Communications

The scandals of geometry and school mathematics: the parallel postulate and the equality 0.99… = 1

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Many mathematicians of the seventeenth and eighteenth centuries considered the intuitive idea of infinitesimals as the true foundation of calculus. But Cantor, Dedekind and Weierstrass defined real numbers as the complete ordered field, which was devoid of infinitesimals and infinite numbers. Since the arithmetization of analysis in the nineteenth century, the rigorously defined real numbers have become the "standard" numbers for describing the continuum. Finally, in the middle of the twentieth century, Robinson constructed a new number system called nonstandard real numbers or hyperreal numbers, which contain infinitesimals and infinite numbers as well as standard real numbers. Thus, just as there exist Euclidean and non-Euclidean geometries which are equally consistent, there exist standard and non-standard real numbers.

It is well-known that the parallel postulate is independent from Euclid's other axioms. However, until the late eighteenth century, many mathematicians firmly believed that the postulate was provable and proposed numerous fallacious proofs of it. D'Alembert called the controversy over the parallel postulate "the scandal of geometry". Since school mathematics introduces real numbers via infinite decimals, students encounter the notorious equality 0.99… = 1. Although some students feel the equality 0.99… = 1 counterintuitive, many teachers hope that common school algebra arguments are sufficient to convince their students. As Beswick (2004) wrote: "at its heart is some profound mathematics, but it is explicable in reasonably simple terms" (p. 7). But much research has reported students' serious resistance to 0.99… = 1 despite teachers' various inculcations of it. So the controversy over 0.99… = 1 could be considered a "scandal of school mathematics".

This communication starts from the observation that there exists an essential similarity between the scandal of geometry and that of school mathematics. We elaborate the parallel problem structure between the proof attempts of 0.99… = 1 in school mathematics and those of the parallel postulate in the history of Euclidean geometry. The comparative analysis of the two problems causing the scandals focuses on the following dimensions: motivations, solving attempts, failure mechanisms, and respective mathematical conclusions (see Table 1).

The problems and solving attempts of the two scandals

For over two thousand years, many professional and amateur mathematicians had thought that the parallel postulate was not a true postulate, but rather a theorem which should be proved from the other intuitive axioms. However, the attempted proofs were invariably discovered later to assume some obvious property equivalent to the parallel postulate itself. The attempted proofs could only "prove" that the hidden assumptions were equivalent to the parallel postulate in the context of neutral geometry (see the discussion of Trudeau, 1987, pp. 123-126).

At the school level, real numbers are not exactly defined as the complete ordered field. To secondary school students, real numbers just mean objects that can be operated according to common school algebra rules about arithmetic operations and inequality. Even though the term ordered field is not used in school mathematics, students are familiar with ordered field axioms as basic rules of school algebra and so easily take them as intuitive number axioms. Unlike the ordered field axioms, students cannot conceive explicitly the completeness axiom of real numbers [1]. Hence, it is reasonable to presume that students accept the following proofs of 0.99… = 1 on the basis of an ordered field, rather than that of a complete ordered field. Nevertheless, the following three common proofs of 0.99… = 1 all assume not only ordered field axioms but also tacitly assume the Archimedean property or the completeness axiom.

- The infinite series proof:

  $$0.99\ldots = \sum_{k=1}^{\infty} \frac{9}{10^k} = \lim_{n\to\infty} \left(1 = \frac{1}{10^n}\right) = 1$$

- The decimal expansion argument:

  1/3 = 0.33…; if we multiply both sides by 3, we get 1 = 0.99….

- Proof by digit manipulation:

  $x = 0.99\ldots$, $10x = 9.99\ldots$; therefore $9x = 9$, $x = 1$.

The infinite series proof shows that 0.99… = 1 is equivalent to:

$$\lim_{n\to\infty} \frac{1}{10^n} = 0 \quad \text{or} \quad \lim_{n\to\infty} \frac{1}{n} = 0$$

Using the formal definition of limit, these limits are in fact the Archimedean property:

> For any $\varepsilon > 0$, there exists a natural number n such that $1/n < \varepsilon$.

As for the decimal expansion argument, decimal expansions such as 1/3 = 0.33… are possible also due to the Archimedean property (Rudin, 1976, p. 11). Furthermore, the notation $x = 0.d_1d_2d_3\ldots$ in digit manipulation already assumes that every infinite decimal converges to a limit, which is equivalent to the completeness axiom in an ordered field (Propp, 2013; Burn, 2000, p. 77; Weir, 1973, pp. 9-10).

The resolutions of the two scandals

While many mathematicians have attempted to directly prove the parallel postulate, Saccheri tried to prove the falsity of the negation of the parallel postulate. Although Saccheri failed to deduce a contradiction from the negation

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	The scandal of geometry	The scandal of school mathematics
Motivations for the problems	The parallel postulate was not a self-evident statement.	0.99… = 1 is not a self-evident equality.
Problems causing the scandals	Can the parallel postulate be proved from the other intuitive axioms of Euclid?	Can 0.99… = 1 be proved from intuitive number axioms?
Attempts to solve the problems	Neutral geometry → the parallel postulate. (Neutral geometry: geometry assuming only the other "intuitive" axioms of Euclid.)	Ordered fields → 0.99… = 1 (Ordered fields: the intuitive number axioms in school mathematics.)
Failure mechanisms	The attempted proofs included hidden assumptions (A) equivalent to the parallel postulate in neutral geometry. Examples of (A): • Parallel straight lines are equidistant. • Through any point only one parallel can be drawn to a given straight line. • Every triangle can be circumscribed.	The common arguments include hidden assumptions (A) equivalent to this equality or even stronger one (B) implying it in ordered fields. Examples of (A)—the Archimedean property: • $\lim_{n\to\infty} \frac{1}{10^n} = 0$ (or $\lim_{n\to\infty} \frac{1}{n} = 0$) • Given any x, there is an integer n such that $x < n$. Examples of (B): • (Equivalent to completeness) every infinite decimal expansion is a real number.
Resolutions	The existence of models for hyperbolic geometry implies that there exists no proof of the parallel postulate in neutral geometry.	The existence of non-Archimedean ordered fields implies that there exists no proof of 0.99… = 1 in ordered fields.

Table 1. The parallelisms in the problem structure of the two scandals.

of the parallel postulate, he denied the logical possibility of the other geometries assuming the negation of the parallel postulate. However, repeated failures to prove the postulate gradually persuaded several mathematicians of the early nineteenth century that the contradictions which Saccheri sought would never be reached. For example, hyperbolic geometry accepts all axioms of neutral geometry and replaces the parallel postulate by the hyperbolic axiom [2]. Finally, Beltrami and Klein cleared the remaining doubts about the validity of hyperbolic geometry by constructing a Euclidean model for it. The existence of this model proved the relative consistency of the hyperbolic geometry, and implies that no proof or disproof of the parallel postulate from neutral geometry will ever be found (Greenberg, 1993, p. 225). It was the final conclusion of a two thousand year scandal caused by the attempts to prove the parallel postulate.

In the same manner as the *reductio ad absurdum* of Saccheri, we can assume that 0.99… is less than 1 in an ordered field. If ε were the difference between 1 and 0.99…, then ε would become an infinitesimal and the Archimedean property could not hold. However, there does exist a non-Archimedean ordered field [3] containing infinitesimals and infinite numbers. Just like the model of hyperbolic geometry, the existence of a non-Archimedean ordered field implies that no proof of the Archimedean property or 0.99… = 1 can be found from the "intuitive" number axioms of ordered fields.

Conclusion

As summarized in Table 1, our communication shows the parallel structure of scandals of school mathematics and geometry. These parallelisms explain why and how teachers' inculcation of 0.99… = 1 cannot succeed in clearing students' doubts. Just as numerous historical attempts to prove the parallel postulate inevitably fell into a circular reasoning, arguments about 0.99… = 1 at the school level cannot avoid a circular reasoning. As mentioned before, the scandal of geometry was completely resolved by the conclusion that there exists no proof of the parallel postulate from neutral geometry. Analogously, the conclusion that there exists no proof or disproof of 0.99… = 1 from intuitive number axioms, ironically puts an end to the ongoing scandal of school mathematics. It also liberates teachers who suffer from the burden of convincing their students that 0.99… = 1 is the only truth.

Notes

[1] Every non-empty set which is bounded above has a least upper bound.
[2] There exist a line l and a point P not on l such that at least two distinct lines parallel to l pass through P (Greenberg, 1993, p. 187).
[3] Consider the set K of rational functions with real coefficients. Using the usual algebraic rules for manipulating rational functions, K becomes a field. Also K is an ordered field by defining that $P(x)/Q(x)$ is positive if the coefficients of the highest-order terms of $P(x)$ and $Q(x)$ have the same sign. But there exists an infinitesimal $1/x$ and an infinite number x in this ordered field (Schramm, 1996, p. 89; Körner, 2003, pp.12-13).

References

Beswick, K. (2004) Why does 0.999...= 1? A perennial question and number sense. *Australian Mathematics Teacher* **60**(4), 7-9.

Burn, R. P. (2000) *Numbers and Functions: Steps into Analysis* (2nd edition). Cambridge, UK: Cambridge University Press.

Greenberg, M. J. (1993) *Euclidean and Non-Euclidean Geometries: Development and History* (3rd edition). San Francisco, CA: W. H. Freeman.

Körner, T. W. (2003) *A Companion to Analysis: A Second First and First Second Course in Analysis.* Providence, RI: American Mathematical

Society.
Rudin, W. (1976) *Principles of Mathematical Analysis.* New York, NY: McGraw-Hill.
Propp, J. (2013) Real analysis in reverse. *American Mathematical Monthly* **120**(5), 392-408.
Schramm, M. J. (1996) *Introduction to Real Analysis.* Englewood Cliffs, NJ: Prentice Hall.
Trudeau, R. J. (1987) *The Non-Euclidean Revolution.* Boston, MA: Birkhäuser.
Weir, A. J. (1973) *Lebesgue Integration and Measure.* Cambridge, UK: Cambridge University Press.

A Luta Continua! An ethnomathematical appreciation of Paulus Pierre Joseph Gerdes

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Shortly after the time we met in Berkeley, California, in 1980, at the fourth International Congress on Mathematics Education [1], until months before his untimely death, Paulus often signed correspondences, letters and later emails, to me, and I assume to others, with the closing expression: "A luta continua!" [The struggle continues!]. I imagine that he adopted this exclamation from the slogan of the *Frente de Libertação de Moçambique* (Front for the Liberation of Mozambique) (FELIMO) [2] because he was in solidarity with its *raison d'être,* the independence of an African people from European domination in the form of Portuguese colonial rule. His anticolonial and antiracist sentiments are, I contend, foundational to appreciating the motivating force of his scholarly activities, that resulted in a vast corpus of publications that contribute uniquely to both mathematics and mathematics education.

The magnitude of Paulus's contribution is impossible to portray adequately in a few words. He read and wrote in several languages, including Dutch, German, French, Portuguese, and English. In some of these languages, he published hundreds of articles and books. Here, I provide only brief biographical information and discuss two areas of his work in ethnomathematics that may interest FLM readers: foundations of the calculus and the mathematics and aesthetics of basketry.

Paulus, as an acutely and politically conscious participant observer of the global social and political revolutions of the late 1960s and early 1970s, successfully fashioned a means to combine his quest for global social justice with his love of mathematics and appreciation of diverse cultures. In 1976, he arrived in Mozambique from the Netherlands, the country of his birth. He was educated at Radbound University Nijmegen, where he earned a bachelor's degree with honors in mathematics and physics. Afterwards, he participated in a solidarity mission in Vietnam, and returned to Nijmegen to complete a baccalaureate in cultural anthropology in 1974. A year later, he earned a master's degree in mathematics. While still in the Netherlands, he became a professor at the Centro do Terceiro Mundo [Center of the Third World], with links to the liberation and anti-apartheid movements in southern Africa [3].

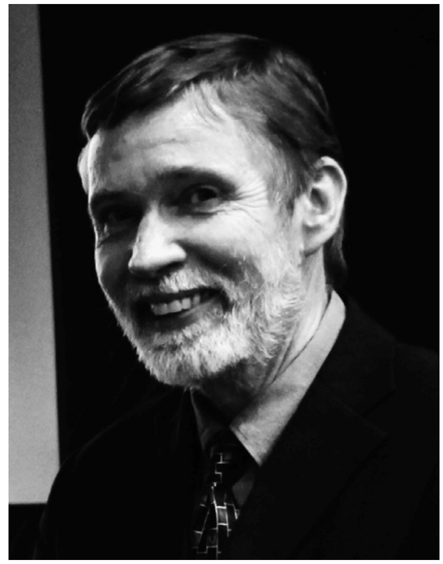

Figure 1. Paulus Pierre Joseph Gerdes (11 November 1952 – 11 November 2014).

In Mozambique, Paulus's mathematics teaching and research represented the incunabula of ethnomathematics. Several of his earliest publications in English appeared in this journal (Gerdes, 1985a, 1986, 1988). In 1986, he completed a doctorate at the University of Dresden, Germany, with a thesis on *O Despertar do Pensamento Geométrico* [The Origins of Geometric Thinking], later published (Gerdes, 2003). In 1996, he earned a second doctorate with a thesis on *Geometria Sona: reflexões sobre tradições de desenhar na areia entre os povos da África ao Sul do Equador* [Sona geometry: reflections on the tradition of sand drawings among the people of Africa south of the Equator], at the University of Wuppertal, Germany, an expanded version of an earlier publication (Gerdes, 1994).

One variant of ethnomathematics as a research program theorizes the emergence of cognitive practices that attend discursively to objects, relations among objects, and relations among relations. These cognitive practices can be shaped by cultural activities such as labor or intellectual stances. Paulus recognized cultural groups defined philosophically as well as geographically. Extending Struik's (1948, 1997) report and analysis, Paulus (1985b) published a book, *Marx demystifies calculus* [4], which he later revised and retitled as *The philosophic-mathematical manuscripts of Karl Marx on differential calculus: an introduction* (Gerdes, 2014). In these texts, Paulus synthesizes references from four different languages to provide an ethnomathematical account of Marx's critique of three theoretical formulations of the differential calculus and his presentation of an alternative perspective that attempts to account for the dialectics of motion and change to which a mathematical function is subjected in the process of differentiation (Marx, 1983). Understanding that adherents of the philosophy of historical and dialectical materialism constitute a distinct cultural group whose intellectual perspective varies from other Western philosophical and social worldviews, Paulus analyzed