Cartesian and Non-Cartesian thinking: Reflections on the Learning of Mathematics

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This paper is prompted by reading John W. Berry's recent paper, "Learning mathematics in a second language: some cross-cultural issues". He drew attention to two types of difficulty that persons learning mathematics in a second language have, those attendant merely upon the student's lack of fluency in the language of instruction, of which everyone knows individual examples, and secondly those resulting from the difference between the cognitive structures natural to the student's own linguistic community and those assumed in the instruction, for which Berry used as examples the people of Botswana, called Batswana collectively (one if a Motswana). While I am far from wishing to deny that there are cognitive differences between urban North Americans and rural Africans, I want to make a contribution to the discussion of Berry's second kind of difficulty by considering what those disparate groups have in common, cognitively speaking, only then considering some differences. I then want to indicate how such differences, not limited to Batswana, might go some way toward accounting for why the Batswana think that mathematics cannot be done in their language. Further applications of these reflections might help to explain why girls so commonly do not take to mathematics, and why the mathematization of knowledge involved in the scientific method seems to some observers as an alienating act of depersonalization. In order to be able to give the promised indications, I need to consider a few aspects of cognition. I distinguish three layers of knowledge that result from an ordinary act of knowing. The first two layers I have called below "cartesian knowing", and the third I have called "non-cartesian". I want to suggest that noticing this structure and making use of the freedom that knowing about it gives may help teachers with the mathematical dfficulties experienced in the third world as well as with math phobia elsewhere. The case that I am trying to make also gives additional support to Berry's practical suggestion of how to tackle the third-world problem, namely by having their own people structure what is to be learned right from the bottom up.

1. CARTESIAN KNOWING

Layer One: Figuration

The first of the two processes that I wish to draw attention to is so fundamental that one can easily not notice it and can even deny it. It is the result of an initial act of noticing that there is something to know. This layer is most easily noticed in sensory perception of things. In sensory perception, the thing noticed is differentiated from the rest of the sensory field. This is done for each sensory field in which the think to be noticed can *be* noticed. Then from those part-things, a visible part, an audible part, a tangible part, a smelly part, a tasty part, is integrated a single whole thing. This succession of differentiation and integration, which I call *figuration,* produces the things that we think about, that we talk about, that we assume others are talking about, all also called figurations. This is the process that gives us something to know. It need not be a sensory process, and the things need not be "things" in the ordinary physical sense of the word. When we read a novel, the process is invoked by the author to give us the characters, the scenes, the actions that are described there. Relations require figuration as much as things do. Figuration produces things in relation. The process is always ready to be invoked, even without current sensory input, for we perform it at will when we imagine a person, thing, or event from the field of our past. Memory of the past and the imagination of what has not been experienced involve at the very beginning one or several acts of figuration. In short, anything that one can later put a name to and say one knows has been figurated before one puts a name to it. The word "figuration" means either the action of the mind in making a thing present or what is present as the result of that action. I have adopted the term from Owen Barfield [1957], who uses it in a slightly looser sense.

I emphasize that figuration is a personal mental act. A feature of human mental activity is that it concerns itself with what it is about. A computer, on the other hand, concerns (if that is an appropriate word) itself with the information that it processes without any regard for the intention of that information. The same is true of most mathematical logic: intention is struck out of the process of reasoning. Neither logic nor computers figurate before dealing with information.

When we deal with those puzzles that involve gestalt switches, we may or may not be figurating differently. In one of the most famous in which a black-and-white picture is seen as either the head or an old woman or a full-lenth picture of a young woman, the difference is in what one makes of the picture rather than on the most fundamental level. The figuration can be said to be the same for both interpretations. In the case of the centre of the Canadian

flag, which is normally seen as a red maple leaf on a white background, a different differentiation is required to put the red into the background and see the pattern as two white faces in profile shouting downwards and towards each other. Figuration is often voluntary, as in the last example. On the other hand, when someone bumps into you the figuration of the tactile object is scarcely an option. Humans must figurate mathematical objects as much as physical objects, and, like others, mathematical objects need a basis of sensory knowledge before they can readily be noticed. The length of time that it took for one and zero to be recognized as numbers suggests that figuration can be a large step. The historical priority of algebra over set theory and of geometry over topology, and quite generally of the special over the more general, argues the same. Figuration is, however, a bigger research hurdle than pedagogical hurdle, since one notices more easily what is pointed out than what one must discover for oneself.

Layer two: Configuration

The second layer of structure, inseparable from the first, is produced by categorizing the figurations of things and relations. Figuration notices them: categorization decides what sort of things or relation each one is. This puts names to them and allows them to be spoken about. The level of cognition that this process operates on is the level of objective knowledge, by which I mean knowledge that can be shared between persons by symbolic (including verbal) descriptions of it. I hypothesize that all such shared knowledge, and in any case what I want to talk about, is sharable by the appropriate choice of ordinary language or the specialized languages of various disciplines imbedded in ordinary language. The objective-knowledge part of ordinary language states (or asks about) relations among things. At its simplest, a configuration will be just a single thing categorized, like the picture of the old woman or the picture of the young woman of the previous paragraph. That example indicates that categorization is not the arbitrary assignment of a name but rather the mental matching that convinces us to assign a particular name. A configuration can be described by the appropriate choice of words for each of the things and for each of the relations. Our notion of configuration is of course a theoretical construct to represent what is in one's consciousness when one utters a sentence that is intended to convey to another the kind of objective knowledge (true, hypothetical, false, or deceiving) that one commonly expresses in assertive sentences in the hope that one's hearer will think a thought more or less like one's own and referred to the same referents, i.e., will form a similar configuration with the same reference and with the figurations of its things and relations similarly categorized.

In a sentence like "the cat sat on the mat", the things that are figurated are the cat and the mat. They, their relations, and the categories to which they are assigned, which I shall discuss a little, are the configuration corresponding to that sentence. One categorizes the things and the relations even in mentioning them, because there are few ways to mention a thing without categorizing it; the demonstrative pronoun "that", where the meaning is obvious from the physical context, is one. The verb "sat", as is quite usual for a main verb, represents the important relation between the cat and the mat. The relation of sitting is analyzable, for instance, into the relations among the limbs of the cat that makes its posture sitting and the relation between the cat and the surroundings that make the cat sitting rather than being a cat in a sitting posture that happens to be lying on its side because it has been knocked over (as a porcelain cat might be). The positional relation between the cat and the mat, touching from above, makes the cat to be sitting *on* the mat. A lot of categorizing is imbedded in the use of the words "sat on". A lot of categorizing is imbedded in the use of almost *every* word. A lot of that categorizing, all of which is a personal activity, is not done in exactly the same way by all hearers of a sentence, and yet the existence of what is hopefully called "a meaning" for the sentence is dependent upon configurations and categorizings being done in sufficiently similar ways by different hearers that the personal meanings of the sentence are adequately similar for them all. What might be glorified as the "objective" meaning of a sentence (note, not of mere words alone) is the meaning for various hearers that is common to them all. Most language use makes the assumption and the demand that a sentence has the same meaning regardless of who hears it. If meanings were not widely held in common, then it would never be possible to speak to more than one person at a time, and one would have to know the interlocutor so well that one would know his or her private meanings. All communication would have to be in the sort of code in which long-married couples sometimes speak on familiar topics, highly allusive remarks based on personally shared experiences. And typical communication of objective knowledge is not in that sort of code. It is in a carefully neutral language making the minimum of special demands on the hearer in order that it can be understood by anyone. As soon as one is communicating to any more specialized audience than newspaper readers, one can use some words or assume some understandings that are not shared by the population at large. But for the population at large we have to stick to the dictionary level of common word meanings in making up our sentences. And the results of this are configurations that anyone can reconstruct from our sentences. This neutrality is a highly impersonal artifice, in spite of the fact that, with a few absurd exceptions like computer-produced poetry, it is all done by persons for persons.

Unless the speaker is himself or herself a part of its configuration, the meaning of a sentence should not depend for its objective or descriptive meaning on who said it. This seems extraordinarily abstract, and so it is. But it is brought into contact wth reality by the fact that most of what we say has not only meaning but reference, giving a triangular relation among the utterance, its meaning, and its referents, represented by Ogden and Richards [1923] as a triangle with the vertices "what is said", "what is meant", and "what it is about". In the above example of the cat and the mat, the context offers only one cat and one mat, as is indicated by the definite article. The choice of referents

frequently depends upon the circumstances of the utterance, which is why the same sentence can apply to different referents by being spoken on different occasions. In order that the meaning, dependent on categorization done more or less commonly, not change with the variance of occasion, we keep separate what the sentence means and what it says to us under our present circumstances. "The door is open" has the same literal meaning always, but which door is open depends on which door the hearer or speaker is near or looking at or interested in. The sentence and the configuration that it evokes apply to non-linguistic things and relations, which change with our circumstances and indeed with who is the hearer. Out of its natural context, "The Kind is dead; long live the King" would pose a serious referential problem.

This feature of natural language is raised to new heights in science, where laws and physical constants are specifically chosen to have a maximum of objectivity and depend minimally on anything personal. An important aspect of the scientific method is devoted to the removal of personal bias. Two remarkable books discuss this matter at length. Theodore Roszak [1973] discusses the attempt of science at objectivity, its history and consequences, calling it an idolatry and much else; this is in fact the theme of the book. More constructively, Michael Polanyi [1964] makes a convincing case that the objectivity after which science has striven since the days of Francis Bacon is not and cannot be complete. It suffices for our present purpose to note that science has striven after objectivity; it has widely been thought to be looking for a way to correlate all pointer readings regardless of what any persons thought or did about anything. What I want to point to in this paragraph is only that this process requires a certain detachment on the part of its practitioners. Not that they cannot or should not or do not care about their scientific results, but what those results are does not ideally depend in any way on those carings. The carings are irrelevant rather than nonexistent, mistaken, or impossible.

This feature of natural language is raised to its pinnacle in mathematics. In mathematics one is at a loss to imagine how one's carings could have any effect on mathematical results, except by leading one into error, which is by definition not correct mathematics. The idea is quite absurd, and there is nothing to discuss about it. The objectivity aimed at by science is and always has been an obvious feature of mathematics, leading many mathematicians and those who have thought about mathematics since Plato to take the view that mathematical objects exist in some quasi-physical sense of "exsist". The detachment that I noted above as a requirement of physical scientists is even more required of one who does mathematics. The appreciation of mathematical beauty is the appreciation of what one does not participate in, like that of sculpture, as Bertrand Russell said. The way to participate in mathematics is to do it, and that is an unusual activity at any but the most routine level. It is scarcely surprising that, in one sense of the word, many persons do not *care* for mathematics since caring matters so little.

Layer two seems to have been what Jack Yates was attending to when he wrote the recent paper, "The content of awareness is a model of the world"[1985]. He makes a good case that awareness contains a model of the world, portions of which can be called from implicit awareness to consciousness as configurations and presumably portions of which are off limits to normal consciousness. Nevertheless, there is more to awareness than just a model of the world, and the third layer of an act of knowing is where that more fits.

Layers one and two: Beginnings of a picture

Before passing on to layer three, which is inextricably attached to layers one and two, I pause to ensure that the reader can visualize as I do what layers one and two look like in a model of lines and points. Each thing, the most neutral word that I can think of, appears in the model of the world as a point. Each relation that each thing (represented by a point) enters into is represented by a graph (in the sense of graph theory) where the vertices are the points representing things related. When a point enters into a binary relation, then that point and the point representing the other things are joined by an edge or a directed edge. When the relation is ternary, then there is a triangle of edges representing that relation among those things. When the relation is *n*-ary, there is an *n*-clique with things' points as vertices. See Figure 1, which shows two things and a relation figurated.

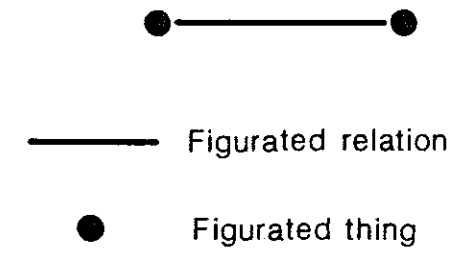

Figure 1

The categorization of a thing is a relation between its point and the personal category to which one assigns it. The categorization of a relation is the relation between the arrangement of edges that represent it and the personal categories to which one assigns it. The categories are used here without themselves necessarily being figurated. The categorizing relations are very rarely figurated. A very early lesson in a first philosophy course leads to the figuration of one's categories of things if not of relations. Most of this apparatus is not figurated but only used implicitly like grammar rules. See Figure 2, which displays the categories of what appeared merely

figurated in Figure 1. This whole model of the world, which includes oneself as a member, can be thought of as a very large two-dimensional map consisting of nothing but points and lines joining them. It is highly abstract and very informative when it is explored. It indicates the closeness of the relation between layers one and two since they are points and the lines of a single gigantic graph, as though the points and lines of figurations were drawn on the back side of a sheet of glass and the lines of categorizing and the categories were drawn on the front. Science, to the extent to which it is a thing in itself rather than existing only in the minds of scientists, is the gigantic graph that the scientists have in common, what Karl Popper calls objective knowledge [1972] and what Ilya Prigogine has called "the vantage point from which God himself surveys the world"[1984]. We each see the world through our own personal graphs (which is why I made them transparent) and in terms of these graphs; I shall refer to these graphs as our windows.

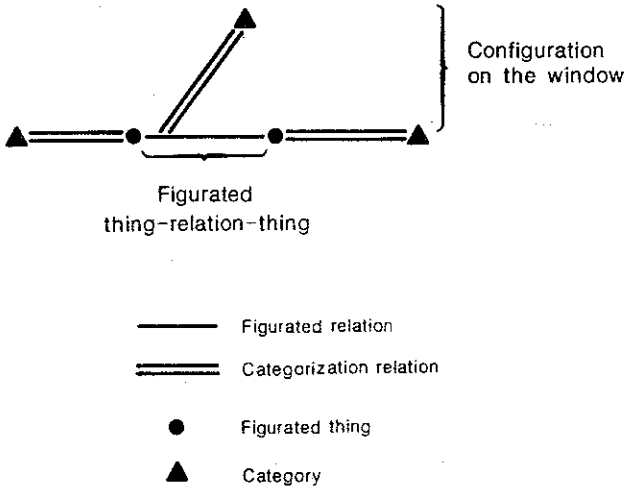

Figure 2

Layer three: The cognitive structure

My image of the third layer puts myself as a knowing subject outside the big graph described in the previous paragraph to make an even larger graph. The third layer, which makes use of a third dimension, connects me as a knowing subject to each of the vertices and edges of the graph representing my knowledge. Everything in that graph is something in my awareness; there can be no doubt that I am related to each point and each line there because I have put them there. How that relation arose varies from configuration to configuration. In some cases of things I have figurated a directly sensed thing, but in others I have seen a picture, in others I have read about the thing, in still others imagined the thing for myself, making it out of parts of other things. In the case of figurated relations, I have learned of them in those same ways and also have deduced some, guessed some. In short the third layer, cognitively speaking, comprises the knowing relations between the knowing subject and the known things and known relations. These relations of awareness must exist, for the knowing subject has integrated the known into his or her awareness. I do not specifically indicate a process of producing this layer, because the layer is primarily a record that the other two layers were produced. Its construction is the construction of layers one and two. See Figure 3 for the three-dimensional view of Figure 2. The descriptions of the structures of an act of knowing and of the knowledge are now as complete as seems necessary for the present purpose. I now add to the picture of the strictly cognitive contents of awareness some of its other contents.

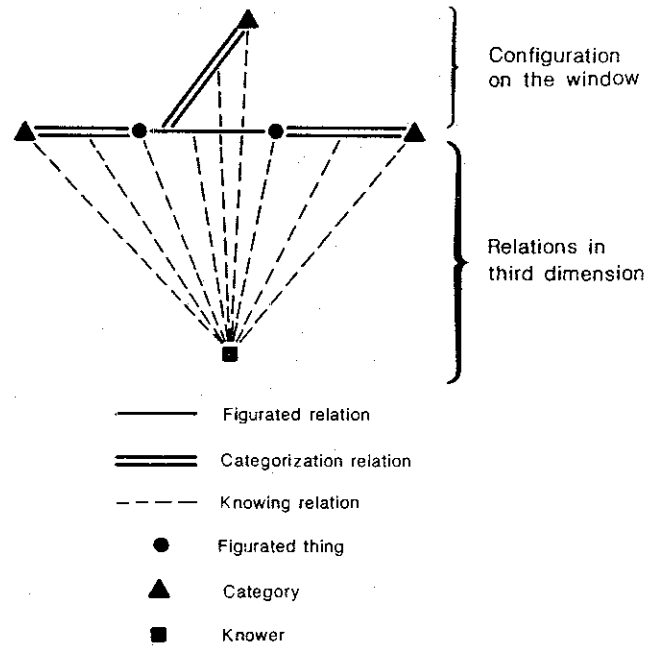

Figure 3

2. KNOWING IN GENERAL

More layer three

The relations of layer three, which I have so far described as being cognitive relations, are overlaid with all of the not primarily cognitive relations that knowers have to what they know, including positive and negative attitudes, notions of importance and unimportance, valuations, love and hatred, boredom and interest. This is where all the non-cognitive aspects of our relations must be placed because they all need a cognitive structure to be attached to; otherwise they would be unattached, and most of the relations that I have just named *are* attached to specific figurations. The third cognitive layer, which makes the model three dimensional, is the part of the structure that makes room for the fact that the knowing subject is a knowing subject and not a mere manipulator of cognitive

representations like a computer

Let us pause to look briefly at the picture I have called the window model (Figure 3). Recall that one's figurations, represented as points and lines on the glass, and the categorizing relations, which I regard as one's knowledge of those things, are spread out as an enormous graph. These relations and things that one knows stretch backwards in time as far as one's memory. Linking oneself, as with many strings, to each of the points and lines of that big graph is a relation of awareness that is a condition for that point or line to be in one's personal graph. Hanging on those strings, like so much laundry, are one's non-cognitive relations with the figurations represented by the points and lines. The result of this is that when one thinks about something, for example sees a person one knows, one sees that person through the window. The person is seen through a web of relations that one knows that person to have with other persons and things. All the laundry hanging on the strings to that person's point and to that person's relations is there in the way too. One's actual relation to the person is not a part of this picture because this is a picture only of what one has constructed, and that actual relation is not a part of one's construction. Only what one knows and feels about the relation is there. But one's actual relation to the person is influenced by the mental relations to the person, one of which is represented by a string that is in the third dimension of the model. The laundry on that string can have a powerful effect on one's dealings with the person. Most of the basis of so-called math anxiety (cf. Mitchell [1984] and literature referred to in the note and references) is laundry, but I am here trying to get a non-subjective basis for mathematical difficulty or disinterest.

Awareness *does* contain a model of the world, as it were a two-dimensional model, but also it contains how the knower knows, what the knower feels and how much the knower cares about what is known. These additions fit into the third dimension of awareness. It is possible for one to think of oneself as primarily a knower with these extras superadded but basically inessential. Or one can think of oneself as primarily a person in non-cognitive relations with other persons and things, the cognitive structure being like laths behind the plaster on a wall, essential perhaps but not the real point, not what is visible. Caricatured this way, both of these extreme views look rather silly, but it might not be too difficult to find persons prepared to speak along those lines. It is certainly not necessary that any human example of either extreme should exist. They remain characteristic of tendencies toward what might be loosely called the cognitive and the affective. So loosely, however, that it has seemed worth spelling out at some length what it might mean with more precision, in particular because the tendency loosely called "affective" involves much more than merely what is affective; it includes motivations and judgements of value that are only remotely connected to simple feelings. All real persons, presumably, have developed in themselves both of these sides of themselves. But there are differences of emphasis between different persons, between different cultures, between different occupations within a single culture, in short between groups that have any reason to approach life differently.

A second culture

I think that, in terms of the above crude model of the cognitive and non-cognitive aspects of awareness, one can give a hypothetical explanation of why Batswana think that one cannot do mathematics in their language, why they think that they must adopt a completely foreign vocabulary and mind-set in order to do mathematics. Like John Berry I think that this is probably a mistake, but I would like to explore what the above reflections suggest about why the mistake is made. In sub-Saharan Africa, English and French are the languages of science and mathematics; they are also the languages of persons with a mind-set that tends to emphasize the cognitive at the expense of the non-cognitive. By this I do not mean that we fail to admit the existence of our non-cognitive side, though some of us almost do. I merely mean that we tend, if I may speak from several decades of daily interaction with predominantly urban white Canadians, to regard the non-cognitive side as incidental to the cognitive. As an example of what I mean by this, let me suggest that one Canadian asks another Canadian whether he or she knows Mr. X; if so, the reply is likely to be something along these lines, "Yes, he's the y", where y could represent "the manager of the Royal Bank branch in some town". That is, the reply is likely to be an objective fact about Mr. X unless Mr. X is a member of the speaker's family, in which case the reply will probably indicate that relation of the speaker to Mr. X, also an object fact but unusually personal. The most obvious reply of the opposite tendency is a simple comment on liking or disliking Mr. X, which one probably does if one knows Mr. X. It is a habit of our society to leave ourselves out of much of what we say. We are always there, but we speak as though we were not. Science and mathematics are the product of this cast of mind, of whole lifetimes of intellectual labour that involve essentially the absence or the elimination of the personal aspect of knowing. Among mathematicians, divided by Poincaré into the geometers and the analysts, the analysts are the more adept at self-effacement.

A part of growing up in North America is to overcome one's tendency to view everything as related to oneself. This change, which we go through as individuals, is related to, indeed recapitulates, the cultural change from myth to history. The flowerchild movement of the sixties, as well as rejecting the establishment, rejected the structured and rational, seeking immediacy and spontaneous relations, the antithesis of the tradition of cartesian knowing. We end up regarding much of the world as unrelated to ourselves, even those parts that *are* related to us. One is scarcely likely to follow an intellectual occupation if one takes the extreme opposite view, but it is possible to regard the cognitive as subsidiary to one's relations to persons and things. Nurses, clergy, and teachers of primary grades probably do a lot more of this than surgeons, computer scientists, and professors. Ideally one should have the whole contents of one's mind at one's disposal, knowing that neither of these two tendencies is appropriate across the board but that each situation requires a mix of the two. Examples of persons interested in the arrangement of things who ignore personal relations in the process are

many and notorious. Examples of the opposite tendency are fewer in my experience and less easy to document. An example that springs to my mind is in a psychiatrist's book, Peck [1978], written to stress the importance of love in psychotherapy. As one might suspect, the author, a person person, has problems with thing-thing relations; he confesses that he had to learn as an adult that simple matters of a mechanical nature, which require the painstaking sorting out of simple objective physical relations, respond only when one takes the time to do the necessary sorting out. The opposite lesson, namely, that objective knowledge does not produce person-person relations, is too frequently learned to require much comment.

I suspect that the culture of Botswana encourages personal relations, in which much that is cognitive is implicit, but to which the cognitive is regarded as incidental. When Berry asked a young Motswana "What is that?" pointing to a dog, the answer he received included the cognitive information that he might have expected, but it was not the main point of the reply, "That is Mofa's dog, from the hut near the well, who chases me when I go to get water." It is almost as though the question had been "Who is that?" asked of a member of the child's family or perhaps "What is that to you?" When I checked my impression that Batswana are sufficiently involved in personal relations to make the cognitive side of their thinking somewhat incidental, John Berry mentioned that entering a shop in Gabarone (the capital of Botswana) involves two natural steps, buying something and greeting the other customers individually. It is no wonder that whites in Africa are regarded as cold and aloof; by comparison with local customs most of us *are* cold and aloof, though to one another as much as to the peoples among whom we go. I should say in defence of what may be regarded as a dangerous generalization that, like John Berry, I have worked at an African university so that my generalizations, while they may be dangerous, are based on first-hand observation. Those few contexts in which I found myself simply a person among persons, rather than being the foreign expert, were overwhelming in their warmth and connectedness. Having taught mathematics with the usual mixture of success and failure in Zambia, I claim that I found no deficiency of ability to learn mathematics, but equally I must confess to having observed no significant interest in learning mathematics. This seems to me to be a matter of grassroots cultural decision about what is important, and not the kind of decision made by the Department of Education.

Corroborating this view is a further difficulty that Berry mentions as being common to third-world mathematics learning, namely that even children that are able to manipulate structured materials with "quite astonishing skill", do not appreciate and apply the abstract concept as is expected by the authors of foreign textbooks. This is no new problem. It is in one sense easier to do pure mathematics than applied mathematics because to do applied mathematics or the corresponding science one must not only work on the abstract relations (the pure mathematics) but must at the same time relate the abstractions to the application or science. Since the mathematics and one's knowledge of the science are both parts of one's window on the world, this operation of relating the abstract to the concrete is done on the window and is obviously done there; there is nothing of oneself in it nor is it directly connected to the world beyond the window. When this discouragement is added to the fact that even for those inured to the process it is difficult, it is little wonder that it causes serious trouble.

Although I am not an advocate of the Sapir-Whorf hypothesis (see Note at end of article) that one's native language determines how one thinks, I believe that one's native language is an indication of how one's culture thinks and a means by which one comes to think likewise. This matters because it means that the problem to which Berry draws our attention is much wider than he says. Rather than being based simply on the "distance" between a third-world language and English, it is based on the multidimensional "distance" between the third-world culture and ours. Berry himself points out that Chinese and English are very distant as languages, and yet Chinese speakers are not prone to the same difficulties as Batswana. Thus the problem can be expected to occur in English-speaking subcultures in so-called developed countries, for instance ghettoized black Americans. This does not mean, however, that I disagree in any way with his proposed solution, to render the whole educational process, from curriculum design up, indigenous. Indeed, I propose an extension of it below.

Conclusion

It may be helpful to mention explicitly a notion that has been influential in my thinking along the above lines. The late Canadian philosopher Bernard Lonergan distinguished four levels of mental activity in his book *Insight*: the level of the senses, the level of understanding, the level of criticism, and the level of action or will. Understanding operates on the data of the senses. Criticism acts on our understandings, most of which are misunderstandings to one degree or another; this level may be absent, of course. Action is based on understanding, whether or not subjected to criticism. Science works in the middle two levels, with experimental forays into the bottom level. It is in this sense that I see the whole cognitive enterprise as subordinated to what persons outside research labs and libraries call the doings of real world. Also in Lonergan's terms, I am proposing an increased exercise of will at the level of understanding; I shall elaborate in the final paragraph.

For teaching effectiveness as for mental health, we must strive for balance, that is, for giving the cognitive pictures and actual relations each their due. In particular, this balance among the next generation will be most effectively achieved if those persons that are not exclusively oriented toward the consideration of relations in detachment are encouraged to study mathematics and subsequently become teachers of it. I find myself arguing for affirmative action not because it will confer a benefit on the affirmed but because it will confer a benefit on the mathematical and larger communities to involve more Batswana and probably more women. The benefit to both communities would be the enlargement, one might say the broadening, of the

mathematical community. This broadening would, at the same time, be a substantial benefit to the individuals involved.

What I am proposing is to attempt breaking the vicious circle of math aversion and math phobia by taking seriously a basis for understanding of such aversion. There are grounds for persuasion based on that understanding (no doubt a misunderstanding) because any understanding at all can lead to sympathy, something that normally those that are adept at mathematics do not have for those that are not. It is perhaps a novel notion to regard an origin for that aversion as respectable. Without such sympathy I see no hope of effective persuasion, and there are not enough of us to get at all the children of primary-school age; it must be done through their teachers. And so I come to a proposal similar to that of Berry: let their own kind teach them. What is needed is to persuade some teachers suffering from math aversion to overcome it (this is where I am thinking of the exercise of the will on understanding) for the sake of enlarging their mental lives, not to mention their teaching effectiveness. No sacrifice of their previous modes of understanding or indeed of anything but limitations is involved. Then those converts would inevitably, if the conversion were successful, spread the good word, not to mention thinking up novel approaches that lifelong mathematical adepts might neither think of nor approve of. This ties in with Berry's proposal of "the development of an alternate mathematics curriculum" adapted to local culture. Most mathematicians are agreed that the notion of mathematics as a meaningless game is a slanderous caricature; nevertheless the beginnings of arithmetic have been approached that way with some success, something that professional mathematicians would neither do nor countenance. Such persuasion as I am suggesting amounts to convincing persons of the possibility and the desirability of *changing their minds* about mathematics. I am not claiming that the preference for relations in themselves with the windshield kept implicit is the result of a rational choice, nor that a rational choice to reverse that preference is likely or even would be effective, only that a choice can be made to operate in both modes. To deny the possibility of this choice is to deny *a priori* a level of human freedom that I regard as fundamental.

Acknowledgement

I acknowledge with gratitude and pleasure the help derived from conversations with John Berry and Jim Romeyn.

Note

B.L. Whorf's quotation from Edward Sapir at the beginning of his paper [1941] was as follows:

> Human beings do not live in the objective world alone, nor alone in the world of social activity as ordinarily understood, but are very much at the mercy of the particular language which has become the medium of expression for their society. It is quite an illusion to imagine that one adjusts to reality essentially without the use of language and that language is merely an incidental means of solving specific problems of communication or reflection. The fact of the matter is that the "real world" is to a large extent unconsciously built up on the language habits of the group . . We see and hear and otherwise experience very largely as we do because the language habits of our community predispose certain choices of interpretation

The hypothesis has come under attack based on psychological research on children too young to be influenced, and on monkeys. See Lumsden and Wilson [1981], Bornstein, Kessen, and Weiskopf [1976], and Sandell, Gross, and Bornstein [1979].

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