *double* because, in my opinion, she uses the *nyaya* model twice.

In the first part:

1. PQRS must be a parallelogram	P(A)
2. because PQ//RS and PQ//QR. Yes.	F(A)
3. When the sides of a square [*quadrilateral*] are two by two parallel then the square [*quadrilateral*] is a parallelogram.	(∀X) [F(X)→P(X)]
Here, for example, [*drawing of a rectangle that should be a generic parallelogram*]	Example
4. In our case it works because PS is parallel to QR and also PQ to SR	F(A)
5. Therefore it is true: PQRS is indeed a parallelogram.	P(A)

The interviewer asks for a reason for the statement PQ//RS. Here starts the second part:

1. I must show that PQ is parallel to RS; yes, parallel, PQ parallel to RS.	P(A)
2. because both maybe, because I think they are parallel to AC. Yes, [...] because there is the triangle ACD and so S and P are those mid-points.	F(A)
3. If two lines [*straight lines*] are both parallel to the same line [*straight line*], then they are also parallel to each other.	(∀X) [F(X)→P(X)]
I can see it here	Example
4. the same for the other two. [...] Yes, they are all parallel, SP, RQ, AC and then SP, PQ, BD.	F(A)
5. But also... I have it! This was the thesis, or not?	P(A)

Without considering the improper use of terms, Giada shows a certain mastery of mathematics; it is well known that in language's oral form it is common to say one thing for another (such as "square" instead of "quadrilateral", "line" instead of "straight line"), but this does not compromise the judgment of her action. Giada fulfils the assignment well, using a way of arguing easily ascribable to *nyaya* behaviour.

Pitto's example

'Pitto' is a classroom nickname for Pietro that everybody uses, including the teacher. This fifteen-year-old boy is one of the most popular students in his ninth grade class. I proposed this assignment to the whole class:

> The sum of three consecutive natural numbers is certainly divisible by three.

This is an easy, traditional exercise that requires various strategies.

I will not give an account of this experience, since I have another specific purpose. Needless to say, as the well-known research literature on the subject confirms, most of the students only proposed examples. (There was also a curious one –1, 0, +1, which did not adhere to the requirement to use natural numbers; this example raises the eternal question as to whether 0 is divisible by 3).

Of all of the interesting interviews, Pitto's stands out as most suitable for this case study.

Pitto starts by writing a + b + c and looks at the researcher.

P: I must perform the sum…

R: Of what?
P: Eh. Of three natural numbers.
R: Any?
P: Yes.
R: Are you sure? Read carefully.
P: Consecutive. As, for instance, it could be 5, 6, 7, like this? [*He writes 5 6 7 spaced out*.]

R: Yes. How can you recognise three consecutive numbers? Think in general... What would you call a number in general?

P: Ah, yes, n. It is like saying [*meanwhile he writes*]: n, then $n + 1$ and then $n + 2$.

He stops and writes the sum: $n + (n + 1) + (n + 2)$, exactly in this way with the correct brackets

R: Ah, yes like this it works out well. So?

P: [*He reads the text again.*] Therefore this [*he indicates $n + (n + 1) + (n + 2)$*] is divisible by 3. Well, 5 plus 6 plus 7 [*and he puts the + signs between 5 and 6 and between 6 and 7 in the previous tex*t] which yields 11 and 7, 18. [*He continues writing = 18*]. 18 is divisible by 3. Because there is a t such that $n + (n + 1) + (n + 2)$ is $3t$, like before with t equal to 6.

Pitto looks at the interviewer who nods.

P: If I always find t it would be [*he writes from the beginning*] $n + (n + 1) + (n + 2) = 3t$ and therefore it is always [*he touches with the point of the pen the first side of this equality*] n plus $n + 1$ plus $n + 2$ divisible by 3. For example 1 plus 2 plus 3 is with t equal to 2.

R: Very well. How can you prove what you want?

Pitto writes one more time $n + (n + 1) + (n + 2)$ and he tries next to perform semiotic transformations. Needless to say, he first writes $n^2 + 1$, but he corrects himself immediately pronouncing a resolute "no" and precisely erasing his attempt. He then writes $2n + 1 + n + 2$ and says:

P: …it yields 3n + 3. I have it, here there is 3.

He writes= $3(n + 1)$. While he is hitting the 3 with the point of his pen he looks at the interviewer.

R: Well we have it.

P: Eh, yes, yes.

On the sheet of paper, starting from the $(n + 1)$ that appears on the second side of the equality, he writes = t sideways. Pitto is satisfied and he remarks:

P: Ah, cool, look at it [*he touches the equality with the point of the pen and says*] It is always 3 for the one in the middle.

Looking more closely at Pitto's work, whose argumentation is undoubtedly appropriate to *nyaya* behaviour:

1.	$n + (n + 1) + (n + 2)$ is divisible by 3	P(A)
	5 + 6 + 7 yields […] 18. 18 divisible by 3	Example of P(A)
2.	Because there is a t such that $n + (n + 1) + (n + 2)$ is $3t$	F(A)
	Like before when t was 6	Example of F(A)
3.	If I always find t then it would be $n + (n + 1) + (n + 2) = 3t$ and therefore $n + (n + 1) + (n + 2)$ is always divisible by 3	$(\forall X)\ [F(X) \rightarrow P(X)]$
	For example 1 + 2 + 3 is with $t = 2$	Example
4.	$[n + (n + 1) + (n + 2)]$ that yields $3n + 3$ […] $= 3(n + 1)$ $[= 3t]$	F(A)
5.	It is always three for the one in the middle $[3(n + 1)]$	P(A)

In this schematically summarised argument, it is evident how Pitto needs to "anchor" his reasoning even more. Examples, typical of *nyaya* reasoning and rejected by the Aristotelian and Megarian-Stoic thought, are used to justify not only step 3 but also steps 1 and 2. This reassures Pitto and leads him to a correct starting point for his argument.

Conclusions

The aim of this work is to show how demonstrative behaviour, or, in general, argumentative behaviour of sufficiently evolved students is not only bound to Aristotelian and Megarian-Stoic typologies, as historical and traditional approaches would suggest.

The use of a scheme at least analogous to *nyaya* by these students does not necessarily mean that they simply do not carry out a proof, but rather that they fail to follow the Aristotelian design. Some of them do perform a proof (Pitto), others not completely (Filippo), but it is interesting to point to the common existence of steps 1 and 2, which give meaning to the quantified general statement (thesis). Is the role played by steps 1 and 2, among the students, the same as in *nyaya*? Or is it solely a question of "anchoring" to examples, and therefore simply an application of the general statement to specific cases? Now that I have used this logical model to interpret behaviour that all too often could be considered incorrect (because it does not adhere to another, more established, model), I can proceed with my analysis.

For the time being, I want to reflect on areas that seem to have some didactic interest. This investigation shows at least one feature, *i.e.*, that the adherence to Aristotelian logic as a model for natural proof cannot be taken for granted and anyway is not unique. My purpose here is in no way to substitute one logical model for another – I want to open the analysis of learning how to prove to other possible schemes.

In fact, I share Luis Radford's (1999, 2004) general idea that a culture's mode of thinking has to pay tribute to the activities shared by its elements, since it is the human activity that generates knowledge. Instead of establishing a specific logical method as a model for human thinking, it is preferable to analyse socio-cultural activities and observe how thinking takes shape as a reflection of what individuals do during such activities.

Moving in the framework of a typical methodological analysis drawn from the highlighted cases of the studied subjects, the shifts from several particular cases to universal quantification appear to be an exaggerated interpretation and anyway, it is not a spontaneous activity of the subject. For example, when Filippo talks about "these two" he certainly refers to two particular cases and the generalisation we see in his words stems from the fact that he analyses the possible cases and not from a process of generalisation. But this fact exactly strengthens the idea of a pragmatic adherence, more to *nyaya* rather than to first-order predicate logic.

Finally, in semiotic terms, there are transformations at the level of denotation (common to the behaviour of all the examined subjects) that the same subjects have not taken explicitly into account. The result is that expressing statements within predicate logic, using quantifiers, could take place without them being aware of this. This kind of reflection requires further study, for example involving students in the analysis of their own demonstrative strategy.

From a didactic point of view, on the one hand we are led to put back into perspective the idea that the only demonstrative model is the Aristotelian enunciative-predicative one; on the other, we are led to provide tools to analyse social and cultural activities shared in the classroom. Since the objective of the didactic action is the control of argumentation and demonstrative skills reached by students during high school, we cannot avoid taking into account the results discussed above.

Acknowledgements

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Notes

[1] Recently, various authors have made contributions to the study of the complex phenomenon of learning what the French refer to as *démonstration* (see Bachaleff, 2004 for an important presentation of the characteristics of various recent research studies in this field). The positions vary significantly, from more formalist (Duval, 1991, 1993; Duval and Egret, 1993) to quasi-empirical, in the manner of Lakatos (Hanna and Janke, 1996). The same terminology gives rise to different interpretations. *Proof* in English and *démonstration* in French express different meanings, such that some researchers prefer to call a 'démonstration' a 'mathematical proof'. For a more detailed examination of the question of use of terminology, see Balacheff (2004) and the relevant bibliography.

Several other works have shown the huge difficulties that students encounter in using quantifiers, even solely in regard to geometry (*e.g.*, Durand-Guerrier, 1999). Blaise Pascal (1656/1985), in referring to what we now would call 'quantification', also noticed that we tend to anchor reasoning to general examples in proofs. The common logical framework in these situations is the propositional calculus and the initial elements of first-order predicate calculus (Durand-Guerrier and Arsac, 2003).

[2] For historical reference, prince Gautama (the Buddha, the awakened one, the enlightened one) lived in the sixth or fifth century BC. Therefore the religion, named after the name of its founder, clearly developed before Aristotelianism in Greece (third century BC). By contrast, the first philosophical book on *nyaya* (Gautama's *Nyaya Sutra*) dates from the first century AD. The philosophy developed in the following centuries with the renowned Vatsyayana's commentaries (fifth century AD), Uddoyotakare's commentaries in the sixth or seventh century AD, up to what could be considered a new evolution of *nyaya*, headed by the philosopher Gangesa, in the thirteenth century AD.

[3] Recall Aristotle's point of view regarding the empty set and Gergonne's solution to this issue (D'Amore, 2001, pp. 17-54).

[4] Triumphant Greek philosophy (Socrates – Plato – Aristotle) not only failed to recognise this, but also deeply rejected it. Greek philosophers continued to reject *doxa* (opinions), supporting instead Parmenides's *aletheia*. Sophists, subdued by Aristotle's triumph and Plato's previous dialogic arguments, deserve a different treatise.

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