REVISITING 0.999… AND $(-8)^{1/3}$ IN SCHOOL MATHEMATICS FROM THE PERSPECTIVE OF THE ALGEBRAIC PERMANENCE PRINCIPLE

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Mathematics is known as a science of forms. As Schwartzman (1994) notes, formalism "is the belief that mathematics consists of 'forms' and symbols that can be manipulated by certain rules" (p. 95) – from which it follows that understanding, investigating, preserving, extending and generalizing forms are essential features of mathematical progress. Our specific focus in this writing has to do with the principle of preserving and extending a form, which is called the "principle of permanence of equivalent forms."

This paper reports an in-depth analysis of the structure of two mathematical curiosities that have provoked interesting discussions in relation to school mathematics: first, recurring decimals in which the digit 9 is repeated infinitely; second, rational exponents of negative bases. The algebraic permanence principle has been an important method of extending numbers and, therefore, might serve as a powerful way of seeing the underlying forms of numbers. Using this principle, then, we reveal the improper structure of these two ideas in school mathematics and discuss how they might be more appropriately framed.

The algebraic permanence principle

The *principle of permanence of equivalent forms* first appeared in the mathematical literature when Peacock published his *Treatise on Algebra* in 1830. He coined the phrase to refer to the process of extending from arithmetic algebra to symbolic algebra whilst preserving the rules of operations. Since then, the principle has been regarded as a powerful notion and has played an important role in the development of mathematics, as illustrated in the emergence of the system of complex numbers (Eves, 1990). Even today, the principle guides similar attempts to extend or generalize a definition.

Freudenthal (1973) used the phrase *the algebraic principle* to refer to the principle of extending algebraic structure whilst preserving basic properties – or more specifically, extending the number system whilst preserving its properties. Later, Freudenthal (1983, p. 435) added the word *permanence*, recalling Hankel's permanence of calculation laws. Applying the principle, numbers are extended. Consider the solutions that satisfy each of these equations:

- $^2/_3$ in the equation $3x = 2$;
- -3 in the equation $x + 3 = 0$;
- $\sqrt{2}$ in the equation $x^2 = 2$;
- $a^{p/q}$ in the equation $x^q = a^p$;
- i in the equation $x^2 = -1$.

Each of these numbers is introduced as x, a new formal subject that is considered to satisfy each equation in the existing arithmetic system. The foundation of this analytic method is to come up with a solution setting the unknown as x, supposing that it satisfies the existing laws of operations.

Dieudonné (1972) demonstrates that the progressive steps of abstraction in the development of algebraic structures involve a separation from concrete particulars (namely, recognition of the indeterminate nature of objects and the significance of their operational rules). He notes:

> In the course of this long effort, we have become more and more clearly aware of the fact that of the two fundamental constituents of every arithmetic – that is to say, on the one hand the objects on which one operates and on the other hand the operational rules – only the latter are really essential. (p. 112)

Dieudonné is describing the process of becoming aware of the structure of operations, of which the foundation is the principle of the permanence of equivalent forms while preserving the rules of operations. MacLane and Birkhoff (1967) add:

> An algebraic system … is thus a set of elements of any sort on which functions such as addition and multiplication operate, provided only that these operations satisfy certain basic rules. (p. 1)

Here we see that the laws of operations are the essential forms of algebraic thinking. Moreover, the permanence principle reaches beyond the realm of algebraic operation. Freudenthal (1983) expands it to the geometric-algebraic permanence principle that preserves geometric properties such as rectilinearity of the half-lines on the Cartesian plane when numbers are extended to negative numbers.

The algebraic permanence principle also plays a role in secondary school mathematics. For example, when we explain the property of equality, such as "if $a = b$ then $a + c = b + c$," a pair of scales is commonly used as a model. Since the objects have positive weight, the values of a, b and c in the equality should be limited to positive numbers. However, the property of equality is soon assumed to be valid for all the numbers, including negatives. Here, the algebraic permanence principle is applied implicitly.

There are many basic forms in school mathematics. First of all, the arithmetic-algebraic operation laws – such as the associative law, commutative law and distributive law – are fundamental in computation and solving equations. The division algorithm and exponential laws, which are the bases of decimal notation and exponential extension respectively, are other examples of basic forms in school mathematics that can be compared to axioms in mathematics. Care needs to be taken when mathematical content that contradicts established basic forms is introduced into school mathematics.

The improper structure of 0.999... in school mathematics

There are many educational discussions about 0.999…, which is treated as one of the recurring decimals. For example, many researchers have noted that students have difficulty in understanding 0.999… = 1 (Cornu, 1991; Sierpinska, 1994; Edward & Ward, 2004). From the perspective of actual infinity, there is no difference between 0.123412341234… = $^{1234}/_{9999}$ and 0.999… = 1. From the perspective of potential infinity, 0.999… gets closer and closer to 1, yet is never equal to 1. It is also the case that 0.123412341234… gets closer and closer to $^{1234}/_{9999}$, yet is never equal to $^{1234}/_{9999}$. According to Sierpinska (1994), however, there are students who accept the argument for expansions such as 0.123412341234… = $^{1234}/_{9999}$, but refuse to accept that 0.999… = 1, even though it is obtained in an analogous way. The fact that there are such students implies that 0.999… may have a unique feature that differentiates it from other recurring decimals, and that this feature may prompt the difficulties involved in students' understanding of 0.999… = 1.

The division algorithm and the standard numeration system in school mathematics

The division algorithm arises in school mathematics in the context of teaching natural numbers, taking the cases where the remainder is 0 (*e.g.*, 8 ÷ 2 = 4) as a starting point and gradually investigating the cases where the remainder is not 0 (*e.g.*, 17 ÷ 5 = 3 *R*2). The algorithm is a process of finding a unique pair of integers, q and r, such that $a = bq + r$ $(0 \le r < b)$. Division with integers is followed by cases of division in which the quotient is not an integer (*e.g.*, 27 ÷ 4 = 6.75). The division algorithm is the basis of decimal expression. In general, based on the division algorithm, a rational number $^{a}/_{b}$ (≥ 0) has the unique decimal expression $^{a}/_{b} = m.a_1a_2a_3\ldots$. The process of finding the decimal expression of a rational number based on the above algorithm can be explained using intervals on a number line. For example, for an arbitrary non-negative rational number x, the unique decimal expression $m.a_1a_2a_3\ldots$ can be obtained by finding $m \in N \cup \{0\}$ that satisfies $x \in [m, m+1)$, cutting up the interval $[m, m+1)$ into ten smaller intervals of equal length to find $a_1 \in \{0, 1, 2, \ldots, 9\}$ that satisfies

$$x \in \left[m + \frac{a_1}{10},\ m + \frac{a_1 + 1}{10}\right)$$

again dividing the interval

$$\left[m + \frac{a_1}{10},\ m + \frac{a_1 + 1}{10}\right)$$

into ten smaller intervals of equal length to find $a_2 \in \{0, 1, 2, \ldots, 9\}$ that satisfies

$$x \in \left[m + \frac{a_1}{10} + \frac{a_2}{10^2},\ m + \frac{a_1}{10} + \frac{a_2 + 1}{10^2}\right)$$

and repeating the process of dividing the interval to find a_i. This procedure can be regarded as the standard decimal numeration system in school mathematics, in that it is based on the division algorithm – one of the core forms to be maintained in the contents of school mathematics.

The decimal expression of school mathematics is based on the numeration system that expresses non-negative real numbers by dividing the number line $[0,\infty)$ into closed-open intervals. In the course of repeating the above process, a rational number $^{a}/_{b}$, where a and b are relatively prime and in which the denominator is a multiple of 2 or 5, will be positioned on a dividing point – and so is often expressed as a finite decimal (*e.g.*, $^{3}/_{4}$ = 0.75000… = 0.75). According to this system, since m, which is decided at the first step of finding the decimal expression of number 1, is not 0 but 1, 1 can be expressed as 1.000… but cannot be expressed as 0.999… . To put it another way, when following the standard numeration system that forms the basis of decimal expression in school mathematics, the expression of the infinite decimal in which 9 is repeated (*i.e.*, 0.999…) cannot appear.

In school mathematics 0.999… appears in two contexts: first, in the part that deals with the relations between recurring decimals and rational numbers; second, in the application of infinite series.

0.999… as a recurring decimal in school mathematics

Recurring decimals are introduced in the context of division, such as 3 ÷ 11 = 0.2727…, 2 ÷ 3 = 0.666… . It is common that the existence of recurring decimals of which the repetend is 9 (*e.g.*, 0.999…) is assumed without any reflection (although 0.999… cannot be obtained by a division), and 0.999… = 1 is demonstrated as follows:

$x = 0.999\ldots$

Multiply both sides by 10: $10x = 9.999\ldots$

Subtract the first equality from the second: $9x = 9$

$\therefore x = {}^{9}/_{9} = 1$

If 1 is represented as 0.999… and 0.5 as 0.4999…, by accepting recurring decimals in which 9 is repeated, the relations between rational numbers and recurring decimals can be simply stated: All rational numbers can be expressed as recurring decimals, and vice versa. However, this relation can be preserved without accepting 0.999… as a valid numeral. Some real numbers, such as integers and finite decimals, have two infinite representations – in one form 0 is repeated, and in the other form 9 is repeated (Bartle, 1976; Wade, 1995). It is proper in school mathematics to choose 1.000…, which preserves the forms of the standard numeration system – a basic form of school mathematics in the light of the algebraic permanence principle.

One might argue that the existence of 0.999… can be accepted by students through the generalization of $^1/_9$ = 0.111… and $^2/_9$ = 0.222… or calculations such as 0.232323… + 0.767676… = 0.999… or 0.333… × 3 = 0.999… . Even so, dealing with the existence of recurring decimals in which 9 is repeated without reflecting on the relation between recurring decimals and standard decimal numeration system does not seem to be completely intellectually honest.

Expressing rational numbers as the infinite decimal in which 9 is repeated is based on forms that are different from the standard numeration system and division algorithm of school mathematics. Though the standard numeration system uses closed-open intervals, this numeration system keeps dividing the number line $(0,\infty)$ into open-closed intervals. In this system, for an arbitrary positive rational number x, the decimal expression $x = m.a_1a_2a_3\ldots$ can be obtained by finding $m \in N \cup \{0\}$ that satisfies $x \in (m, m+1]$, cutting up the interval $(m, m+1]$ into ten smaller intervals of equal length to find $a_1 \in \{0, 1, 2, \ldots, 9\}$ that satisfies

$$x \in \left(m + \frac{a_1}{10}, m + \frac{a_1 + 1}{10}\right]$$

again dividing the interval

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into ten smaller intervals of equal length to find $a_2 \in \{0, 1, 2, \ldots, 9\}$ that satisfies

$$x \in \left(m + \frac{a_1}{10} + \frac{a_2}{10^2}, m + \frac{a_1}{10} + \frac{a_2 + 1}{10^2}\right]$$

and repeating the process of dividing the interval to find a_i.

Equations such as 0.999… = 1 and 0.74999… = 0.75 just mean that a decimal expression obtained from the division of closed-open intervals and a decimal expression obtained from open-closed intervals are different expressions of the same thing. Accepting 0.999… as a decimal expression in school mathematics means accepting the numeration system that adopts the division into open-closed intervals in school mathematics.

The numeration system based on the division into open-closed intervals is not a standard numeration system used in school mathematics – and it has defects. Above all, 0 cannot be expressed with this numeration system. The fact that there is a number that cannot be expressed with a numeration system means that the system has a serious deficiency, since a numeration system is for expressing numbers. And notation such as 0.999… does not preserve the one-to-one correspondence between real numbers and infinite decimals. In addition, since the expressions of integers or finite decimals such as 2 or 0.75 cannot be displayed with this system, the beautiful structure made by embedding relations between integers, rational numbers, and real numbers cannot be maintained. All of these results come from notation that goes against the algebraic permanence principle.

The above non-standard numeration system is based on the following property (Artmann, 1988): for any $a \in Z$ and $b \in N$, there exists a unique pair of integers q and r such that $a = qb + r$ $(0 < r \leq b)$. If we follow this different division algorithm, 3 ÷ 3 is calculated as a division of which the quotient is 0 and the remainder is 3, the quotient is 9 and the remainder is 3 again, and so on. This calculation, which excludes those cases where the remainder is 0, is completely different from the traditional division algorithm adopted in school mathematics. Consequently, accepting 0.999… prevents us from maintaining the standard division algorithm in school mathematics. The method of expressing decimals by excluding those cases where the remainder is 0 is therefore not sound within school mathematics.

Finally, recurring decimals in which 9 is repeated are against the method of comparing the size of decimals, currently used in school mathematics, that was induced from the method of comparing the size of natural numbers. When comparing the sizes of two natural numbers, the place values of each number expressed in accordance with the standard numeration system are compared from the front. Extending this strategy, the relative sizes of two decimal numbers can be determined by comparing the digits of each number, starting from the front.

0.999… as an expression of the sum of infinite series in school mathematics

Following Wade (1995), an infinite decimal can be interpreted as an infinite series such as

$$0.a_1a_2a_3\cdots = \sum_{k=1}^{\infty} \frac{a_k}{10^k}$$

According to this interpretation, 0.999… is the infinite series 0.9 + 0.09 + 0.009 + … . By using the formula

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad (-1 < r < 1)$$

it can be shown that the sum of infinite series is 1. This means that the sequence of partial sums of the series <0.9, 0.99, 0.999, …> converges to 1.

Here, we need to note that 0.999… is an expression of the infinite series 0.9 + 0.09 + 0.009 + … itself, as

$$\sum_{k=1}^{\infty} a_k$$

is an expression of the infinite series $a_1 + a_2 + a_3 + \ldots$ itself. This is different from acknowledging 0.999… as a standard decimal expression like 1, 0.75, and 0.333…, which are expressions in accordance with the standard numeration system. This difference needs to be taken into account when 0.999… is introduced in the context of infinite series in school mathematics.

The improper structure of $(-8)^{1/3}$ in school mathematics

Even and Tirosh (1995), while investigating the conceptions of Israeli secondary mathematics teachers of mathematical operations such as $^4/_0$ and $(-8)^{1/3}$, imply that $(-8)^{1/3}$ is undefined. Goel and Robillard (1997) argue that the assertion of Even and Tirosh is flawed since $(-8)^{1/3}$ is well defined as -2. Tirosh and Even (1997) present the counter-argument that, although it is possible to define $(-8)^{1/3} = -2$ and start many of the theorems related to powers with the preface 'the base is positive', this approach is unnecessarily complicated. More recently, Choi and Do (2005) argue that, since $(-8)^{1/3}$ is the solution of the equation $x^3 + 8 = 0$, it has to be understood in the complex number system. Choi and Do criticize the arguments of Even and Tirosh (1995, 1997) and Goel and Robillard (1997) as incomplete and inadequate in that they seek a solution in the real number system. For all these insightful arguments on the definition of $(-8)^{1/3}$, it does not seem to have been analysed and discussed in relation to the structure of school mathematics.

In school mathematics, the exponent is first introduced as a natural number exponent. The exponential laws are then induced and, with a view to maintaining the laws, the exponent is extended through the application of the algebraic permanence principle. Considering this process, the use of rational exponents with negative bases, which do not preserve the exponential law, a basic form in exponential extension, is not proper in the structure of school mathematics.

Exponential extension in accordance with the algebraic permanence principle

A natural exponent can be understood as repeated multiplication, which is assumed in the recursive definition of the natural exponent, $a^1 = a$, $a^{n+1} = a^n \times a$. However, a student who views an exponent as repeated multiplication might regard 2^{-1} and $2^{1/2}$ as nonsensical (Weber, 2002). Though the meaning of repeated multiplication that is contained in the form $a^{n+1} = a^n \times a$ cannot be extended to 0 or negative integer exponents, the form itself can be extended.

Generally, if the natural exponential law $a^m \cdot a^n = a^{m+n}$ also holds for $m = 0$ and $n = 1$, $a^0 = 1$ comes out by dividing $a^0 \cdot a^1 = a^{0+1} = a$ by a. And again, if the above exponential law still holds for $m = -n$ $(n > 0)$,

$$a^{-n} = \frac{1}{a^n}$$

comes out by dividing $a^{-n} \cdot a^n = a^{-n+n} = a^0 = 1$ by a^n. Since the divisor must not be zero, it cannot be guaranteed that $a^0 = 1$ and $a^{-n} = 1/a^n$ holds for $a = 0$. In fact, 0^{-n} seems to be undefinable from $a^{-n} = 1/a^n$, and 0^0 seems to be indeterminate from $a^0 \cdot a^1 = a^{0+1} = a$. Therefore, it is proper to exclude the cases where the base is 0 in the process of extension based on the algebraic permanence principle that preserves the exponential law.

Now, if the exponential law $(a^m)^n = a^{mn}$ holds also in rational exponents, $a^{1/n}$ can be defined using

$$(a^{\frac{1}{n}})^n = a^{\frac{1}{n}\cdot n} = a$$

as the number whose n-power is a. Furthermore, from

$$(a^{\frac{m}{n}})^n = a^{\frac{m}{n}\cdot n} = a^m$$

$a^{m/n}$ can be defined as the number whose n-power is a^m. Let us call it the definition (i) of $a^{m/n}$. Or, from

$$(a^{\frac{1}{n}})^m = a^{\frac{m}{n}}$$

$a^{m/n}$ can be defined as the m-power of $a^{1/n}$. Let us call it the definition (ii) of $a^{m/n}$. If the natural exponential law $(a^m)^n = a^{mn}$ holds also in rational exponents, the two definitions should represent the same object, since

$$(a^{\frac{1}{n}})^m = a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}}$$

Although in many cases the algebraic permanence principle produces productive results, it does not necessarily follow that the results obtained from applying the principle are correct (Arcavi, Bruckheimer, & Ben-Zvi, 1982). The two definitions above, obtained by applying the algebraic permanence principle to $(a^m)^n = a^{mn}$, are candidates for which validity needs to be confirmed.

The definition of $a^{1/n}$

Let us consider the validity of the definition of $a^{1/n}$, which is claimed to be the number whose n-power is a. In the set of real numbers, $a^{1/n}$ cannot be defined when n is an even number and a is a negative number. If n is an even number and a is a positive number, there are two of values of $a^{1/n}$ (*e.g.*, $16^{1/4} = \pm 2$). Of these two, let us represent 2 as $\sqrt[4]{16}$ and -2 as $-\sqrt[4]{16}$. However, if $a^{1/n}$ is set to represent the two values $\pm\sqrt[n]{a}$ at the same time, it goes against the exponential law $(a^m)^n = a^{mn}$, because and

$$4^{\frac{1}{2}\cdot 2} = 4^1 = 4$$

It is, therefore, necessary to make an adjustment so that the

$$(4^{\frac{1}{2}})^2 = 4^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} = \pm 4$$

exponential law, which forms the basis of the candidate definitions, is not violated. One possible adjustment is to define $a^{1/n}$ as either $\sqrt[n]{a}$ or $-\sqrt[n]{a}$, but not both. That is, $a^{1/n}$ can be defined as $\sqrt[n]{a}$ (not as $-\sqrt[n]{a}$) to be able to include $a^1 = a$. In the case where n is an odd number, only one number that becomes a by multiplying n times exists in the set of real numbers, for example, $8^{1/3} = 2$ and $(-8)^{1/3} = -2$. These can be represented as $2 = \sqrt[3]{8}$ and $-2 = \sqrt[3]{-8}$.

After going through the above process, $a^{1/n}$ can be tentatively defined as follows:

If $a > 0$, $a^{1/n} = \sqrt[n]{a}$.

If $a < 0$ and n is an even number, $a^{1/n}$ does not exist.

If $a < 0$ and n is an odd number, $a^{1/n} = \sqrt[n]{a}$.

According to this, we can define $(-8)^{1/3}$ as -2. However, in the educational process, as in the history of science, knowledge accepted as true at one point in time can be revealed as false at a later point in time. The same applies to $(-8)^{1/3} = -2$. This matter will be dealt with later.

The definition of $a^{m/n}$

In the case where the base is a positive number, (i) $(a^m)^{1/n}$ and (ii) $(a^{1/n})^m$, the candidate definitions of $a^{m/n}$ can be well defined respectively and the values that the two represent are the same. If the base is a negative number, however, the situation is different. According to definition (ii), $(-8)^{2/6}$ is $((-8)^{1/6})^2$. However, in the earlier definition of $a^{1/n}$, $(-8)^{1/6}$ is undefined. Therefore, this candidate definition is improper when applied to a negative base. According to definition (i), $(-8)^{2/6}$ is $((-8)^2)^{1/6}$, or in other words, $\sqrt[6]{(-8)^2} = 2$. However, since the definition $(-8)^{1/3} = -2$ was made earlier, the problem $(-8)^{1/3} \neq (-8)^{2/6}$ occurs. If considered in the context of extending exponents based on the exponential law $(a^m)^n = a^{mn}$, this problem is a serious one since

$$(-8)^{\frac{1}{3}} = ((-8)^{\frac{1}{3}})^{\frac{2}{2}} = (-8)^{\frac{1\times2}{3\times2}} = (-8)^{\frac{2}{6}}$$

according to the law $(a^m)^n = a^{mn}$. This means that a definition that was made based on the exponential law becomes inconsistent with the law that forms its own basis. The candidate definition (i) is therefore also judged to be improper when generally applied to a negative base.

Back to the definition $(-8)^{1/3} = -2$

This situation, where an ordinary rational exponent $a^{m/n}$ cannot be well-defined when the base is a negative number, raises a question about the definition of $(-8)^{1/3}$, which earlier was regarded as proper. The statement

$$(-8)^{\frac{1}{3}} = ((-8)^{\frac{1}{3}})^{\frac{2}{2}} = (-8)^{\frac{1\times2}{3\times2}} = (-8)^{\frac{2}{6}}$$

creates a contradiction that $(-8)^{1/3}$, a term that can be defined equals $(-8)^{2/6}$, a term that cannot be defined. Because the definition of $(-8)^{1/3}$ also violates the exponential law that forms its foundation, it is revealed that defining $(-8)^{1/3}$ is improper. Consequently, the definition of rational exponents, which was extended according to the algebraic permanence principle, collides with the exponential law on which it is based when the base is a negative number. Accordingly, in this context of extension in school mathematics, it is proper not to make a definition in the case where the base is a negative number.

Extension into real and complex exponent

Limiting the base to positive numbers in rational exponents is also proper when considering the extension to real exponents, the final step of exponential extension in secondary school mathematics. By limiting the base to positive numbers, the possibility of extending exponents to irrational numbers can be accepted intuitively. For example, $3^{\sqrt{2}}$ can be defined as a limit of the sequence $3^1, 3^{1.4}, 3^{1.41}, 3^{1.414}, 3^{1.4142}, \ldots$, which has each term of the rational number sequence 1, 1.4, 1.41, 1.414, 1.4142, … that infinitely approaches $\sqrt{2}$ as an exponent. In this way, for every real number x, one real number a^x $(a > 0)$ is well defined; exponential functions become continuous functions and various kinds of exponential laws hold.

Extending the base and exponent to complex numbers transcends the scope of school mathematics – which is to say, it cannot be accomplished through the application of the algebraic permanence principle to exponential laws, as it is typically developed. However, from

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

we can define e^z as $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$. Using this and the logarithmic function, $z^\alpha = e^{\alpha[\mathrm{Log}\, r + i(\theta+2n\pi)]}$ is defined for any complex number $z = re^{i\theta} (\neq 0)$ and α (Silverman, 1975). For $\alpha = {}^m\!/_k$,

$$z^{\frac{m}{k}} = r^{\frac{m}{k}} \cdot e^{i\frac{m}{k}(\theta+2n\pi)}$$

is obtained from

$$z^\alpha = z^{\frac{m}{k}} = e^{\frac{m}{k}\log z} = e^{\frac{m}{k}[\mathrm{Log}\, r + i(\theta+2n\pi)]}$$

Hence, we obtain $(-8)^{1/3} = (-8)^{2/6}$. Since $(-8)^{1/3} = (-8)^{2/6}$, this definition satisfies the requirement that "a definition should not depend on the representatives of the numbers involved in the operation" (Borasi, 1992; Even & Tirosh, 1995). But in applying this definition, exponential laws become untenable because $(-8)^{1/3} = (-8)^{2/6} \neq ((-8)^2)^{1/6}$. The application of exponential laws are restricted if the base and exponent are extended to complex numbers. Generally, $z^\alpha \cdot z^\beta$ and $z^{\alpha+\beta}$ are not the same; neither are $z^{\alpha\beta}$ and $(z^\alpha)^\beta$. The exponential law $(z^m)^{1/n} = (z^{1/n})^m$ holds when n and m are relative primes (Silverman, 1975).

Because the extension of complex exponents is not dependent on exponential laws, unlike the extensions up until rational exponents, definitions do not have to be abandoned on the grounds that they do not satisfy exponential laws. In the extension process from natural exponents to rational or real exponents in school mathematics, exponential laws served as a foundation that made the extension possible. In complex exponents, exponential laws fail to occupy the basic position; they are put in a position where their application is limited within the extended system. As the idea of forsaking the form of the commutative law of multiplication served as the motive for the creation of the quaternion (Eves, 1990), the idea of abandoning forms in the existing system is significant.

A mathematical concept can be evaluated differently according to the level of the point of evaluation. $(-8)^{1/3}$ might be well-defined in the level of complex exponents, but defining it in the knowledge structure of secondary school mathematics, in which exponents are extended by applying the algebraic permanence principle to exponential laws, is improper.

Concluding remarks

In this paper, we have analysed recurring decimals in which 9 is repeated infinitely and rational exponents with negative bases from the viewpoint of the algebraic permanence principle. A recurring decimal like 0.999… for which the repetend is 9 is an improper expression that is not consistent with the division algorithm and standard numeration system that are the basic forms of decimal notation in school mathematics. Rational exponents with negative bases are also improper in the structure of school mathematics, where exponents are extended applying the algebraic permanence principle to the exponential laws.

The algebraic permanence principle is, as Freudenthal (1973) emphasizes, a natural requirement in extending numbers and a didactically sound principle for this extension. As we argue in this paper, it is also a powerful concept with which to understand the structure of numbers in school mathematics. There may be a range of different standpoints on the content of school mathematics and on the way it is approached. However, the basic forms and principles underlying the structure of school mathematics should be consistent and firm.

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