CAN PEDAGOGICAL CONCERNS ECLIPSE MATHEMATICAL KNOWLEDGE?

MARK A. CREAGER, ERIK D. JACOBSON, FETIYE AYDENIZ

To teach well, mathematics teachers need to know more than a few general principles of teaching and the content they will teach. Shulman (1987) identified pedagogical content knowledge (PCK) as another important kind of teacher knowledge and defined it as the "special amalgam of content and pedagogy" (1987, p. 8). Over the last decade, there has been considerable interest among mathematics educators in the mathematical knowledge that teachers use in their work, and a variety of definitions and emphases have been proposed (Ball, Thames & Phelps, 2008; Davis & Simmt, 2006; Turner & Rowland, 2010). Although the definitions are built from different theoretical foundations, each is consistent with the assertion that content knowledge on its own is not enough, and each advances the notion of a transformed, mutually-influencing mixture of content and pedagogy. For example, Ball and colleagues (2008) grouped PCK with content knowledge calling the new whole *mathematical knowledge for teaching* (MKT)—a term we adopt. Vivid examples of MKT illustrate how mathematical and pedagogical concerns are coordinated at the level of teachers' individual, job-embedded decisions (*e.g.*, Ball *et al.*, 2008), not by a broad synthesis of high-level categories.

The perspective we take in this article is that teaching for conceptual understanding requires the teacher to have at least conceptual understanding (Silverman & Thompson, 2008), which we consider to be a connected network of mathematical ideas and consistent ways of thinking. Teachers also need to consider the needs of their students as learners, meaning that even though a particular representation, statement, or process may be mathematically accurate, it may not be accessible to the learners. Thus, the teacher needs to be able to integrate these two kinds of considerations to make a decision about whether or not a representation, statement, or process is both mathematically accurate and accessible to learners.

Mathematical knowledge can be transformed into MKT when a teacher is able to "decenter" and reflect on their mathematical knowledge (Silverman & Thompson, 2008). Doing so allows a teacher to visualize how a student might come to understand the concept and the types of actions that would facilitate the students' learning. To integrate pedagogical concerns and mathematical knowledge from the opposite direction requires a teacher to be able to disembed themselves from the context of a teaching scenario and anchor her decisions in the discipline of mathematics. Just as teachers who decenter set aside but do not lose hold of what they know about mathematics, we conceive of teachers who are able to disembed as setting aside but not losing hold of the social and pedagogical context of their work.

Research has revealed several instances where mathematical knowledge and pedagogical knowledge have failed to be transformed into MKT. For example, Thompson and Thompson (1996) described a teacher whose deep conceptual understanding of division and proportionality left him insensitive to the conceptual obstacles faced by the learner. In this article, we discuss a complementary way that pedagogical and content knowledge may fail to form MKT that is transformed and well integrated. We present the case of Emma, a prospective elementary school teacher, whose views on students' learning needs trumped her content knowledge. We found Emma's case interesting because she based her decision of whether hypothetical student solution methods were mathematically valid on whether they would be easy to understand and be useful for children. Thus, Emma sometimes answered MKT items incorrectly by inadvertently giving more weight to pedagogical than to mathematical concerns.

We discovered Emma's case in the context of designing an MKT survey measure. This context emphasizes the significance of the case for the field. Measures of MKT are valuable because they quantify a consequential dimension of teacher knowledge. Scores on MKT assessments account for differences in observed classroom instruction and student achievement (Hill, Rowan & Ball, 2005; Tchoshanov, 2011). Assessing MKT is not without its challenges (*e.g.*, Fauskanger, 2015; Hill, Ball & Schilling, 2008), however, and positive correlations at large scale do not mean that MKT surveys accurately reflect individuals' knowledge as it might be used in the classroom (*e.g.*, Hodgen, 2011). MKT items work to measure whether teachers are able to coordinate and apply their pedagogical and mathematical knowledge by posing questions that describe specific instructional scenarios teachers might face while teaching.

The case of Emma highlights the challenges of using characterizations of MKT based on experienced teachers to inform developmental trajectories for novice teachers, and it raises several issues for researchers studying MKT and how to measure it. We describe the data that brought the phenomenon of pedagogical knowledge *eclipsing* mathematical knowledge to our attention and discuss the implications for teacher education. Our larger purpose is to illuminate the challenges pre-service teachers face in developing the transformed mixture of content and pedagogical knowledge that characterizes MKT. We argue that existing descriptions of MKT may obscure critical challenges, and that differences between preservice and experienced teachers have implications for how MKT should be assessed.

For the Learning of Mathematics **36**, 2 (July, 2016)
FLM Publishing Association, Fredericton, New Brunswick, Canada

A context of MKT measure design

We draw on data collected as part of a study to design an MKT survey measure focused on multi-digit addition and subtraction for use with pre-service elementary school teachers. We focus on 5 items from one task (Figure 1). The data we report came from a corpus of hour-long, item-response interviews that we conducted with 15 pre-service teachers. Each one-on-one interview was videotaped. To explore how pre-service teachers interpreted and solved the items, we asked them to explain what they understood each question to be asking and why they selected their answer.

During the analysis, we noticed that five of the pre-service teachers seemed to rely on pedagogical reasoning to answer items about the mathematical validity of non-conventional student solution methods. Even when the interviewers pressed the pre-service teachers to consider the question from a mathematical perspective, these pre-service teachers maintained that the strategies were not valid. We present data from the interview with Emma because she had relatively strong mathematical knowledge and clearly articulated her thinking. Data from the Subtracting Large Numbers Task were particularly helpful for understanding how Emma's pedagogical concerns influenced her activity. Each item consisted of written work from a hypothetical student solving 748 – 264, and each item asked whether the student's solution method could be used to subtract any two whole numbers. Pre-service teachers replied by selecting one of three options: "Works for any two whole numbers", "Doesn't work", or "Not sure." All five hypothetical student methods are mathematically valid.

The items were designed to understand the pre-service teachers' flexibility with alternative algorithms for multi-digit subtraction. In our view, the MKT assessed in this task is important for teachers because student learning is associated with classrooms that ask students to do mathematics (Boaler & Staples, 2008). By "doing" mathematics we mean: creating, evaluating, and justifying conjectures; defining terms; inventing algorithms; and so forth. If teachers are expected to support their students in doing mathematics, then they need to be able to decide whether unfamiliar algorithms are valid.

Pedagogical concerns eclipse mathematical knowledge

We briefly introduce Emma and describe her work on the MKT instrument to provide context. Then we use her work on items from the Subtracting Large Numbers Task to illustrate that Emma did not correctly answer mathematical items despite displaying significant understanding because she chose to focus on students' learning needs—a pedagogical concern.

Emma was a first-year student enrolled in an inquiry-based mathematics content course for elementary education majors focused on number and quantity. The course aimed to improve pre-service teachers' mathematical understanding and to help them understand how young students think about mathematics. Based on her interview, Emma had a strong knowledge of place value and multi-digit addition and subtraction. She worked on twelve items during the interview, answering seven correctly. Four of her five incorrect answers were the result of her answering mathematical questions for pedagogical reasons.

During her interview, Emma worked quietly for just over five minutes on the Subtracting Large Numbers Task and spent another 12 minutes explaining her reasoning to the interviewer. She answered methods A and C correctly, and answered B, D, and E incorrectly. At first, she skipped method B, coming back to finish it at the end of her work. After debriefing her answers in the original order with the interviewer, she switched her answer for method D from "Does not work" to "Works for any two whole numbers" and said she was "between" the options "Works for any two whole numbers" and "Not sure" for methods B and E.

Based on her actions and her interview responses, we observed that Emma solved the items by looking for general strategies that had been discussed in her number and quantity course, such as "chunking" and "rounding-and-adjusting." For Emma, chunking was a process of removing the subtrahend from the minuend in several parts or "chunks" that were easy to compute mentally, and rounding-and-adjusting was a process of adding values to both the minuend and subtrahend to avoid regrouping. Next Emma tried to understand why a student would perform the

Ms. Jones was working with her class on subtracting large numbers and recording their work. Among the students' papers, she noticed the following solutions:

Method A
```
 748    264    +6
-264    270   +30
        300  +400
        700   +48
        748   484
```

Method B
```
 748       748
-264     - 200
           548
         -  60
           488
         -   4
           484
```

Method C
```
 748    754     784
-264   -270   - 300
                484
```

Method D
```
 748   +51     799
-264         - 264
               535
               -31
               504
               -20
               484
```

Method E
```
 748           748
-264   -20   - 244
               504
               -20
               484
```

Which of these students is using a method that could be used to subtract any two whole numbers?

Figure 1. The Subtracting Large Numbers Task.

process. For example, when describing method A she said, "They're just trying to fill placeholders and then add the rest of the numbers they know." She described method C as rounding-and-adjusting.

Although Emma worked hard to make sense of the non-standard solution methods and why someone might use them, she also weighed how easily students would be able to understand and adopt them—a pedagogical consideration not intended by the task design. We provide a more detailed description of her reasoning for methods B, D, and E to describe how her pedagogical concerns eclipsed her mathematical knowledge.

Emma recognized that method B would work, gave a mathematical justification for why it would work, and acknowledged that the method made sense to her. However, Emma suggested that the method would only work "some of the time." Her rationale was that "lower level" students would not be able to do the required mental computations, a pedagogical consideration [1]:

Emma: This one, I was kind of confused on. Because what they're doing, it makes sense, is that they're removing each part, so at first they remove the third place holder then the second place holder and then the ones placeholder, but like here some kids wouldn't be able to subtract the six from the four [...] What I would've done was chunk this. I would have subtracted forty, and got them five-oh-eight and then subtracted twenty instead of subtracting the sixty altogether. So I think that depending on the age so if they're a higher level and can do that in their head but if they're a lower level I don't think they'll be able to compute that so I wasn't sure. [...]

Interviewer: You talked from a student's perspective, but how about your perspective as a teacher? Do you think it works for whole numbers?

Emma: Yeah I think it would work. [...] I see how it would make sense and I would think that it would work most of the time, but I don't know if they'd be able to compute.

Emma displayed her knowledge of the situation by capturing essential mathematical features when she summarized the method as, "removing each part" by "placeholder." She also suggested a valid alternative method for solving the problem, chunking. She was also concerned, however, that the method would be difficult for younger students because of the subtraction step that requires regrouping. In our view, her pedagogical concern with regrouping is warranted because regrouping is difficult for many students when they first encounter it, but that concern seemed to blind her to the mathematical nature of the question, that is, whether the implied algorithm was general.

When Emma explained method D to the interviewer, she said, "I don't think it will work all the time. Depending on the numbers that you have." The interviewer then asked Emma to suggest another two numbers and Emma interjected by asking, "And see if it will work?" Her subsequent work is presented below (Figure 2). She created the problem 826 – 479, and said she chose these values because the tens and ones digits in the subtrahend were larger than the tens and ones digits in the minuend. To solve the problem, Emma first used a standard algorithm to subtract 826 from 899 to obtain 73. She then wrote "+73" and then performed the subtraction steps in the middle column. To check her work, she added 347 and 479 using a standard algorithm for addition. Emma's work aligned with method D because she subtracted the initial extra by subtracting first 20 and then 53 to avoid subtracting across 100.

After working through the method, Emma marked "doesn't work" and she and the interviewer had the following exchange:

Emma: I'm assuming what they were trying to do by adding the fifty-one was get it so that that all the digits that each placeholder was higher than the placeholder they were subtracting. So like originally seven is bigger than two so that's fine but four is not bigger than six so my guess is by adding that number they were just trying to get that number as large as possible, but you can't just add fifty-one or some random number every time to get to whatever number you want. I think it has to be more systematic than that. So I think adding to make a base would make more sense or rounding to a different, you know maybe rounding the second placeholder or something like that, but I don't think that it'll be easy to do to just add a number to get them to be all whole then try to subtract and adjust for that.

Interviewer: So again you are thinking for students, right?

Emma: Yeah. [...]

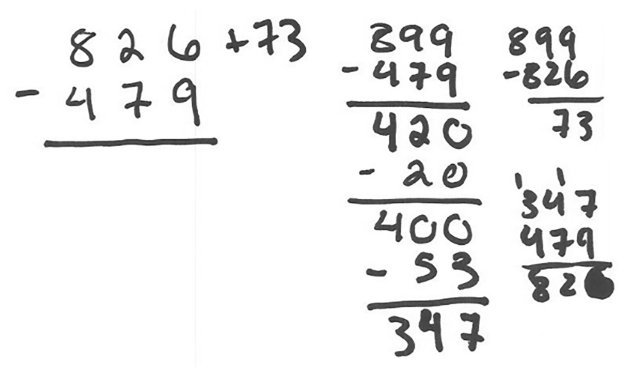

Figure 2. Emma's work to check method D from the Subtracting Large Numbers Task.

Interviewer: How about this method? Does, it should work for all whole numbers? Or or does it not work for all whole numbers?

Emma: It makes sense how they broke this part so they are adjusting for it, so I guess it would work. […] So I think that a lot of mistakes can be- I mean I think that it's plausible but I think that a lot of mistakes could be made. […]

Interviewer: So as I understood you said this works for numbers but it could be kind of hard for students, right?

Emma: Yeah, so I mean I guess I'm kind of in between the "yes it works" and "not sure" because I just, the circumstances.

After discussing all the other items for the Subtracting Large Numbers Task, Emma came back to method D and changed her answer to "works for any two numbers." As she switched her answer, she said, "I can't say it doesn't work because it worked in these cases, so I guess it does work."

We argue that Emma's decision about the generality of method D was based primarily on pedagogical considerations, similar to her decision about method B. Her reason for rejecting the method was that "a lot of mistakes could be made" implying that she was anticipating that students could make mistakes if they used this method. Moreover, Emma seemed to have sufficient mathematical knowledge to understand the method for three reasons. First, she was able to articulate a purpose for adding 51 as making the values in the minuend larger than the values in the subtrahend. Second she called this method rounding-and-adjusting, which she used to describe method C—a method she classified as, "Works for any two whole numbers." Third, the alternative method she suggested followed the same form as method D, but used a value that she saw as more understandable for students ("adding to make a base would make more sense").

Emma compared method E to a method that she said would work (method C), but in the end she said that method E would not work all the time. Emma focused on how this method could be confusing for students just as she had with method B and D:

Emma: And then for this one what they did was round so here [pointed to method C] they rounded up and here [pointed to method E] they rounded down so it's rounding-and-adjusting again and they round it down so that the numbers are easier to compute. So the first set of placeholders is greater than the second set of placeholders. […] by subtracting this twenty here they do know that um it is, you're subtracting too few so they subtract more but once again with this [pointed to method D] if they were to do it with a different number then I think that that would get confusing with the adding or subtracting whether too few or too many. So it's similar to this one [pointed to method D]. This number [pointed to the –20 in method E] makes more sense to add computationally, but I still think that it wouldn't work all the time.

Again, Emma justified the method mathematically ("you're subtracting too few so they subtract more"), and even provided a reason someone might use it ("the numbers are easier to compute"). For these reasons, we believe that Emma did understand the method and had sufficient content knowledge to answer the MKT question correctly.

She rejected the method, however, by arguing, "If they were to do it with a different number then I think that that would get confusing." In the first and third excerpts, Emma uses the phrase "they" when rejecting the methods because "they" might find it confusing or difficult. We take Emma's meaning for "they" to imply that she was not considering this problem for herself, but instead was speaking of the confusion that some students might encounter when using this method. We take this to be compelling evidence that Emma had been considering students' needs throughout her work on the Subtracting Large Numbers Task. Emma focused on her pedagogical concerns for students, but these pedagogical concerns were not well integrated with her available content knowledge. She ended up using pedagogical reasoning to make essentially mathematical decisions.

Before moving to the implications for teacher education and for research on MKT, we discuss two alternative explanations for Emma's activity and explain why they are less plausible to us than the account we have offered. First, Emma may not have understood the question we intended to ask. In the first excerpt, Emma's initial response appeared to answer a question different than the one intended, and the interviewer asked Emma to reconsider the problem, "from your perspective as a teacher." In the second excerpt, Emma again seemed to answer a different question, and the interviewer pressed her more directly by saying, "Should it work for all numbers?"—a probe that emphasized the central mathematical issue. In the third excerpt, Emma's statement, "If they were to do it with a different number, then I think that that would get confusing" also gives the impression that she was answering a different question than the one intended. Rather than weakening our claim, this fixation on non-mathematical issues epitomizes the need for pedagogical knowledge and content knowledge to be a transformed, mutually-influencing mixture. Although she was asked with frequent clarification whether a method would work with all numbers, Emma consistently answered whether the method would work for all students. The contrast between the intended question and the understood question demonstrates how pedagogical concerns shaped Emma's reasoning from the beginning of her interaction with the item. Despite significant encouragement, Emma seemed unable to disembed from the teaching scenario described as the context of the question and recognize that she was faced with a mathematical question that (we argue) she had sufficient knowledge to answer correctly.

A second alternative explanation is that Emma may not have understood the content well enough to make a correct decision. In other words, she may be a familiar case of insufficient mathematical knowledge (see Ball, 1990) rather than a case of pedagogical knowledge eclipsing content knowledge as we have argued. We discuss two aspects of content knowledge: proof and conceptual understanding of number and subtraction.

Emma did not prove that any of the methods worked in general, and we next consider whether her responses could have come from not knowing how to prove. Emma checked the generality of most of the methods by using two or more examples. In fact in the second excerpt, Emma chose an example that she thought might challenge the method because it required regrouping across two digits. Reasoning this way is aligned with an empirical proof scheme (Harel & Sowder, 2007), which is common for elementary pre-service teachers (Martin & Harel, 1989), meaning these examples would have been sufficient evidence for her that the method worked in general. Moreover, if she held an empirical proof scheme, she might not find other proofs (*e.g.*, a symbolic proof of the method) as convincing as her empirical "proofs." Therefore, Emma likely "proved" that the methods were valid for all numbers by checking examples (*i.e.*, convinced herself in the sense of Harel & Sowder, 2007).

We next consider whether she lacked conceptual understanding. Especially pertinent were Emma's understanding of number and of subtraction. There is limited evidence that Emma lacked understanding. For example, Emma's primary concern for method B was subtracting 60 from 40 without regrouping or "chunking." Similarly, her concern with method E stemmed from her preference to add versus subtract when rounding-and-adjusting. On this basis, one might argue that Emma did not have conceptions of subtraction and negative numbers that connected subtracting a value and adding the opposite. Davis and Simmt (2006) would refer to this as having "access to the web of connections that constitute a concept" (p. 301). Had Emma held this type of connected knowledge for the methods she was evaluating, the argument goes, she might have been able to see how the methods were connected to the skill the students were supposed to learn and been less reticent about endorsing them.

There is more evidence that Emma did understand number and subtraction than evidence to the contrary. Emma asserted that all the methods made sense to her, and she justified how they worked mathematically. On another MKT item, we asked Emma to make sense of a student-invented strategy for 63 – 27 where the student wrote "–4 + 40 = 36". Emma answered the item correctly and she demonstrated conceptual understanding both by describing "–4" as "negative four" and by invoking an appropriate metaphor ("they owe four more"). Rather than being insufficient, Emma's mathematical knowledge was not well integrated. It did not form MKT, because pedagogical concerns trumped evident content knowledge.

Implications

What does Emma's case imply for efforts both to measure novice teachers' MKT and to improve it? We begin by considering how these critical challenges for novice teachers are illuminated or obscured by existing descriptions of MKT. Shulman's (1987) articulation of teachers' knowledge base delineated categories of teacher knowledge and identified PCK as an important area of investigation. Subsequent work to build psychometric instruments (*e.g.*, Ball *et al.*, 2008) has refined these knowledge categories, where each category is distinct. Moreover, knowledge in each category is assumed to be quantifiable. Examining expert practice to delineate different knowledge categories helps articulate what teachers (should) know, but it neglects the important issue for novices developing MKT—the transforming integration of different kinds of knowledge to address the problems of teaching mathematics.

The focus on integration is more apparent in the frameworks developing Shulman's ideas to focus on teacher education. Davis and Simmt (2006) studied teacher knowledge in the context of long-term, school-based professional development, and they emphasized the inextricably intertwined nature of mathematics-for-teaching. Rowland, Huckstep and Thwaites (2005) developed the Knowledge Quartet in their work with preservice teachers, in which three of the four categories deal explicitly with the transforming integration of knowledge in different teaching contexts. Reports from these projects have illustrated both (1) how experienced teachers use integrated knowledge of mathematics and mathematics learning, and (2) how novice teachers' mathematics knowledge was not necessarily available to inform their mathematics teaching. Although the accounts with the lenses of mathematics-for-teaching and of the Knowledge Quartet help us understand why integrated knowledge is important and how a lack of integration is problematic, they do not provide a universal, genetic account. We are left to ask: how do novice teachers develop transformed, integrated knowledge?

With the goals of teacher education in mind, we return to the case of Emma from the perspective of the Knowledge Quartet (Rowland *et al.*, 2005). Both Emma's understanding of subtraction and place value as demonstrated by her explanation of the different student methods and her apparent belief that students need only a small number of optimal strategies to learn mathematics can be categorized as foundational knowledge, "[a] teacher's theoretical background and beliefs" (Turner & Rowland, 2010, p. 200). The scenario in the Subtracting Large Numbers Task, however, seems to require contingency knowledge—knowledge needed to react productively in the moment to unplanned events (albeit in the very different situation of responding to a survey rather than interacting with children). So perhaps we can conclude that Emma's response reveals a lack of contingency knowledge. The Knowledge Quartet suggests the lack of integration in foundation knowledge is the problem, but does not identify what must be integrated. By drawing on the categories of teacher knowledge highlighted in other frameworks, we have argued Emma needed to integrate her mathematical knowledge and pedagogical beliefs. Some frameworks fail to emphasize the integration at the heart of MKT by emphasizing categories, yet other frameworks fail to identify what must be integrated by ignoring categories.

Current MKT assessments used with novices are based on research with experienced teachers and essentially compare novices with experts. An incorrect response indicates "less" knowledge in a particular category, but does not help identify how teacher educators might help teachers gain "more" of such knowledge. These measures are not designed to identify or characterize gaps in integrated knowledge. Teacher educators would benefit from an MKT framework and assessment for novice teachers that focused on knowledge integration yet also identified how integration goes wrong, perhaps by leveraging our understanding of categories of teacher knowledge.

Whether teachers' knowledge of pedagogy and content are integrated into MKT is not differentiated by current assessments. Moreover, whether teachers' decisions are a result of mathematical knowledge or pedagogical knowledge overshadowing the other requires different responses from teacher educators. To illustrate this point, we compare the case of Emma with several teachers from a similar study of responses on multiple-choice MKT items.

Fauskanger (2015) asked novice teachers if different decompositions of the number 456 were correct. Aware that the non-standard decompositions gave the right value, many teachers nonetheless answered the item incorrectly because they believed the standard decomposition ($4 \times 100 + 5 \times 10 + 6 \times 1$) was the most appropriate. Both these teachers and Emma showed evidence of having the mathematical knowledge the item was designed to assess and both responded incorrectly to the item; however, the two cases are qualitatively different. Fauskanger argued the teachers were likely unaware of the importance of non-standard decompositions in the learning progression for multi-digit arithmetic (see, for example, Jones *et al.,* 1996), whereas Emma objected that the student strategies were too hard for students to perform or would be confusing. Like the teacher in Thompson and Thompson's (1996) study, the first is a case of prioritizing content over students' learning needs, whereas Emma's is a case of pedagogical concerns eclipsing her content knowledge.

The significance of the difference between cases is clear if one considers how a teacher educator might intervene. A discussion about the pedagogical utility of flexible decomposition may help teachers like those in Fauskanger's study transform their preference for the standard decomposition. Given that these teachers are aware of and recognized the relevance of the alternative decompositions, they could decenter and appreciate the role that alternative decompositions might play in developing students' knowledge and thus develop MKT. By contrast, an analogous discussion with Emma of the pedagogical utility of multiple methods would likely not develop her MKT because it would contradict rather than support the transformation of her underlying belief that good mathematics instruction avoids student difficulty. In order for Emma to use her knowledge of the mathematics supporting the methods to recognize their potential utility for students in spite of what she believes about their difficulty, she must somehow learn to disembed herself from the situation.

Emma's case came to our attention as we analyzed data for a larger project developing a survey to measure MKT. Our study was not designed to investigate the integration of mathematical knowledge and pedagogical knowledge as such, and we recognize this limitation. However, we raise Emma's case both to describe a challenge faced by mathematics teacher educators and to call for more focused research on novice teachers' MKT. When assessing novice teachers' MKT, it may be more useful to understand to what extent knowledge is integrated than it is to judge novices relative to the standard of experienced teachers. Teacher educators already know *that* novices fall short. Knowing *why* novices fall short will suggest what can be done.

Notes

[1] Parts of the transcript have been omitted due to space restrictions and to aid in readability.

Acknowledgments

We would like to thank Mike Daiga and Erol Uzan for their helpful comments on an earlier draft of this paper.

References

Ball, D. L. (1990) The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal* **90**, 449-466.

Ball, D. L., Thames, M. H. & Phelps, G. (2008) Content knowledge for teaching: what makes it special? *Journal for Teacher Education* **59**(5), 389-407.

Boaler, J. & Staples, M. (2008) Creating mathematical futures through an equitable teaching approach: the case of Railside School. *The Teachers College Record* **110**(3), 608-645.

Davis, B. & Simmt, E. (2006) Mathematics-for-teaching: an ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics* **61**(3), 293-319.

Fauskanger, J. (2015) Challenges in measuring teachers' knowledge. *Educational Studies in Mathematics* **90**(1), 57-73.

Harel, G. & Sowder, L. (2007) Toward comprehensive perspectives on the learning and teaching of proof. In Lester, F. K. (Ed.) *Second Handbook of Research on Mathematics Teaching and Learning,* pp. 805-842. Charlotte, NC: Information Age Publishing.

Hill, H. C., Ball, D. L. & Schilling, S. G. (2008) Unpacking pedagogical content knowledge: conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education* **39**(4), 372-400.

Hill, H. C., Rowan, B. & Ball, D. L. (2005) Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal* **42**(2), 371-406.

Hodgen, J. (2011) Knowing and identity: a situated theory of mathematics knowledge in teaching. In Rowland, T. & Ruthven, K. (Eds.) *Mathematical Knowledge in Teaching,* pp. 27-42. Dordrecht, The Netherlands: Springer.

Jones, G. A., Thornton, C. A., Putt, I. J., Hill, K. M., Mogill, T. A., Rich, B. S. & Van Zoest, L. R. (1996) Multidigit number sense: a framework for instruction and assessment. *Journal for Research in Mathematics Education* **27**(3), 310-336.

Martin, W. G. & Harel, G. (1989) Proof frames of prospective elementary teachers. *Journal for Research in Mathematics Education* **20**(1), 41-51.

Rowland, T., Huckstep, P. & Thwaites, A. (2005) Elementary teachers' mathematics subject knowledge: the Knowledge Quartet and the case of Naomi. *Journal of Mathematics Teacher Education* **8**(3), 255–281.

Shulman, L. S. (1987) Knowledge and teaching: foundations of the new reform. *Harvard Educational Review* **57**(1), 1-23.

Silverman, J. & Thompson, P. W. (2008) Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education* **11**(6), 499-511.

Tchoshanov, M. A. (2011) Relationships between teacher knowledge of concepts and connections, teaching practice, and student achievement in middle grades mathematics. *Educational Studies in Mathematics* **76**(2), 141-164.

Thompson, A. G. & Thompson, P. W. (1996) Talking about rates conceptually. Part 2: mathematical knowledge for teaching. *Journal for Research in Mathematics Education* **27**(1), 2-24.

Turner, F. & Rowland, T. (2010) The Knowledge Quartet as an organizing framework for developing and deepening teachers' mathematical knowledge. In Rowland, T. & Ruthven, K. (Eds.) *Mathematical Knowledge in Teaching,* pp. 195-212. Dordrecht, The Netherlands: Springer.