Office for Standards in Education (Ofsted). In my conversations with experienced Ofsted inspectors, it is whether decisions about lesson structure are defended using a clear and well-argued sense of pedagogy that is valued rather than whether one particular form (*e.g.*, the three-part lesson) is being rigorously followed So, if a lesson is observed that takes on a different form to the three-part lesson, then the issue is not so much that it is non-compliant but whether there is a story for why the lesson was constructed how it was. As far as I am concerned, I would like to see whether such a story exists for three-part lessons as well (other than "I was following the NNS", which for me is no story at all).

Local considerations

Within any lesson form there are local considerations. These local considerations are often temporal in nature, in that they concern a particular moment in time or part of a lesson. Two such examples are (a) the use of stressing and ignoring; and (b) the use of time.

I suggest that these two are techniques that all teachers use. The only issue is the level of awareness in the way that they are being used

Stressing and ignoring.

Stressing and ignoring is something we all do every moment of every day. At any one moment in time there is so much available information for us to attend to through all our senses, we cannot attend to everything. We make choices – for example, we turn our eyes in particular directions and focus on particular things By doing this, certain visual opportunities are stressed and as a consequence others are ignored. Furthermore, what we learn during our day is partly determined by those decisions as to what we stressed and what we ignored Two students can be sitting in the same lesson and end up learning completely different things during that lesson due partly to the fact that they chose to attend to different things during that lesson So, where a student's attention is placed at any particular moment in time is an important consideration for a teacher It is not just a matter of whether the student is attending to me, the teacher, and not talking to their friend, but it matters which words and phrases are noted by students and how they relate to each student's existing awareness of mathematics For example, when working on adding fractions there may be a time when I would like my students to attend to the numerators within fractions. To use their knowledge of one plus two equals three to say that one fifth plus two fifths equals three fifths. For example, use might be made of voice in order to stress certain aspects: *one plus two equals three*; *one* chair *plus two* chairs *equals three* chairs; *one* fifth *plus two* fifths *equals three* fifths.

At other times, I would like students to attend to the denominator, such as when being asked to add fractions with different denominators. For example: one *chair* plus two *chairs* equals three *chairs*; one *chair* plus two *pencils* equals ... hmmm, a problem; one *fifth* plus two *thirds* equals ... hmmm, a problem. Thus, where I, as teacher, would like a student's attention to be placed is subject specific, and so the techniques I use to try to stress certain aspects and ignore others have the potential to be subject-specific teaching methods. I state "potential" in the last sentence deliberately, as, again, I feel it is a matter of the awareness of a teacher as to their deliberate use as to whether something is a teaching technique or whether it is a habit. For example, a teacher when talking with friends might have a habit of speaking in a certain way whereby, for example, verbs tend to get emphasised. If this habit is continued within a mathematics lesson, then I do not consider such stressings to be part of a pedagogic technique At some level of awareness the particular stressings need to relate to both the subject matter being taught and the student, if the use of stressing is to be considered a pedagogic technique.

So, some questions relevant to consider are:

1. When speaking to a class about a mathematical topic, are there words which are stressed? Which words and why?
2. When a drawing appears on the board are there parts of the drawing to which you point?
   i. When did you point to a particular part?
   ii. Is the timing important?
   iii. Might you point to another part of the same drawing at a different time?
   iv. Was it the mathematics at that moment in time that affected where you pointed? Or was it due to a particular student's comment? Or was it a mixture of both?
3. What might a lesson plan look like which attends to the issue of what is to be stressed and what is to be ignored at different moments in a lesson?

Timing

The issue of timing is also significant for someone's learning, although I will not go into much detail here. I offer two examples where time is significant. I suggest that, as a learner, I do not prove anything unless I am given time to consider carefully. I also suggest that I would not have learnt my multiplication tables very well unless some form of time pressure was placed upon me So, as a teacher, when I give my students time and when I choose to restrict my students' time is a pedagogical issue, and the deliberate use of time whilst working with students can be another example of a local subject-specific teaching method.

Dynamic representation, connections and meaning in mathematical problem solving

MANUEL SANTOS-TRIGO

A comment after reading 'Developing and using symbol sense in mathematics', Arcavi, 25(2): Making sense of mathematical expressions and results and looking for meaning while comprehending and solving problems seem to be crucial aspects that permeate all mathematics activities Polya

(1945) suggests that students need to make sense of the information given in a problem as a way to understand it. Similarly, designing and carrying out a solution plan makes demands on students examining and making sense of the mathematical resources and strategies needed to approach the problem. In this process, students need to interpret and make sense of distinct mathematical operations that lead them to solve the problem. At this stage, they need to revise, in terms of meaning, the response or solution given to the problem.

How can students be aware that making sense of mathematical ideas, relationships, problem solutions or concepts is a crucial activity in understanding mathematical content and solving problems? What tools do students need to use in order to develop and appreciate the relevance of searching for meaning or making sense of data and results during their mathematics learning? To what extent does the use of technology help students construct problem representations to explore meaning and sense of mathematical relationships? These types of questions are part of the research agenda in mathematics education and have been central in designing and implementing distinct research programs (Schoenfeld, 1985; Hiebert *et al.*, 1997; NCTM, 2000; Guin, Ruthven and Trouche, 2005).

Arcavi (2005) identifies and discusses important elements that characterize and can help structure and develop students' competence in symbol sense He focuses on identifying particular students' mathematical behaviours and dispositions that highlight the importance for students to develop and appreciate symbol sense to deal with situations or problems:

> [. B]eing competent in school algebra would imply, among other things, the opportunistic, flexible back and forth transition from the use of meaningless actions (such as the automatic application of rules and procedures) to sense making In other words, competence would include, [.] the interruption of an automatic routine in order to question, reflect, conclude, relate ideas or create new meaning (Arcavi, 2005, p. 45).

The aspects identified by Arcavi include: friendliness with symbols; reading and manipulation of symbolic expressions; construction of symbolic expressions for a desired graph; selection of proper symbolic representation, checking for symbolic meaning and the recognition of distinct roles in using symbols in different contexts. Based on this list, teachers can think of problems or learning activities in which students have the opportunity to reflect on and value the importance of developing symbol sense in mathematics.

Arcavi's contribution sheds light on ways to structure a framework that enhances mathematical thinking and the search for meaning beyond symbol sense. However, little attention is paid, at least explicitly, to the role played by the use of technology in developing a disposition, resources and problem-solving strategies associated with that way of thinking Hiebert *et al* (1997) state that:

> Not only is technology making some conventional skills obsolete - such as high levels of speed and efficiency with paper-and-pencil calculation - it is also underscoring the importance of learning new and flexible ways of thinking mathematically (p. 1)

In this context, an example, taken from Arcavi's article, is used to illustrate that the use of technology becomes important to generate a dynamic representation of the problem in which students can pose and pursue questions that might lead them to identify and examine connections and extensions of that problem. In this process, it is argued that representing the problem dynamically might offer students the opportunity to be engaged in a line of thinking in which they constantly need to search for mathematical relationships and utilize the knowledge they possess in order to explore and support the pertinence and validity of those relationships.

> *The problem*: Q is a point on the graph of the function $f(x) = 1/x$ (in the first quadrant). A tangent line to the graph through Q creates (with the axes) a right-angled triangle. What should be the coordinates of Q in order for the hypotenuse of that triangle to be maximum/minimum? (Arcavi, 2005, p 44).

Students may initially think of ways to represent the function $f(x) = 1/x$ graphically. What is the domain of $f(x)$? How can we represent and relate elements of the domain with their corresponding function values? For discussing these questions it becomes important to construct a dynamic representation of the problem. Thus, using dynamic geometry software (Cabri-Geometry), students can rely on the Cartesian system, situating point P on the x-axis and locating the corresponding value $1/x$ on the y-axis to determine point Q (Figure 1).

At the outset, the use of the software demands that students introduce series of symbols that will eventually be important to distinguish properties of objects and to communicate mathematical relationships. The y-coordinate of point Q is defined in terms of the x-coordinate of point P. What is the locus of point Q when point P is moved along the x-axis? The software becomes a powerful tool to determine that locus (Figure 2).

Initial observations: Students may notice that when point P is moved along the x-axis the inclination of line PR varies (Figure 3). How can we measure that line inclination? How can we calculate the slope of a line? What is the relationship between the slope of a line and the derivative of a function? How can we draw a tangent line to $f(x)$ at point Q? Here, students can observe that, for any position of $P(x, 0)$ then the position of R will be $(0, 1/x)$, that is, the slope of line PR will be $-1/(x^2)$. This slope corresponds to the tangent line to $f(x) = 1/x$ at point $Q(x, 1/x)$. This is because $f'(x) = -1/(x^2)$ (geometric interpretation of the derivative). Thus, to draw a tangent line to $f(x)$ at point Q, it is enough to draw a parallel line to PR passing through point Q (Figure 3).

Looking for mathematical relationships: This dynamic representation of the problem may become an inspiration for students to identify and examine particular relationships. Are there any relationships among points O, R, and S (Figure 4)? By using congruence criteria, students can show that triangles RQS, OPR and PTQ are congruent and therefore R and P are midpoints of segments SO and OT respectively. With this information, it holds that ST = 2RP. There is then the opportunity to use a theorem previously studied: A mid-segment of a triangle is parallel to the third side and one half the length of the third side.

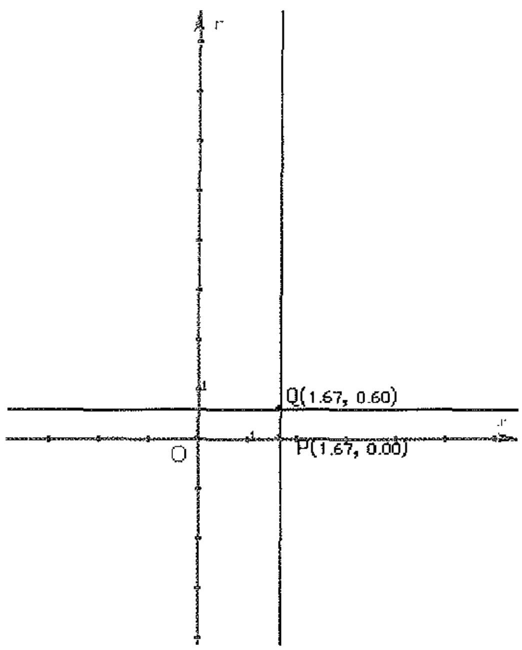

Figure 1: representing Q with coordinates (x, 1/x)

Examining variation graphically: The recognition that for any position of point Q the length of segment ST is twice the length of segment RP might lead students to conclude that to explore the variation of segment ST is equivalent to analysing the variation of segment PR when point P is moved along the x-axis. With the use of the software, students can represent the relationship between the position of point P and the corresponding value of the hypotenuse PR graphically (Figure 5).

By moving point P along the x-axis, at one point the rectangle OPQR becomes square and it is at that position of P

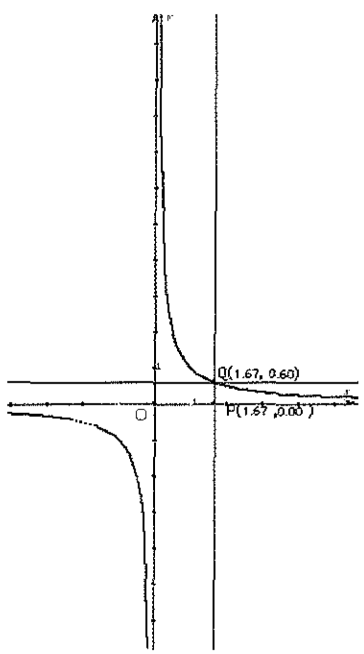

Figure 2: Locus of point Q when point P is moved along the x-axis.

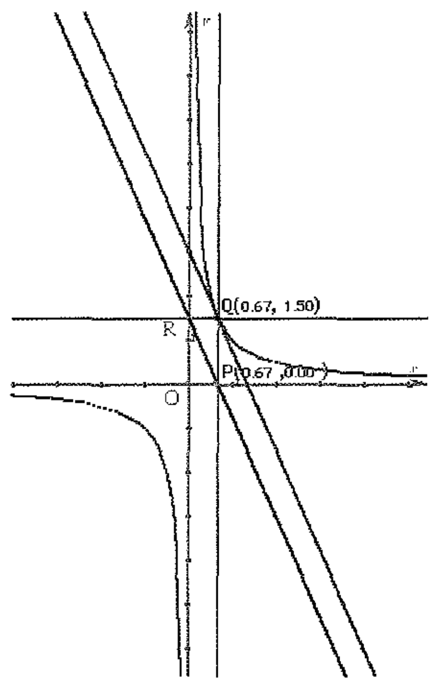

Figure 3: Tangent line to F passing through Q is parallel to line PR

(which determines the position of Q) when the length of hypotenuse PR reaches its minimum length (Figure 6). That is, at $Q(1, 1)$ the length of ST is minimum.

Connections: Students can also notice that for any position of point Q, the area of the rectangle OPQR is always one squared unit. This can be justified algebraically by observing that the dimensions of rectangle OPQR are x and $1/x$ respectively. Therefore, the area of rectangle OPQR is $(x)(1/x) = 1$. As a consequence, the area of rectangle OTUS is always 4 squared units for any position of point Q.

An interesting result emerges here:

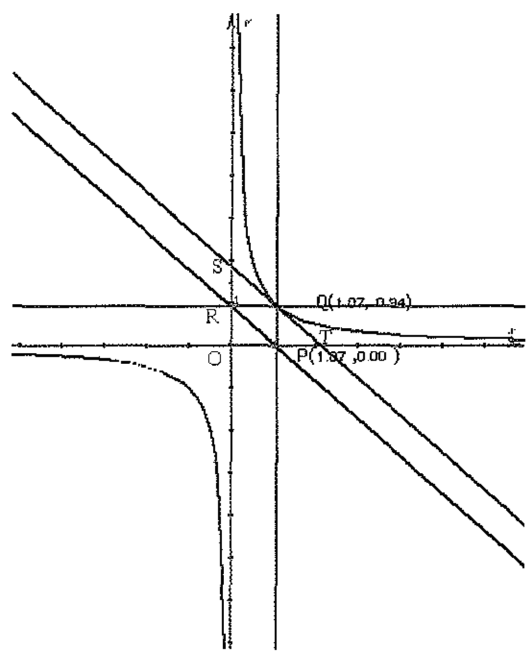

Figure 4: Searching for mathematical relationships within this representation.

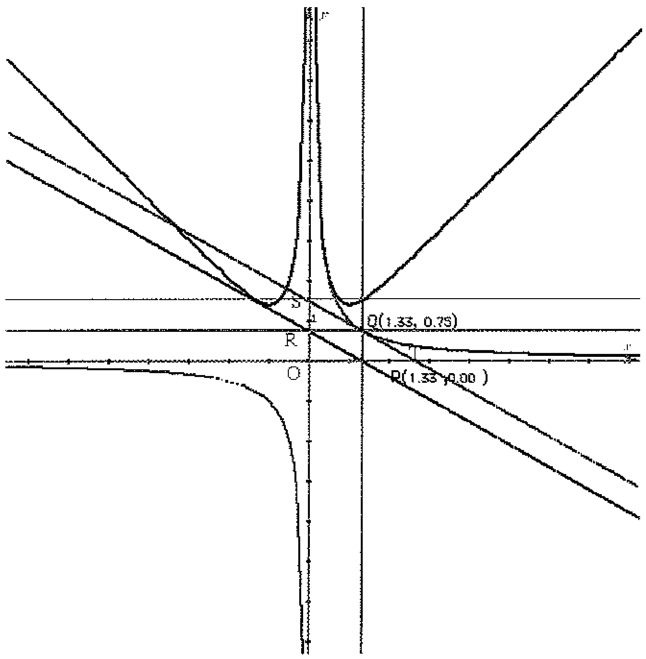

Figure 5: Graphic representation of the variation of PR when P is moved along the x-axis

> Q is a point on the graph of the function $f(x) = 1/x$ (in the first quadrant). A tangent line to the graph through Q creates (with the axes) a right-angled triangle For each position of Q, a right triangle is generated and the area of each of those triangles is always two squared units

What happens to the area of those triangles when we consider $f(x) = n/x$ for $n = 2, 3, \ldots$? Again, the use of the software allows students to examine some cases and observe a pattern in the values of the corresponding areas (Figure 8 shows the value of the area of the triangle for $f(x) = 3/x$).

Thus, students can conjecture that the area of the right triangle is twice the area of the rectangle formed by drawing perpendicular lines from point Q to both axes. This conjecture can be proved by observing that the coordinates of point Q on $f(x) = n/x$) will be $Q(x), n/x$) and the corresponding area of the rectangle formed by drawing perpendicular lines to both axes is $x(n/x) = n$ Again, the use of the software provides useful information for students to generalize the result.

Remarks: Making sense and looking for meaning in relationships makes students conceptualise their mathematical learning as an inquiry process in which they constantly pose questions and identify dilemmas that need to be resolved in terms of mathematical resources In particular, questions like: What is the unknown? What are the data? Can you draw a picture? What is the condition? Do you know a related problem? Could you solve a part of the problem? Can you check the result? Have you taken into account all essential notions involved in the problem? Can you derive the result differently? Or can you use the result, or the method, for some other problem? (Polya, 1945), associated with the distinct problem solving phases, can be addressed from diverse angles or perspectives when students use dynamic software. For instance, representing mathematical objects dynamically provides students with the opportunity of searching for particular relationships and looking for arguments to support and communicate their results. In this process, the use of the software (Cabri-Geometry) becomes an important tool to represent and examine mathematical objects and their relations in terms of mathematical properties Identifying properties of those objects requires students to develop and use particular notation, including symbol sense, to express and comprehend what they observe while exploring the behaviour of parts of the representation via the use of the software From this perspective, the use of the tool seems to offer a natural environment for students to develop symbol sense

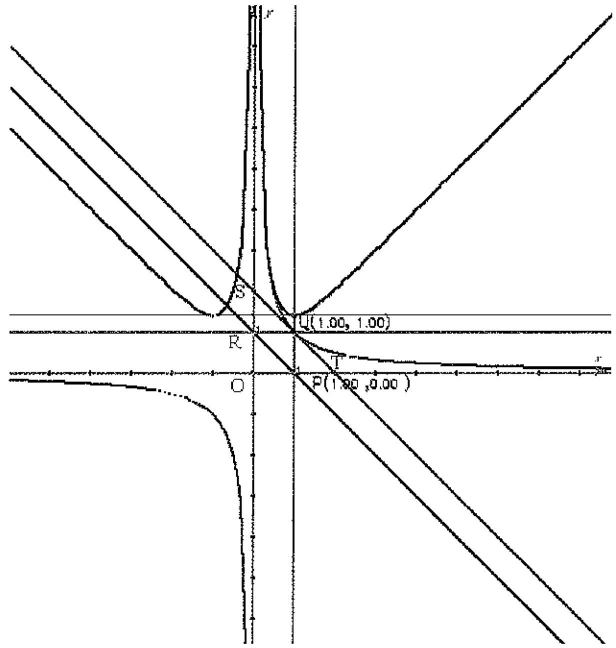

Figure 6: Finding the position of point P to determine the minimum length of segment PR

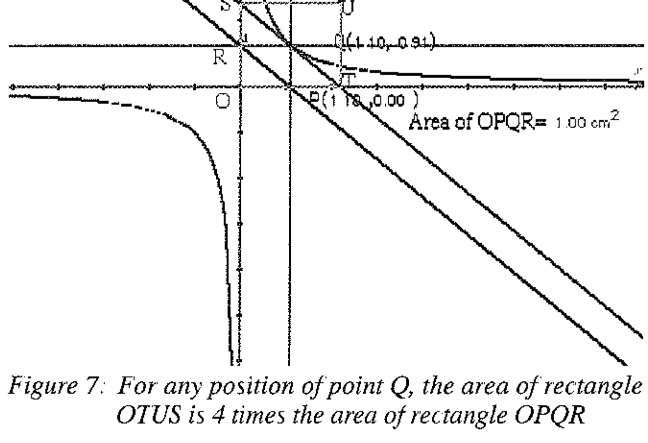

Figure 7: For any position of point Q, the area of rectangle OTUS is 4 times the area of rectangle OPQR

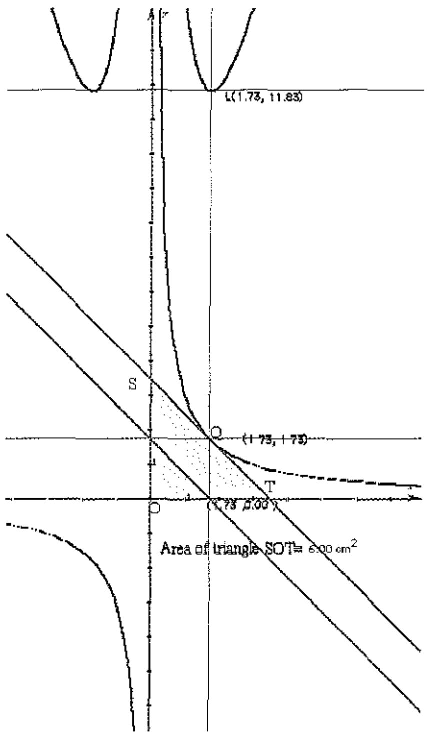

Figure 8: Exploring the behaviour of the area of triangle OTS when f(x) = 3/x)

- to express in writing the representations of the problem
- to explore and communicate properties of mathematical objects that emerge while moving parts or analysing relationships within the dynamic representation.

Here, students often are asked to think of the problem in general terms, that is, beyond the particular case being studied. Again, a proper notation is needed to express and support the validity of those results (Santos-Trigo, 2004). In addition, it seems apparent that with the use of the tool students may consider the dynamic representations of problems as a platform to launch and pursue new and relevant questions that eventually become a source to formulate and generate other problems or results.

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[These references are from the article "Hidden mathematics curriculum: a positive learning framework" that starts on page 12. (ed.)]

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