TRANSITIONING FROM INTRODUCTORY CALCULUS TO FORMAL LIMIT CONCEPTIONS

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The limit concept is a fundamental mathematical notion both for its practical applications and its importance as a prerequisite for later calculus topics. Past research suggests that limit conceptualizations promoted in introductory calculus are far removed from the formal epsilon-delta definition of limit. In this article, I provide an overview of research on limit conceptions and compare the competing theories for how the formal definition is constructed. The findings and assumptions of these research designs are discussed, leading to discussion of an alternative theory, corresponding pedagogical practices, and future research needs.

Introductory calculus conception of limit

Research describes an introductory calculus conception of limit in which students depend on dynamic conceptions, hold a variety of misconceptions, and focus on procedures with little conceptual understanding of a limit. In order to consolidate past research on limits and discuss traditional instructional practices, consider the experiences of a hypothetical student named Kyle. Although Kyle encounters everyday uses of the word *limit* from a young age (*e.g.,* speed limit and weight limit), he is first introduced to the mathematical meaning of the term in introductory calculus. Kyle's teacher copies the book's definition of limit on the board:

> As x approaches a, the **limit** of $f(x)$ is L, written
> $$\lim_{x \to a} f(x) = L$$
> if all values of $f(x)$ are close to L for values of x that are sufficiently close, but not equal, to a. The limit L must be a unique real number. (Bittinger, Ellenbogen & Surgent, 2012, p. 95)

Kyle copies the definition into his notes, wondering what this definition has to do with the non-mathematical meaning of limit as a boundary or barrier, such as in a speed limit. Kyle never learns what "sufficiently close" means since the instructional focus quickly turns to finding limit values.

Like many teachers, Kyle's teacher places a great deal of attention on determining limit values using dynamic approaches. Dynamic approaches to finding limit values have received a great deal of attention by researchers (Boester, 2010; Cottrill *et al.,* 1996; Núñez, Edwards & Matos, 1999; Swinyard, 2011; Williams, 1991, 2001). The term *dynamic* is used to describe any limit interpretation based on motion. For example, a functional emphasis on outputs *approaching* (getting close to) a value L as inputs approach (get close to) a value c builds a dynamic view, as does a graphical emphasis on finding limit values by tracing along a graph to see how outputs change for inputs close to c (Boester, 2010; Williams, 1991). Although dynamic approaches vary, the term is used throughout this article to indicate any such approach based on motion. Kyle develops a dynamic approach to limits by learning to evaluate f at several inputs that move closer to a and by examining what happens to the corresponding outputs. For instance, to find

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

Kyle evaluates f at several values and finds the outputs shown in Figure 1. Since the outputs appear to be getting closer to six as the inputs get closer to 3, Kyle determines that

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$$

In order to determine the limit, Kyle focuses on determining the outputs associated with given inputs, referred to as a "forward" or x-first approach (Swinyard & Lockwood, 2007).

Kyle also learns to use the graph of a function to determine

$$\lim_{x \to a} f(x)$$

Tracing along the path of the graph, Kyle considers what happens to the y-values as the x-values move closer to a. For the function

$$f(x) = \frac{x^2 - 9}{x - 3}$$

Kyle examines the graph depicted in Figure 2, tracing his pencil along the graph of the function to see what happens to the point of the pencil near an x-value equal to three. His teacher reminds him that it does not matter what the graph does when x equals three, meaning that in this case the limit value is six. Despite his teacher's persistence, Kyle struggles to ignore the hole in the graph, sometimes succumbing to the temptation of saying the limit does not exist.

After developing motion-based notions of limit, Kyle is introduced to many algorithms that allow for fast calculation of limit values. Kyle learns that sometimes

$$\lim_{x \to a} f(x)$$

can be found by evaluating the function at $x = a,$ such as the case of

$$\lim_{x \to 2} (3x^2 + 1) = 3(2)^2 + 1 = 13$$

Input	2.9	2.99	2.999	3	3.001	3.01	3.1
Output	5.9	5.99	5.999	?	6.001	6.01	6.1

Figure 1. Table of values used to determine $\lim_{x\to 3}\frac{x^2-9}{x-3}$

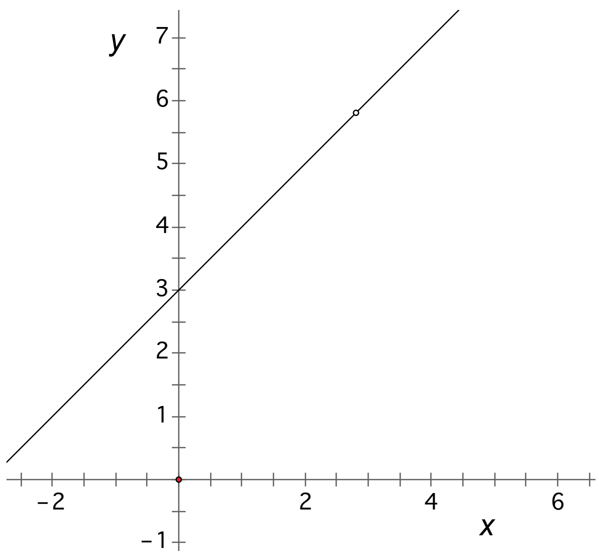


Figure 2. Graphical representation used to find $\lim_{x\to 3}\frac{x^2-9}{x-3}$

For the previously considered example

$$\lim_{x\to 3}\frac{x^2-9}{x-3}$$

Kyle learns to try direct substitution first but notices that it leads to an "indeterminate form" of $\frac{0}{0}$. When this happens, Kyle tries various algebraic techniques. By factoring and cancelling a factor of $x - 3$ before using direct substitution, the limit becomes

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+3)}{(x-3)} = \lim_{x\to 3}(x+3) = 3+3 = 6$$

Kyle learns many other "tricks" for finding limits, including multiplying by the conjugate, L'Hôpital's Rule, and using inequalities to "squeeze" an unknown limit value between two equal limit values.

Students often learn the various algorithms for finding limits independently from the definition of limit (Cetin, 2009; Ferrini-Mundy & Graham, 1994). As a result, students like Kyle develop three independent, unconnected conceptions of limit: (1) the introductory definition involving notions of arbitrary closeness (hereafter referred to as an informal static definition); (2) a dynamic conception of values approaching or graphs approaching a particular value; and (3) a procedural conception including algorithms that produce a numerical value. Because these notions of limit are not necessarily connected, students can develop a limited conceptual understanding of the limit concept (Cetin, 2009; Cornu, 1991; Davis & Vinner, 1986; Ferrini-Mundy & Graham, 1994; Juter, 2005; Tall & Vinner, 1981). Although students can "do" limit problems, they do not grasp the definition and when asked to explain what a limit is, they rely on explaining the procedures used to calculate the value (Cornu, 1991; Davis & Vinner, 1986; Tall & Vinner, 1981). As a result, Kyle cannot transfer knowledge of limits to other contexts. For instance, despite being able to determine

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = 6$$

Kyle cannot approximate the value of

$$f(x) = \frac{x^2-9}{x-3}$$

at 2.99 (Cetin, 2009). Kyle fails to understand that the limit value indicates that the function produces outputs close to six for inputs close to three. Classroom assessments that emphasize skills and procedures often mask students' inability to explain or apply the limit concept (Ferrini-Mundy & Graham, 1994).

In addition to the procedural emphasis of their concept image of limit, students often show several common misconceptions of limits while taking introductory calculus. For example, students may develop misconceptions of limit as a bound (Cornu, 1991; Davis & Vinner, 1986; Monaghan, 1991; Oehrtman, 2003, 2009; Williams, 1991), limit as an approximation of the function value (Oehrtman, 2003; Przenioslo, 2004; Williams, 1991), and limit value as equal to the function value (Davis & Vinner, 1986; Juter, 2006; Przenioslo, 2004). Research indicates that these misconceptions are persistent and rather than replacing earlier notions with notions learned during formal instruction, students often retain both sets of ideas (Davis & Vinner, 1986; Tall & Vinner, 1981). Thus, misconceptions impact students' concept images of limit throughout their introductory calculus experiences and beyond.

Formal calculus conception of limit

Unlike the emphasis of introductory calculus instruction, the formal calculus conception of a limit relies heavily on a conceptual understanding of the formal epsilon-delta definition[1]. The formal epsilon-delta limit definition is based on static terminology, abandoning the motion-based terminology of the dynamic interpretation. The fundamental idea of the formal definition of limit is that for any small interval or neighborhood chosen around the limit value L on the y-axis, another neighborhood can be found around a on the x-axis so that the images of all the points in the x-axis interval (possibly excluding the image of a itself) are contained in the y-axis interval (see Figure 3, overleaf). The definition itself is a static notion of limit since it refers to motionless intervals (although one can envision dynamic expanding and contracting intervals as described later). Since Kyle's introductory calculus instruction largely ignored static notions of limit, he struggles with the formal definition.

The distribution of static, dynamic (motion-based), and algorithmic notions related to the introductory calculus and formal notions of limit are depicted in Figure 4. Notice that Kyle's introductory calculus conception of limit was built on a small and unstable introduction to limits via the informal static definition with the majority of instruction focused on dynamic approaches and algorithms. However, an advanced calculus conception of limit requires a solid foundation on formal static ideas as stated in the epsilon-delta definition.

Once Kyle can identify an x-axis interval that maps to a given y-axis interval, dynamic and algorithmic interpretations of limit can extend this understanding. For instance, a dynamic view of an expanding or contracting y-axis interval and the corresponding expanding or contracting x-axis interval can elaborate the formal static view of a limit. A formal conception of limit also includes knowledge of various algorithms for finding limits but those algorithms are supported by proofs grounded in the formal static definition. For instance, Kyle is expected to prove that

$$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

before this algorithm is used to calculate limits. Proving this result requires the epsilon-delta definition. Thus, while a formal concept image of a limit should include static, dynamic, and algorithmic notions, it is the static epsilon-delta definition which forms the foundation for the dynamic and algorithmic interpretations.

Theories for transitioning to formal limit ideas

Many researchers have acknowledged the inconsistency between students' introductory calculus and formal calculus conceptions of limit. Described above as an imbalance between a dynamic emphasis in introductory calculus and a static foundation in formal calculus, the inconsistency between the introductory and formal limit definitions has also been described in terms of *process* versus *object, x-first* versus *y-first, discrete* versus *continuous, practical* versus *theoretical*, and *proceptual-symbolic* versus *axiomatic-formal* dichotomies. Together, these findings suggest the formal calculus conception of limit does not naturally follow from students' introductory experiences with limits. As seen in Table 1, the inconsistency between the introductory and formal notions of limit is well-documented by multiple researchers. Several theories have been offered to describe the stages required to advance from the introductory calculus to formal limit definition. These approaches are summarized in Figure 5.

The "process-object" approach illustrated in Figure 5 offers a genetic decomposition (*i.e.,* a description of how the limit concept is learned) based on APOS theory and utilizes seven steps to help students gradually transition from a process-oriented calculus conception to a formal conception of limit as a static object (Cottrill *et al.,* 1996). The seven steps include (1) evaluating f at a single point close to a, (2) evaluating f at several points which get successively closer to a, (3) coordinating a domain process where x approaches a with a range process where y approaches L to obtain a process where $f(x)$ approaches L, (4) encapsulating the process of step 3 as an object, (5) reinterpreting step 3 in terms of inequalities and intervals, (6) incorporating quantifiers to describe the process in step 5, and (7) a complete epsilon-delta conception (Cottrill *et al.,* 1996).

For the example of

$$f(x) = \frac{x^2 - 9}{x - 3} \text{ and } a = 3,$$

Kyle may begin by evaluating f at 4 to find $f(4) = 7$. Next, he evaluates f at several values that get closer to 3. For instance, Kyle may determine that $f(4) = 7$, $f(3.5) = 6.5$, $f(3.1) = 6.1$, and $f(3.01) = 6.01$. Next, Kyle must coordinate the process of x approaching 3 with the process of the outputs approaching 6 to view the limit as a process where $f(x)$ approaches 6 as x approaches 3. In the fourth step, Kyle views this process as an object and identifies the resulting object as

$$\lim_{x \to 3} f(x)$$

Many students struggle to move beyond this step (Cottrill *et al.,* 1996). In step 5, Kyle can relate the proximity of $f(x)$ to L with the proximity of x to 3. For instance, since $f(3.5) = 6.5$ and $f(2.5) = 5.5$, Kyle begins to recognize that if

$$0 < |x - 3| < 0.5 \text{ then } |f(x) - 6| < 0.5$$

In step six, Kyle begins to quantify the previous relationship by recognizing that for all neighborhoods around six, there exists a neighborhood around three so that all the images of the inputs fall in the neighborhood of outputs

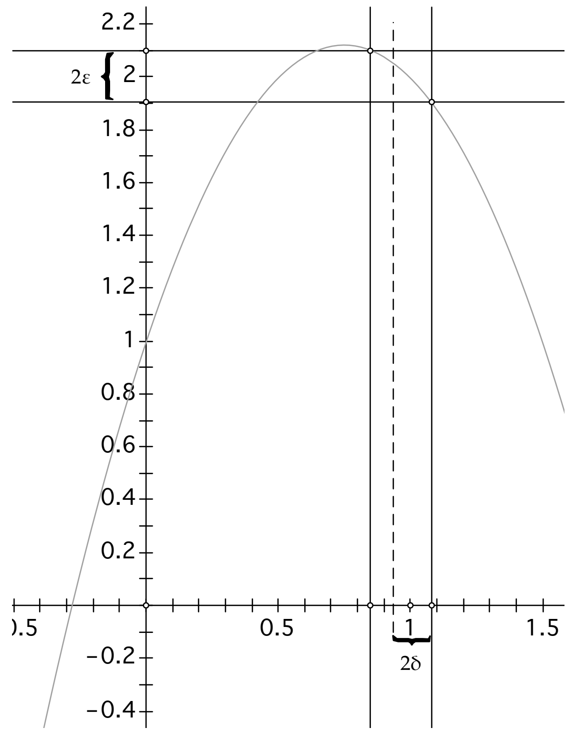


Figure 3. Graphical representation of the epsilon-delta definition.

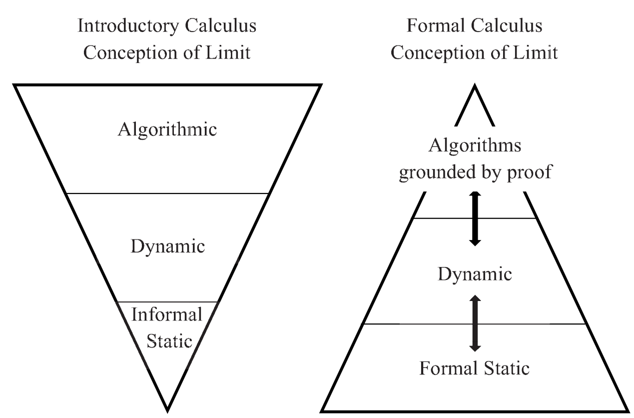


Figure 4. Introductory versus advanced calculus conception of limit.

Introductory Calculus Conception	Formal Calculus Conception	Authors
process	object	Cottrill *et al.* (1996)
x-first (forward)	*y*-first (backward)	Swinyard (2011); Swinyard & Larsen (2012)
dynamic	static	Boester (2010); Williams (1991, 2001); Lakoff & Núñez (2000)
discrete	continuous	Kidron (2008)
practical	theoretical	Barbé *et al.* (2005)
proceptual-symbolic	axiomatic-formal	Tall (2008)

Table 1. Descriptions of the introductory calculus and formal calculus limit conceptions.

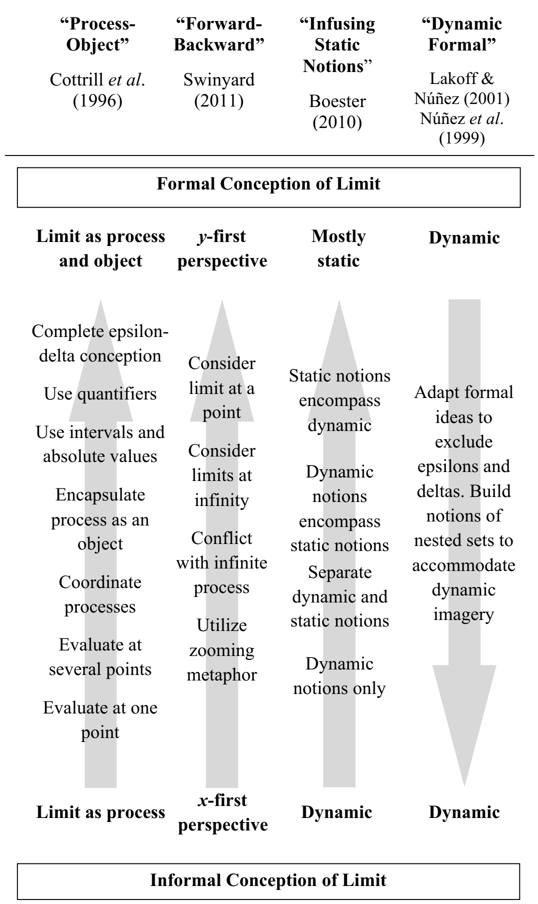


Figure 5. A comparison of theories regarding how to close the gap between informal and formal conceptions of limit.

around six. This leads to the final stage, the construction of the formal epsilon-delta definition of limit.

The "forward-backward" theory similarly suggests the transition involves a series of steps to move students from an introductory calculus conception of limit to a formal conception of limit (Swinyard, 2011). This approach does not conflict with the "process-object" approach, but provides additional detail for how students advance to the later stages of this process (Swinyard & Larsen, 2012). According to this theory, there are two key cognitive changes that students must make to transition to a formal understanding of limit. Students often learn a "forward" approach in introductory calculus that moves from x-values to corresponding y-values, but they must learn to adopt a "backward" approach that moves from y-values to corresponding x-values to understand the formal definition. While introductory calculus focuses on finding a candidate for the limit of a function, the formal definition of a limit focuses on validating the chosen candidate using a "backward" approach (Swinyard & Lockwood, 2007; Swinyard & Larsen, 2012; Swinyard, 2011). Another key cognitive shift that must occur is conceptualizing the infinite process of approaching a value (Swinyard, 2011). Since conceptualizing the formal definition of

$$\lim_{x\to a} f(x) = L$$

requires understanding the process of x approaching a at the same time $f(x)$ approaches L, it is suggested that Kyle first consider what it means for

$$\lim_{x\to\infty} f(x) = L$$

Once Kyle can verbalize what it means for a function to approach a value as x gets large, he can apply similar reasoning to describe the case where x approaches the real number a (Swinyard, 2011).

The "Infusing Static Notions" approach acknowledges that a formal conception of limit includes both dynamic and static notions of limit (Boester, 2010). Like Cottrill *et al.* (1996), this theory suggests students must build static notions of limit from a strong dynamic foundation. Results from a small study involving eight calculus students indicated students were reluctant to adopt static terminology, and suggested that static and dynamic notions of limit were distinct in students' minds (Boester, 2010). According to this theory, instruction should gradually build static notions from dynamic notions in four stages: (1) dynamic only conception; (2) dynamic and static conceptions separate; (3) dynamic conceptions dominate but are integrated with static notions; and (4) static conceptions dominate but are integrated with dynamic notions (Boester, 2010). Furthermore, instruction should motivate the need for static ideas through real-world problems that require static terminology. Without meaningful problems that encourage the use of static terminology, this theory suggests students are unlikely to infuse static ideas into pre–existing dynamic notions of limit.

The "dynamic formal" theory differs from the other theories by suggesting that the epsilons, deltas, and quantifiers traditionally used in the formal definition are not necessary for a formal understanding of limit (Lakoff & Núñez, 2000). Unlike the other theories that suggest students must adapt their informal notions of limit to accommodate the static ter-

	"Process-Object" Cottrill *et al.* (1996)	"Forward-Backward" Swinyard (2011)	"Infusing Static Notions" Boester (2010)	"Dynamic Formal" Lakoff & Núñez (2000), Núñez, Edwards & Matos (1999)
Weaknesses	Students struggled to reach the last three stages, failing to incorporate inequality and quantifier notation to reach the formal epsilon-delta definition	Students showed difficulty adopting the important backward or y-first approach; students were also reluctant to engage in graphical representations	Only 2 of 8 students in the study advanced to the final stage—a predominantly static limit conception; students clung to dynamic notions	The primary definition of limit offered only works for monotone functions; the definition is quite difficult, particularly for functions that are non-monotonic
Assumptions about introductory limit conceptions	Limit as an Action or Process	Emphasizes a Forward, x-first approach to finding limits	Dynamic (motion-based) notions only	Dynamic (motion-based) notions only
Key contributions	Highlights the importance of moving from limit as an action, to a process, to an object, and finally to a schema	Details the importance of students developing a y-first approach to advance to the later stages of Cottril *et al.'s* theory	Suggests that static ideas may be gradually added to a dynamic foundation rather than a switch from purely dynamic to purely static	Recognizes the potential for a formal definition that incorporates dynamic views of limit
Implications for alternative introductory calculus instructional approach	Build introductory notions of limit as an object	Include backward or y-first interpretations of limit in introductory calculus through different requirements for closeness to L (and finding corresponding required closeness to c); include both graphical and numerical representations when limits are introduced	Incorporate static terminology in introductory instruction, including discussions of closeness through exploring fixed neighborhoods of output values, around L, and corresponding fixed neighborhoods of input values around a.	Do not neglect to build dynamic notions in introductory calculus; when formal ideas are introduced, dynamic imagery of expanding and collapsing intervals can enhance static definition

Table 2. Overview of weaknesses, important contributions, and implications for calculus instruction.

minology of the formal definition, this theory adjusts the formal definition to include dynamic terminology. This is represented in Figure 5 by the arrow moving from the formal conception to meet the informal conception. This approach uses the Basic Metaphor of Infinity to define the formal concept of the limit of a function in a dynamic way using the imagery of expanding and contracting sets to first envision limits of sequences on the range and domain of the function (see Lakoff & Núñez, 2000, for a complete description). Although this approach makes the formal conception of limit more dynamic by infusing ideas of motion as the terms of sequences approach a value, the resulting definition is quite complex. In particular, while monotonic functions can be defined naturally in this way, even simple non-monotonic functions require the use of "teaser sequences" and significant adaptations to the initial definition (Lakoff & Núñez, 2000). Since such functions are more the rule rather than an exception, the complexity of dealing with these cases is a cause for concern. In an attempt to make the definition of limit more natural for students, the definition becomes unnecessarily complicated.

Analyzing the research

Each of the theories I have described recognizes the difficult task of building the formal limit definition from introductory calculus notions of limit. This difficulty is supported both by past research that shows that students return to informal limit ideas and misconceptions after instruction on the formal definition (Davis & Vinner, 1986; Tall & Vinner, 1981; Williams, 1991) and by the research of those presenting the theories themselves. For instance, Cottrill *et al.* (1996) admitted that "only a few of the students that we observed gave any indication of passing very far beyond the first four steps of this genetic decomposition" (p. 186) and no students progressed to the point that the last two steps of their decomposition could be explored. In Boester's (2010) study, students were reluctant to adopt static terminology and only two of the eight students advanced to Boester's final stage of a predominantly static limit conception. Although Swinyard and Larsen (2012; Swinyard, 2011) did have one set of student participants successfully reinvent the formal limit definition, they described many roadblocks and hesitancies along the way. For instance, the students involved in their teaching experiment were hesitant to engage in graphical representations and strongly preferred an x-first perspective. At best, each of the theories describes a difficult path for students to successfully advance to formal limit reasoning. The weaknesses of each of the four theories are outlined in row 1 of Table 2.

When interpreting these findings, one must also take into consideration the participants' initial limit conceptions. The studies did not investigate how students initially came to understand informal notions of limit, but rather accepted those informal notions as given and explored how formal notions developed from the existing foundation. For instance, the students in Swinyard's work (Swinyard, 2011; Swinyard & Larsen, 2012) were third semester calculus students who were selected based on their prior ability to (a) provide examples and non–examples of limits and (b) provide an informal definition of limit based on dynamic terminology of outputs approaching L as inputs approached a. The issue, then, is not the findings of these studies. In particular, I believe that each interpretation of how a formal notion of limit might be built from the informal foundation of the participants in the studies has merit. However, by choosing participants with existing informal notions of limit similar to those described for Kyle, these studies have assumed the necessity of an informal limit conception that is dynamic, x-first, or process-oriented. The implicit assumptions regarding students' initial limit conceptions for each theory are summarized in row two of Table 2.

Before we can confidently describe a genetic decomposition for students' development of a formal definition of limit, we must also describe developments that occur as students first encounter the meaning of limit. Perhaps we have incorrectly assumed the necessity of an informal limit conception that is at odds with the formal definition. While empirical evidence is needed to explore how students might first develop these earliest notions of limit during introductory calculus, the findings discussed above point to some suggestions for ways introductory calculus instruction might better equip students to later learn the formal limit definition. Namely, if introductory calculus instruction incorporates both dynamic and static, x-first and y-first, and process and object-oriented interpretations of limit, the difficult transition from informal to formal limit notions might be eased. Tall (1992) suggested that beginning with a richer conceptualization of a concept could reduce later cognitive conflict. In looking for ways to build a formal understanding of limit, we have overlooked the possibility of adapting introductory calculus instruction to build an initial conceptualization of limit that is a better fit with the formal notion. The main contributions of each of the theories and corresponding implication for adapting introductory calculus instruction on limits are provided in rows three and four of Table 2. The pedagogical sequence proposed in the next section incorporates these ideas to suggest how introductory calculus instruction might be improved to ease the transition.

An alternate pedagogical approach

Contrast Kyle's experiences with those of another hypothetical student Lisa, who learns about limits according to an alternative model. Prior to being introduced to limits, Lisa is challenged with the task of predicting a missing or hidden function value. In the example presented in Figure 6 (overleaf), Lisa uses the graphical and tabular representations of a function to predict $f(2)$. Notice that in the graphical representation, a black bar is placed over the graph to remove the distraction of the actual function value and to promote an emphasis on the predicted value (Nagle, 2009). Lisa is encouraged to engage in discussions regarding her confidence in the given prediction, including ideas of "zooming in" on the function in order to examine the outputs for input values that are "closer" to 2. Lisa examines the sample graphs and tables in Figure 7 that result from this zooming process, leading her to make a prediction that $f(2) = 4$. Lisa also encounters examples such as those shown in Figure 8 where the zooming techniques indicate that a distance remains between outputs even for very close inputs. Lisa decides that she cannot make a prediction for this particular input because the function acts very differently on each side of zero.

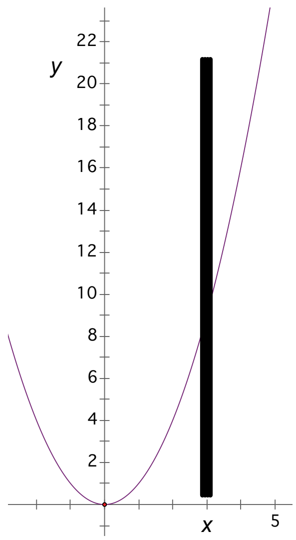
After several examples of predicting function values using the ideas of zooming and closeness, Lisa is introduced to the mathematical meaning of a limit. In particular, Lisa learns that the limit of a function is the intended value (or predicted value) and she is introduced to the mathematical notation. Notice that this definition emphasizes the limit as an object (namely, the predicted value) rather than a process, consistent with the implications of Cottrill *et al.*'s (1996) work. Because the actual function value is never revealed, Lisa learns that the limit value does not depend on the function value. (Later, when learning about continuity, Lisa learns to focus on whether her prediction was correct). Thus, Lisa understands from her previous experience that in Figures 6 and 7,

$$\lim_{x \to 2} f(x) = 4$$

while in Figure 8,

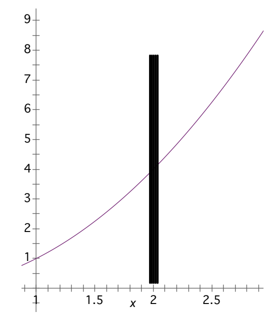
$$\lim_{x \to 0} f(x)$$

does not exist. In addition to functions that behave differently on each side of the value a (Figure 8), Lisa sees examples of functions that oscillate rapidly near a and that increase or decrease without bound near a. By experiencing the difficulty of predicting the function value for these function types, Lisa can conjecture that the limit value does not exist in these situations.

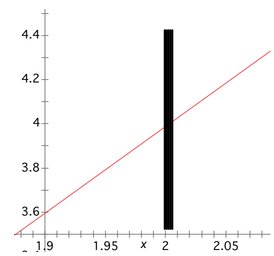


x	-1	0	1	2	3	4	5
y	1	0	1	?	9	16	25

Figure 6. A sample problem encountered by Lisa in introductory calculus.



x	1	1.33	1.67	2	2.67	3
y	1	1.78	2.78	?	7.11	9



x	1.9	1.95	1.99	2	2.01	2.05	2.1
y	3.61	3.80	3.96	?	4.04	4.20	4.41

Figure 7. Lisa zooms in on the input of 2 to examine output.

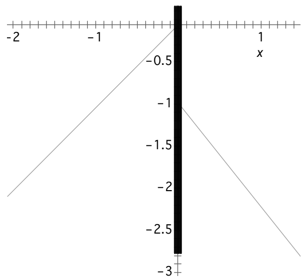
In addition to the dynamic strategies of zooming used in the previous example, Lisa also develops static notions of limit. Classroom discussions emphasize satisfying the “closeness” demands of all classmates. For instance, encouraged by the teacher to determine how close to four the output value must be in Figure 6 before being convinced of the prediction, one classmate says that the outputs would have to be between 3.5 and 4.5, while another classmate says the outputs would have to be between 3.99 and 4.01. Lisa and her classmates are challenged to zoom in on the graph to find that when the inputs are “close enough” to 2, that even the most rigorous student’s criteria of outputs “close enough” to four can be met. Lisa’s teacher emphasizes that it is not sufficient to satisfy just any student’s requirements for closeness to L. For the limit to exist, every possible requirement for closeness to L must be met by making the x-coordinates close enough to a. Thus, Lisa begins to develop static notions of a limit in terms of finding a “snapshot” of a graph or table of a function that provides inputs close enough to a to guarantee that the outputs are within some predetermined (arbitrary) distance of L. These early static notions are supported by Boester’s (2010) findings that students were reluctant to build static notions after a solely dynamic foundation. Notice that by finding a range of inputs that satisfy a given range of outputs, Lisa is also developing a y-first or “backward” approach to limits (Swinyard, 2011). In this way, Lisa’s earliest conceptions of limit include ideas described at the advanced stages of the genetic decompositions described by Cottrill *et al.* (1996), Swinyard (2011) and Boester (2010).

Although the previously described models by Cottrill *et al.* (1996), Lakoff and Núñez (2000), Boester (2010), and Swinyard (2011) described different formal understandings of limit (object, y-first, static, dynamic), each one assumed that students began with a restricted view of limit (process only, x-first only, dynamic only). While the typical instructional sequence (recall Kyle) may introduce limits in such a limited way, the potential for a more robust initial conception of limit has not been explored. Could informal static ideas be fostered at the introductory calculus level, including y-first or “backward” interpretations of limits (using conditions for closeness to L) and views of limit as an object (the predicted value of a function)? If so, then students need not develop introductory conceptions of limit that are void of static notions. As a result, the transition to the formal definition may only require formalizing introductory static, object-oriented, or backward notions rather than building these from solely dynamic, process-oriented, and forward foundations (see Figure 9).

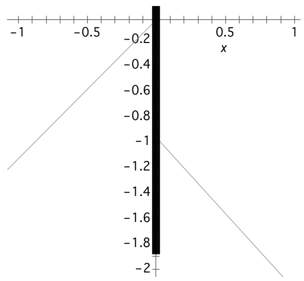
Discussion

Synthesizing past research on limits illuminated widespread agreement on various weaknesses in students’ introductory calculus concept images and the difficult task of transitioning from an introductory calculus conception of a limit (described as dynamic, process-oriented, x-first, discrete, practical, and proceptual-symbolic) to a formal calculus conception of a limit (described as static, object-oriented, y-first, continuous, theoretical, and axiomatic-formal). Researchers have suggested a variety of theories for how students advance to a formal conception of a limit, though each of these assumes an introductory calculus conception that is void of static, object-oriented, and y-first approaches to limits and describes the necessity of building such ideas prior to formal limit reasoning.

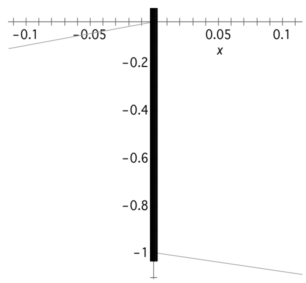
Viewed from a different perspective, the findings of these studies can also be interpreted as recommendations for



x	-3	-2	-1	0	1	2	3
y	-3	-2	-1	?	-1	-3	-5



x	-1	-0.5	-0.1	0	0.1	0.5	1
y	-1	-0.5	-0.1	?	0.8	0	-1



x	-.10	-.05	-.01	0	.01	.05	.10
y	-.10	0.05	-.01	?	0.98	0.9	0.8

Figure 8. *A case where a prediction cannot be made (the limit does not exist).*

improving limit instruction in introductory calculus. The alternative pedagogical approach offered in this article does not recommend earlier exposure to the formal epsilon-delta definition of a limit, but it does suggest that students can begin to formulate the underlying meaning of this definition together with more traditional introductory calculus approaches for understanding limits. By motivating instruction using prior knowledge of making predictions, students are able to build confidence by finding limit values before ever hearing the term limit used mathematically. Furthermore, focusing students' attention on a limit as the predicted value of a function builds an understanding of limit as an object long before a formal epsilon-delta definition is introduced. Discussions regarding the proximity of the output values to the predicted value encourage students to develop a backward approach without dealing with the difficult epsilon-delta notation. Students also develop a more static approach as they deal with a single requirement for proximity of outputs to the limit value to determine the corresponding requirement for proximity of inputs to the value a. In addition to the benefits of building an introductory calculus conception more conducive to the formal definition of a limit, the integration of the static and dynamic, process and object, and x-first and y-first approaches may also help to address misconceptions research has shown are prevalent among students and provides a conceptual background for the algorithms learned during introductory calculus.

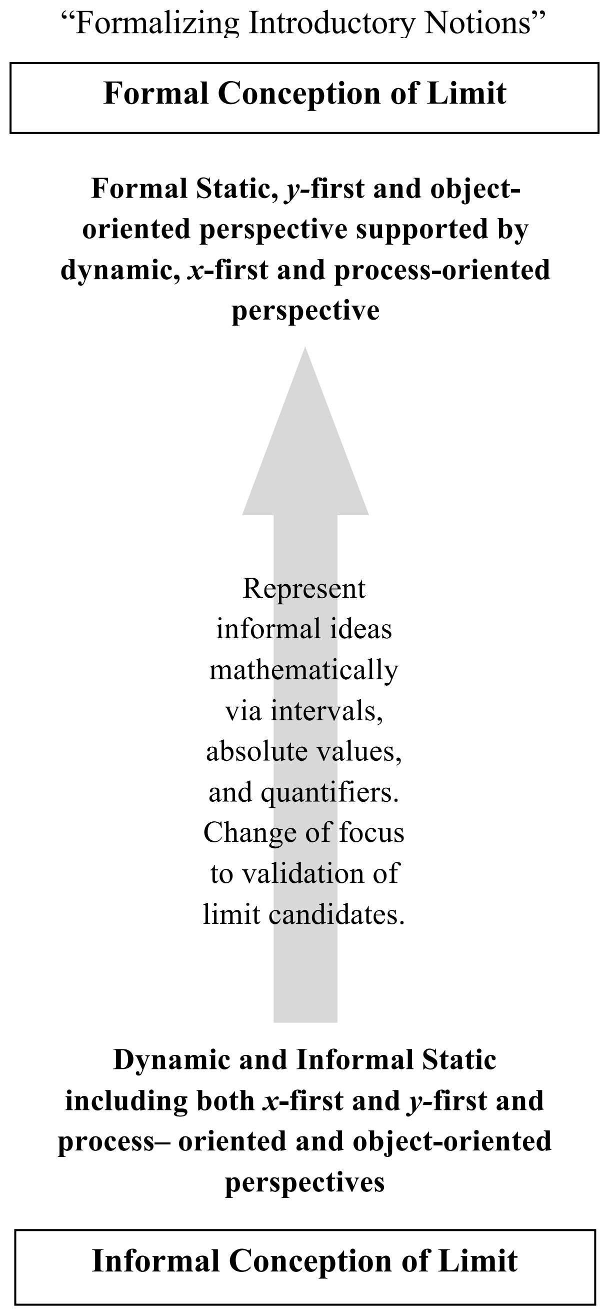


Figure 9. *The alternative model begins with a more robust introductory calculus conception and focuses on formalizing rather than replacing those ideas.*

The suggested alternative approach offered in this article is based on synthesis and evaluation of past research on students' difficulties building a formal concept image of limit. I have taken a new perspective by questioning whether introductory calculus instruction can be revised to build a more robust informal limit conception in order to ease the transition to the formal definition. Past research suggests that a more robust introduction to limits might include static, y-first, and object-oriented approaches to limits. The brief description of an alternative pedagogical approach offers suggestions for how calculus teachers may infuse these typically more advanced interpretations of limit into an introductory lesson on limits. Together, these results indicate the potential for a genetic decomposition of limit ideas different from those previously offered.

In order to test these claims, future research should investigate students' initial limit conceptions as they develop in calculus. This research should examine the initial stages of limit reasoning as developed under the traditional approach often described in introductory calculus textbooks, as well as under the alternative pedagogical model I have described. If students do indeed form different initial conceptions of limit under these two models, further studies of how students in the alternative model advance to a formal limit definition could lead to a new genetic decomposition of limit ideas. Ultimately, this path of research could improve limit instruction to make the formal limit definition more attainable.

Notes

[1] Let f be a function defined on an open interval containing a (except possibly at a) and let L be a real number. The statement

$$\lim_{x \to a} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$ (Larson, Hostetler & Edwards, 2007, p. 72).

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