On the Essence of Multiplication

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Wittgenstein (1953) warned that:

> if you want to pronounce the salutation 'Hail!' expressively, you had better not think of hailstones as you say it. (p 175)

Likewise, if you want to 'multiply', you had better not think of 'adding'. The repeated addition model of the operation of multiplication has been justifiably challenged in contemporary mathematical didactics research. First of all, such a model does not account for all multiplicative situations. For example, according to Greer (1992):

> Cartesian products provide a quite different context for multiplication of natural numbers. [...] This class of situations corresponds to the formal definition of *m* x *n* in terms of the number of distinct ordered pairs that can be formed when the first member of each pair belongs to a set with *m* elements and the second to a set with *n* elements (p 277)

Also, as Davydov (1992) remarked, the repeated addition definition of multiplication presents further difficulties in the case of multiplying by one:

> if given 2x1, then what is added to what? (p. 13)

However, a survey of this literature reveals that in each case either the traditional concept is (perhaps unintentionally) reintroduced surreptitiously into the new theory or the theory requires a fickle concept of multiplication, one that varies according to the different multiplicative situations. Nonetheless, it is possible to formulate a uniform concept of multiplication that is not simply another arithmetical operation in disguise, such as that of addition. In fact, I will offer below just such an account, one that also recognises anew the importance of the roles of multiplier and multiplicand and relies on the asymmetry between them. However, before doing so, I present and criticize current theories of multiplication in order to pave the way for an alternative foundation for the concept of multiplication.

Current theories of multiplication

In their attempt to replace the traditional theory of multiplication as simply repeated addition, Nantais and Herscovics (1990) introduce the following physical procedure, for which they claim multiplication is an arithmetic reflection and quantification:

> A situation is perceived as multiplicative when the whole is viewed as resulting from the repeated iteration of a one-to-one or a one-to-many correspondence. (p. 289)

They arrived at this working definition of multiplication by considering Piaget and Szeminska's (1967) identification of a physical operation of multiplying:

> Une multiplication arithmétique est une équi-distribution, telle que si *n* x *m* on ait *n* collections de *m* termes, ou *m* collections de *n* termes, qui correspondent bi-univoquement entre eux (p. 253)

Although the physical procedure that they offer differs from the one of equal groups usually illustrating repeated addition (Greer, 1992, p. 280), Nantais and Herscovics (1989) reimport the traditional model of repeated addition at the mathematical tier of their analysis of the concept of multiplication. This is not surprising since Piaget and Szeminska (1967), on whose work this physical procedure rests, are explicit in their analysis of arithmetical multiplication as repeated addition. There, they claim that:

> Dès lors l'addition $A_1 + A_2 = 2A$ est par cela même une multiplication, laquelle signifie que la collection A_1 est doublée par une autre collection A_2 lui correspondant de façon bi-univoque et réciproque (p. 253)

Furthermore, there are two other fundamental problems with this definition. First, the action of repeated iteration of the one-to-one or one-to-many correspondence seems more like an action for division than one for multiplication; it does not capture what sets multiplicative thinking apart from additive thinking. Second, the attempt to define the concept of multiplication by way of defining the physical action proper to it is putting the cart before the horse. Multiplication is an operation on numbers and not on physical objects

> Outside the context of obtaining a product [...] real operations themselves do not have meaning. These operations are *in unity (in the same system)* with operations consisting of writing down multiplication formulas. (Davydov, 1992, p. 18)

In other words, that multiplication can be illustrated concretely does not imply the operation is necessarily born of a real operation performed on concrete objects.

Although the Nantais and Herscovics (1989, 1990) theory (derived from Piaget and Szeminska's earlier work) offers a uniform concept of multiplication, it still inherits the inadequacies of the traditional theory. In contrast, most other researchers, while alive to the failings of the repeated addition theory, offer alternatives that require different senses of multiplication for different multiplicative situations. For example, Anghileri (1989) accounts for the different multiplication situations by assigning many different aspects to the concept:

> It has been found in previous research [...] that the cartesian product aspect of multiplication is more difficult for children to understand (in that, related tasks are solved less easily) than tasks reflecting the repeated addition or rate aspects of multiplication The present study implemented tasks relating to six different aspects of multiplication: equal grouping/repeated addition, allocation/rate 1, array, number line, scale factor/rate 2, cartesian product. (p. 369)

A similar proliferation of senses is found in Greer (1992, p. 280), where he provides a more exhaustive classification of multiplicative situations into ten types: equal groups, equal measures, rate, measure conversion, multiplicative comparison, part-whole, multiplicative change, Cartesian product, rectangular area and product of measures. And, although Vergnaud (1983, p 128) does not offer an explicit theory of the operation of multiplication, his classification of multiplicative problems into the three different subtypes (isomorphism of measures, product of measures and multiple proportion other than product) also suggests a *non-uniform* theory of multiplication, in that each subtype requires a *distinct* concept of multiplication.

Considering multiplication via the classification of multiplicative situations or problems certainly does go beyond the scope of the repeated addition analysis. However, discerning the concept of multiplication by examining multiplicative situations or problems is a circular endeavour. What is it that these situations or problems all have in common by virtue of which they are characterized as 'multiplicative'? In other words, if the concept of multiplication is. in fact, multifaceted, then what binds these different facets into a single common concept?

A different view

Lebesgue's (1975) remark concerning multiplication inadvertently provides the common thread connecting such diverse multiplicative situations as those highlighted above:

> Toute question qui conduit à une multiplication est un problème de changement d'unité, ou d'objets: 5 sacs de 300 pommes; 3m.75 d'étoffe à 28fr.45 le mètre. (p. 13)

In fact, it was this remark, as well as the ideas of Fridman (1963), which inspired the work of Davydov (1992). Fridman notes that:

> A major improvement in the operation of counting objects [...] is the introduction of *group count* (for example, cigarettes can be counted as packs). Each group appears as a singular *unit of count* (all groups contain, of course, an equal number of objects). (cited in Davydov, 1992, p. 15)

In light of this idea of group count, Fridman then specifically advances the thesis that multiplication as an arithmetic procedure is:

> a reflection of a transfer operation from a larger unit of count to a smaller one. (p. 15)

Davydov, while agreeing with the idea of the transfer of units of count as the foundation of multiplication, rejects Fridman's claim that such a transfer is from the larger- to the smaller-scaled unit. He argues that:

> Such a description of multiplication is not inconceivable, provided that known and established standards of large-scale units of count, which are already being actively utilized by people in counting small objects (for example, it is already accepted to count jars of preserves *by the box*, and other objects *in pairs, by the dozen*, etc.), are adopted beforehand. (p 16)

In other words, how can the smaller unit be determined by the larger if and when the units are not standard? Davydov claimed that the very nature of the problem in which multiplication arises determines that the transfer is *naturally* performed from the smaller- to the larger-scaled unit:

> We believe that [...] the problem in which sets were counted out with units which were then designated as 'small' was an original and thorough problem. The inconvenience or the impossibility of a practical solution to this problem required a change of units, a transfer to larger-scaled, bigger units [...] In the course of this operation standard group units were then worked out, the use of which more or less turned into an autonomous problem with time (p 17)

Clark and Kamii (1996) provide a good example of the distinction between thinking in terms of repeated addition and that of multiplicative thinking, as well as exemplifying Davydov's view by considering the group count involved in the multiplication 3 x 4, as shown in Figure 1 below. They claim:

> additive thinking involves only one level of abstraction in that each unit of three that the child adds is made of ones, that is, three ones. The child also makes inclusion relations on only one level; he or she includes one in two, and two in three, and so on up to twelve. [...] In contrast, multiplication necessarily involves the making of two kinds of relations not required in addition: (a) the many-to-one correspondence between the three units of one and the one unit of three; and (b) the composition of inclusion relations on more than one level. (pp 42-43)

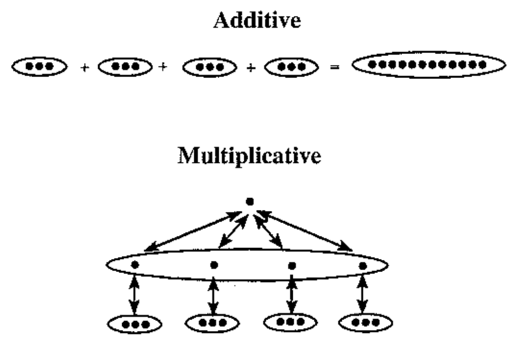

Figure 1 Additive vs multiplicative thinking

This inclusion relation involves a transfer of units in that the multiplicand '3' counts the number of items while the multiplier '4' counts the number of triples of these. Thus, Davydov's question regarding the problematic repeated addition definition in the case of multiplication by one is easily answered: in the case of 2 multiplied by 1, '2', Davydov's smaller unit of count, is counting the number of items while the '1', the larger unit of count, is counting the number of pairs of these.

An extended theory

My own account takes as its point of departure the work of Davydov (1992), which differs in important ways from the ideas discussed in the first section above While he provides a uniform theory of multiplication, it is restricted to the domain of natural numbers. With Davydov (1992), I accept that:

> the assumption (and it is more acceptable from the psychological-genetic point of view) that primary and fundamental operations, which are at the basis of multiplication, consist of transferring from a smaller to a larger unit and finding their relationship is fully justified (p. 17)

However, can such a concept of multiplication be made universal, i e apply in the domain of real numbers?

The negative numbers

> Multiplication is a little more bothersome than addition in the present context, since products of negative rationals are positive (Rudin, 1976, p 19)

Consider first the case of negative numbers

> The conceptual introduction of the negative numbers has been surprisingly slow. It has lasted more than fifteen hundred years. (Glaeser, 1981, p. 304, trans)

In spite of this introduction alluding to the unveiling of the result of the long, historical quest for a concept of negative number underlying its formal mathematical existence, Glaeser comes to the conclusion that the rule of signs (i e. a negative times a negative yields a positive) is an example of just those mathematical phenomena for which a concrete-oriented pedagogy is actually damaging to learning (p 344).

Following in the footsteps of Glaeser (1981), Coquin-Viennot (1985, p. 183) also claims that although concrete models for representing additive structures with negative numbers, such as that of the popular debt and asset, one can in some instances help the beginning learner, they become utterly useless for learning multiplication. Of course, if the *negativity* of the number is being confounded with the operation of *subtraction* (e.g -1 stands for "take away 1"), then the rule of signs (i e -a x -b = +ab) can indeed be misinterpreted as the miraculous acquiring of assets by the multiplication of debts! As it is, therein lies the very source of the historical problem that troubled many great mathematicians over the years (MacLaurin, d'Alembert and Euler, to name a few): conceiving of negative numbers as representing negative quantities.

Kline (1962), on the other hand, claims quite the opposite of Glaeser and Coquin-Viennot in stating that:

> The operations with negative numbers and with negative and positive numbers together are easy to understand if one keeps in mind the physical significance of these operations (p 67)

And, he even promotes the familiar debt and asset model, with an important modification. He can contend with multiplication (and division) equations involving negatives simply by introducing the feature of *direction in time* into the model.

> Suppose a man goes into debt at the rate of 5 dollars per day. Then in 3 days after a given date he will be 15 dollars in debt. If we denote a debt of 5 dollars as -5, then going into debt at the rate of 5 dollars per day for 3 days can be stated mathematically as 3 x (-5) = -15. [...] In the very same situation in which a man goes into debt at the rate of 5 dollars per day, his assets three days *before* a given date are 15 dollars more than they are at the given date If we represent time before the given or zero date by -3 and the loss per day as -5, then his relative financial position 3 days *ago* can be expressed as -3 x (-5) = +15; that is, to consider his assets three days ago, we would multiply the debt per day by -3, whereas to calculate the financial status three days in the future, we multiply by +3. Hence the result is +15 in the former case compared to -15 in the latter. (p. 68)

If negative numbers differ from positive numbers only in so far as they are numbers with direction (Freudenthal, 1983), then multiplication formulas involving such numbers can be reduced to the multiplication of positive numbers with the additional feature of direction. For example, on the number line, a product such as that of '3 x -4' can be interpreted as "three leaps of four hops to the left, arriving at the position -12". (Note here the clarity of the transfer of units from the smaller *hops* to larger *leaps*) Consequently, the charge (+ or -) indicating the direction of the number is of no primary concern in a theory of multiplication in so far as the concept of multiplication itself need not be altered to accommodate the additional feature of direction In other words, the idea of a transfer of units of count remains central to the operation of multiplication, whether the numbers involved are positive or negative.

It should be kept in mind that the product of any two integers is larger than either factor and this is so even in those cases in which one of the factors is negative. Indeed, recalling Kline's debt and asset example, I can safely say that a debt of 15 dollars (-15) is larger than a debt of 5 dollars (-5) It would also be difficult to convince people to consider the temperature -30°C to be "much less hot" (i.e , smaller in heat) than say -2°C; it is more natural to think of -30°C as being "much colder" (i e , larger in cold) than -2°C. In other words, it is the magnitude of the product, not its direction, that is important here.

Of course, the integers constitute an ordered set, but as Rudin (1976) has pointed out:

> The statement '$x < y$' may be read as 'x is less than y' or 'x is smaller than y' or 'x precedes y'. (p. 3)

That is, it can make sense to consider *size* (x is smaller than y) as the ordering criterion on the natural numbers in so far as they can represent quantities, but on the set of all integers, the criterion must be *position* (x precedes y) Therefore, it is still possible to think in terms of a 'large' negative product in relation to the factors of the multiplication without compromising the order defined on the set of integers I will return to this distinction between the product's magnitude and its direction in my concluding remarks.

The rational numbers

Next, there are four different types of multiplication formulas possible in the domain of positive rational numbers. A first kind, as is discussed in Davydov (1992), is one in which both the *multiplier* (which he calls the larger unit) and the *multiplicand* (the smaller unit) are integers. I note that in such a formula the only operation performed is that of a transfer of unit of count (a particular kind of measure). Consequently, in this sense, I shall say that the product of two integers is *pure* multiplication. The following excerpt taken from Davydov's teaching experiment provides an excellent example of such pure multiplication:

> With what measure (unit) did we measure out the water?
> We measured out the water with a mug.
> Yes, we worked with a mug But when we took one mug of water how many little glasses did we thereby take all at once?
> When we took the mug once this was five little glasses all at once.
> But did we really take the water with the little glasses poured out one, two and up to five little glasses?
> No! This we found out earlier we measured out and found out that there are five little glasses in one mug
> And then with what did we measure out the water?
> Then we worked with the mug.
> How many mugs of water turned out to be in the jar?
> Six mugs altogether
> How many times does that mean we took the water with the mugs?
> Six times
> You know there are five little glasses in one mug How many times did we take five little glasses when we measured the water out with mugs?
> We took five little glasses six times.
> That is right! We took five little glasses and all at once six times! (p 25)

However, the product of an integer and a fraction, although still an operation of multiplication, invokes the operation of division by virtue of the very nature of the fraction involved in the operation. The asymmetry characteristic of this extended theory is prominent here in that numerical expressions are treated differently according to their roles as either multiplier or multiplicand This distinction between the two roles is necessary if multiplication is an operation that involves a transfer of units, from the smaller one to a larger one, a distinction that may be blurred by the commutative law.

However, it must be kept in mind that the commutative law merely states that the *products* of the two multiplications $a \times b$ and $b \times a$ are equal, not that the two multiplications themselves are identical In light of this, depending on whether the fraction assumes the role of multiplier or multiplicand, this results in two different formulas. If the multiplier is a whole number and the multiplicand is fractional, then the product is still predominantly an operation of multiplication, but one in which the smaller unit of measure (the multiplicand) must first be determined by division. For example, in Figure 2 you can see that, in '3 times 1/2', the multiplicand '1/2' is first obtained by dividing the smaller unit of measure into two parts.

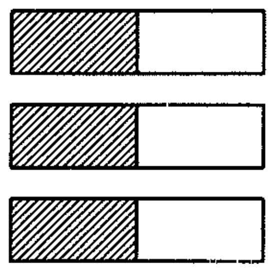

Figure 2 Three half units

On the other hand, if the multiplier is a fraction, the operation becomes predominantly one of division, but one in which the operation of multiplication is also invoked For example, in Figure 3 you can see that, in '1/2 of 3', it is the group of three units that is divided into two parts.

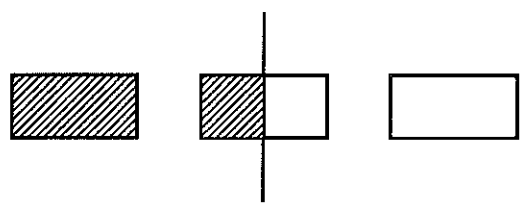

Figure 3 One half of three units

The last possible case of multiplication in the domain of positive rational numbers is the one consisting of the product of two fractions. This is simply a combination of the cases of fraction-multiplier with that of the fraction-multiplicand as treated above: that is, in spite of the symbolization for multiplication used in writing the equation, the predominant operation here is that of division. In fact, division is the only operation involved in those cases where both the multiplier and the multiplicand are unit fractions For example, in Figure 4 you can see that, in '1/2 of 1/3', it is the unit that is first divided into three parts followed by the division of one of these parts into two parts.

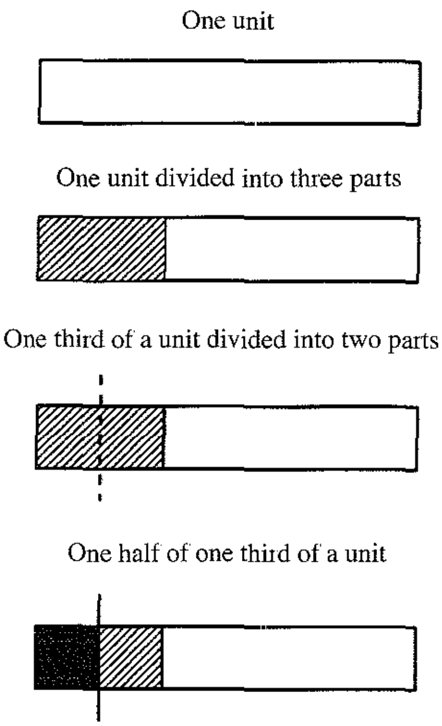

Figure 4 One half of one-third unit

When at least one of the factors in the multiplication is a non-unit fraction, the predominant operation is still that of division, but multiplication is also called upon Consider the multiplication '2/3 of 1/2' in Figure 5

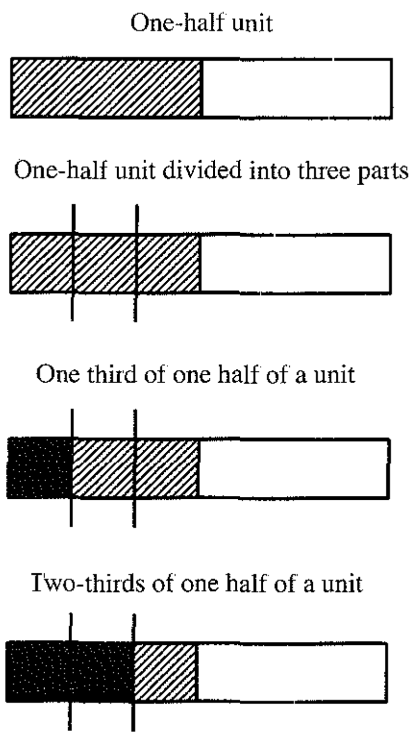

Figure 5 Two thirds of one-half unit

These various cases of multiplication with rational numbers are summarized in Table 1

MULTIPLIER	MULTIPLICAND	OPERATION
Integer	Integer	Pure multiplication (No other operation is involved)
	Fraction	Multiplication (Some division is involved)
Fraction	Integer or fraction	Division (Some multiplication may be involved)

Table 1 Rational multiplication

Before considering the case of multiplying with irrational numbers, Table 2 in the appendix indicates how each of Greer's (1992, p 280) ten classes of multiplicative situations can be given a uniform analysis simply in terms of the transfer of units of count or measure involved in the domain of rational number multiplication

The irrational numbers

> The geometric representation of irrationals and of operations with irrationals was, of course, not practical. It might be logically satisfactory to think of $\sqrt{2} \times \sqrt{3}$ as an area of a rectangle, but if one needed to know the product in order to buy floor covering, he would not have it. (Kline, 1972, p 49)

While it has been shown here that the transfer from a smaller- to a larger-scaled unit of measure can be the foundation of multiplication in the domain of all rational numbers, can it be extended to all real numbers? In other words, can multiplication involving irrationals, either as multipliers, or as multiplicands, or both, still make sense?

Davydov's (1992) definition of the concept of multiplication does not in any way require that the operation be performed only in practical, finite situations. One way to see that multiplication involving irrationals is indeed possible without compromising the definition is to consider irrational numbers as expressed in their infinite decimal expansion. More specifically, any irrational number can be expressed as an infinite sum of rationals. For example,

$$\sqrt{2} = 1\ 4142\ldots$$

$$= 1 + 4/10 + 1/100 + 4/1000 + 2/10000 + \ldots$$

Consequently, the case of multiplying with irrational numbers is nothing more than an infinite iteration of rational multiplication For example,

$$2 \times \sqrt{2} = 2 \times (1 + 4/10 + 1/100 + 4/1000 + 2/10000 + \ldots)$$

$$= (2 \times 1) + (2 \times 4/10) + (2 \times 1/100) + (2 \times 4/1000) + (2 \times 2/10000) + \ldots$$

It is obvious that any case in which the multiplier, the multiplicand or both is irrational decomposes into one or more of the four cases of multiplication involving only rationals

However, Fowler (1985a) clearly issues a caution, in cases such as 1.4142 . x 1.7321 ... = 2.4495 :

> Increasing familiarity with the new techniques then leads to the belief that any arithmetic sum is possible, and so to the confidence to use the arithmetic in bolder and bolder ways; and nagging worries like those about the meaning of $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ recede in the face of a technique that is so successful in practice. (I can easily explain the idea of addition to anybody, including myself, but multiplication sums like this often seem much more mysterious.) (p 20)

But just what is it that is so 'mysterious' here? Is it the design of an effective multiplication algorithm to obtain a product or is it the concept of multiplication itself? Although Fowler (1985b) presents an example of the complexity, even the impossibility, of carrying through the arithmetic operation of multiplication with such numbers, there is no reason to believe that somehow the concept of multiplication is itself the source of the problem The feasibility of obtaining a product should not be confused with the essence of the concept of multiplication One must distinguish between features dependent on the algorithm and those dependent upon the underlying concept

In light of these comments, I can say that this theory of multiplication is uniform in that it requires just one definition of the concept of multiplication in the domain of real numbers. This concept of multiplication applies independently of whatever number set is involved in the operation. It is asymmetrical because the roles of multiplier and multiplicand cannot always be interchanged. In fact, as presented in Table 1, it is these very roles that determine the operations actually at play in the problems expressed by multiplication formulas For example, in the case of a fractional multiplier, the operation of division is necessarily involved.

Conclusion

The mathematical didactician's first task is similar in some respects to that of a philosopher's in that he or she must get clear about concepts, in this case mathematical concepts My attempt here has been to make a start at doing just that in defending a uniform theory of multiplication. A survey of the literature indicates that the concept of multiplication seems to have been defined mainly in terms of its diverse applications, the definition consequently changing with each application (e.g. equal groups, Cartesian product, rates, etc.).

As Clark and Kamii (1996) remind us:

> If we want to teach multiplication, we must first understand the nature of multiplicative thinking. (p. 50)

And a single definition of the concept is a propaedeutic to a clear, unified understanding of such a nature Moreover, without a uniform theory, studies on the learning of multiplication result in scattered bits of information dealing with specificities rather than being an elucidation of the learning of the concept in general.

Davydov's (1992) Lakatos-like approach in his analysis of multiplication procedures was the key to his success in discerning a single definition of multiplication that is applicable in all natural number situations. Lakatos (1976), in criticizing formalism, argued that mathematics cannot be dissociated from its history, since the very development of mathematical thought recapitulates in brief the whole history. Rather than getting lost in the diversity of multiplication situations in search of an explanation, Davydov went directly to the source: why did we need the concept of multiplication in the first place? In other words, what sort of problem was originally encountered for which the solution led to the development of this particular concept?

In answer to this, Davydov developed and experimented with a theory of multiplication of natural numbers that considers the transfer of units of counts as its foundation. For example, it has been tempting to consider the arithmetic procedure of repeated addition for the frequently encountered situation of equal groups, which according to Fischbein *et al.* (1985) corresponds to "features of human mental behavior that are primary, natural, and basic" (p 15), to be elemental to the concept of multiplication. However, whether or not there are such mental features, it is the transfer of units of count (the number of objects in each group to the number of groups) that actually permits such a situation to be conceived as multiplicative in the first place.

A uniform foundation for the concept of multiplication such as that advanced by Davydov (1992) is not entirely radical. Consider the fact that the decimal system of numeration arises from forming groups of units that then become themselves new units of count, and so on. In fact, this link to numeration is evidence that defining multiplication as a transfer of units of count inserts itself coherently into the network of mathematical notions usually taught at the elementary level. The link it provides among multiplicative situations involving only natural numbers is also of the utmost importance to the learning of multiplication, especially since children's difficulties reportedly vary in accordance with the situations For instance:

> It has been found in previous research [...] that the cartesian product aspect of multiplication is more difficult for children to understand (Anghileri, 1989, p 369)

But, if the children came to understand that the same concept is called upon in situations that on the surface may seem different and new (cf Table 2), then surely such difficulties could be alleviated. In the case of the Cartesian product, we can say that each member of the first set is treated as the larger unit of count, since it must be paired with every member of the second set.

In addition to linking all natural multiplicative situations, the extendability to all real multiplicative situations accounts for the force of the account adumbrated in this article In particular, simply by considering the more general sense of *measure* rather than the particular sense of *count*, Davydov's definition becomes foundational in the domain of all real numbers, no longer confined simply to the set of natural numbers. In fact, it was shown above that multiplication in the real number system is nothing more than rational multiplication or the iteration of rational multiplication. That is, the concept of multiplication is preserved and it is the nature of the numbers that determine the outcome of the operation.

Unlike other non-uniform theories of multiplication in which the extension to rational numbers requires various extensions, each peculiar to a specific *aspect* of the concept (e.g. "rectangular area can be generalized to rectangles the measures of whose sides are fractional" - Greer, 1992, p. 278), this new theory of rational multiplication simply relies on the old, unjustly forgotten roles of multiplier and multiplicand in determining the operation performed (cf. Table 1), without having to change the meanings of either operation of multiplication or division. In fact, this uniform theory offers the insight that only those cases in which both the multiplier and the multiplicand are integers are considered purely multiplicative; the other cases, although presented in the form of multiplication, actually invoke the operation of division by virtue of the nature of the numbers the operation is performed on.

In so doing, it also explains away another common difficulty children have in learning multiplication with fractions: accepting a product that is smaller than either factor. In each case of rational multiplication as outlined above, either the product is larger than either multiplier or multiplicand or, when at least one fraction is involved, some part of the overall operation invokes division by a whole number rendering the result smaller. So *pure* multiplication always yields a larger product. And as has already been shown, this is so even when at least one of the factors is negative

As can be seen, the impact of such a uniform theory of multiplication is far reaching. For the past decade, researchers have considered children's over-generalisations and over-simplifications as a source of many obstacles to their learning and, indeed, have used these obstacles to analyse mathematical concepts into lists of ever-more-discrete sub-concept particles. Maybe it is time to look at it differently: can researchers instead appreciate these innate capacities to over-generalise and over-simplify and learn from them to capture the essential? In the present case, the common sense impression "Multiplication makes bigger" should have been considered a clue for understanding multiplicative thinking rather than be labelled as one of "the most notorious *mis*conceptions about mathematics" (Graeber, 1993, p. 408).

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References

Anghileri, J. (1989) 'An investigation of young children's understanding of multiplication', *Educational Studies in Mathematics* **20**(4), 367-385

Clark, F. B. and Kamii, C. (1996) 'Identification of multiplicative thinking in children in grades 1-5', *Journal for Research in Mathematics Education* **27**(1), 41-51

Coquin-Viennot, D. (1985) 'Complexité mathématique et ordre d'acquisition: une hierarchie de conceptions à propos des relatifs', *Recherches en Didactique des Mathématiques* **6**(2-3), 133-192.

Davydov, V. V. (1992) 'The psychological analysis of multiplication procedures', *Focus on Learning Problems in Mathematics* **14**(1), 3-67

Fischbein, E., Deri, M., Nello, M. S. and Marino, M. S. (1985) 'The role of implicit models in solving verbal problems in multiplication and division', *Journal for Research in Mathematics Education* **16**(1), 3-17

Fowler, D. (1985a) '400 years of decimal fractions', *Mathematics Teaching* **110**, 20-21.

Fowler, D. (1985b) '400.25 years of decimal fractions', *Mathematics Teaching* **111**, 30-31.

Freudenthal, H. (1983) *Didactical Phenomenology of Mathematical Structures*, Dordrecht, D. Reidel

Fridman, L. M. (1963) *Arifmetika, ch. 1 [Arithmetic, part I]*, Moscow, Uchpedgiz

Glaeser, G. (1981) 'Epistémologie des nombres relatifs', *Recherches en Didactique des Mathématiques* **2**(3), 303-346

Graeber, A. O. (1993) 'Misconceptions about multiplication and division', *The Arithmetic Teacher* **40**(7), 408-411.

Greer, B. (1992) 'Multiplication and division as models of situations', in Grouws, D. A. (ed.), *Handbook of Research on Mathematics Teaching and Learning*, New York, NY, Macmillan, pp. 276-295

Kline, M. (1962) *Mathematics: a Cultural Approach*, Reading, MA, Addison-Wesley

Kline, M. (1972) *Mathematical Thought from Ancient to Modern Times*, New York, NY, Oxford University Press.

Lakatos, I. (1976) *Proofs and Refutations*, Cambridge, Cambridge University Press.

Lebesgue, H. (1975, Nouveau tirage) *La Mesure des Grandeurs*, Paris, Librairie Scientifique et Technique Albert Blanchard

Nantais, N. and Herscovics, N. (1989) 'Epistemological analysis of early multiplication', in Vergnaud, G., Rogalski, J. and Artigue, M. (eds), *Proceedings of the 13th Conference of the International Group for the Psychology of Mathematics Education*, Paris, Université de Paris, vol. 3, pp. 18-24

Nantais, N. and Herscovics, N. (1990) 'Children's pre-concept of multiplication: logico-physical abstraction', in Booker, G., Cobb, P. and de Mendicuti, T. N. (eds), *Proceedings of the 14th Conference of the International Group for the Psychology of Mathematics Education*, Oaxtapec, Mexico, vol. 3, pp. 289-296.

Piaget, J. and Szeminska, A. (1967) *La Genèse du Nombre chez l'Enfant*, Neuchâtel, Delachaux and Niestlé.

Rudin, W. (1976, 3rd edition) *Principles of Mathematical Analysis*, New York, NY, McGraw-Hill.

Vergnaud, G. (1983) 'Multiplicative structures', in Lesh, R. and Landau, M. (eds), *Acquisition of Mathematics Concepts and Processes*, Orlando, FL, Academic Press, pp. 127-174

Wittgenstein, L. (1953) *Philosophical Investigations*, New York, NY, Macmillan

Appendix

CLASS	MULTIPLICATION PROBLEM	TRANSFER OF UNITS OF COUNT/MEASURE	TYPE OF RATIONAL MULTIPLICATION
Equal Groups	3 children each have 4 oranges. How many do they have altogether?	Child 1 Child 2 Child 3	Pure multiplication (Each of the three children has four oranges)
Equal measures	3 children each have 4 2 litres of orange juice How much orange juice do they have altogether?	Child 1 Child 2 Child 3 4 2 litres 4 2 litres 4 2 litres	Multiplication/division (Division of the fifth litre into ten parts is involved to obtain the 2/10 of a liter in 4 2 litres of orange juice that each of the three children gets)
Rate	A boat moves at a speed of 4 2 metres per second. How far does it move in 3 1 seconds?	1st sec 2nd sec 3rd sec 0 1 of 4th sec 4 2 m 4 2 m 4 2 m 0.42 m* * 0.42 m = 1/10 of 4 2 m	Multiplication/division (Division of the fifth metre is involved to obtain the 2/10 of a metre in 4 2m travelled in each of the seconds and division of the length 4 2m into ten parts is involved to obtain the length covered in 1/10 of the fourth second)
Measure conversion	An inch is about 2 54 cm About how long is 3 1 inches in centimetres?	1 in 1 in 1 in 0 1 in 2 54 cm 2 54 cm 2 54 cm 0 254 cm* * 0 254 cm = 1/10 of 2 54 cm	Multiplication/division (Division of the third centimetre into 100 parts is involved to obtain the 54/100 of a cm in 2 54 corresponding to each of the three inches and division of the length 2 54 into ten is involved to obtain the length in centimetres covered by 1/10 of the fourth inch)
Multiplicative comparison	Iron is 0.88 times as heavy as copper If a piece of copper weighs 4.2 kg how much does a piece of iron the same size weigh?	1st time 2nd time 88th time 0 042 kg* 0 042 kg 0 042 kg * 0 042 kg = 1/100 of 4 2 kg	Multiplication/division (Since the weight of iron is 88 hundredths that of the weight of copper, the division of the 4 2 kg into 100 parts is involved to obtain the measure in kg of one of those hundredths that is then counted 88 times)
Part/whole	A college passed the top 3/5 of its students in an exam. If 80 students did the exam how many passed?	1st time 2nd time 3rd time 16* students 16 students 16 students *16 students = 1/5 of 80 students	Multiplication/division (Division of the 80 students into 5 parts is involved to determine 1/5 of the 80 students then that number is counted 3 times in order to find 3/5 of the 80 students)
Multiplicative change	A piece of elastic can be stretched to 3.3 times its original length What is the length of a piece 4.2 metres long when fully stretched?	1st stretch 2nd stretch 3rd stretch 0 3 stretch 4 2 m 4 2 m 4 2 m 1 26 m* * 1 26 m = 3/10 of 4 2m	Multiplication/division (Division of the fifth metre into ten parts is involved to obtain the 2/10 of a metre in 4 2 m and division of the 4 2 m into 10 parts is involved to obtain the 3/10 of that length in 3/10 of a stretch)
Cartesian product	If there are 3 routes from A to B, and 4 routes from B to C, how many different ways are there going from A to C via B?	B to C B to C B to C B to C 3 routes A to B 3 routes A to B 3 routes A to B 3 routes A to B	Pure multiplication (For each of the four routes from B to C, there are three possible routes from A to B)
Rectangular area	What is the area of a rectangle 3 3 metres long by 4 2 metres wide?	1 m width 1 m width 1 m width 0 3 m width 4 2 m long 4 2 m long 4 2 m long 1 26 m long* *1 26 m = 3/10 of 4 2 m	Multiplication/division (Division of the fifth metre into ten parts is involved in obtaining the 2/10 of a metre in the length 4 2 m of the rectangle and division of the fourth metre into ten parts is involved in obtaining the 3/10 of a metre in the width of 3 3 m)
Product of measures	If a heater uses 3.3 kilowatts of electricity for 3 2 hours. how many kilowatt-hours is that?	1st hour 2nd hour 3rd hour 0 2 of 4th hour 3 3 kW 3 3 kW 3 3 kW 0 66 kW* * 0 66 kW = 2/10 of 3 3 kW	Multiplication/division (Division of the fourth kilowatt is involved in obtaining the 3/10 of a kilowatt in 3 3 kW and division of the 3.3 kW into ten parts is involved in obtaining the measure of kilowatt of electricity consumed in 2/10 of an hour.)

Table 2 Analysis of Greer's ten classes of multiplicative situations