The Mathematics Classroom: Mature Reflections

GILAH C. LEDER

> Didactics has something to do with learning, and the inevitable complement of the theory of teaching is the theory of learning [Freudenthal, 1980, p 164]

The vast literature on research on teaching testifies to the difficulty of identifying and replicating those aspects of the teaching process most likely to facilitate student learning. In mathematics descriptions of effective lessons — operationally defined in terms of gains in student achievement — have been built up particularly within the process-product research framework. Extensive reviews of this work have been undertaken by Brophy and Good [1986] and Doyle [1986] Exemplar lessons, according to this approach, are characterized by an appropriate balance of review of work previously studied, development of new work, a period of independent seat work, and the setting of relevant homework exercises. Various strategies for enhancing student motivation to learn have also been postulated. [Brophy, in press]

In recent years considerable attention has been focussed on strategies employed by expert teachers [Leinhardt & Greeno, 1986] and to contrast those with the methods favoured by novices. Brooks and Hawke [1985], for example, compared the ways in which experienced, effective and inexperienced, ineffective teachers began their mathematics lessons. It is, however, difficult to structure the wealth of observations available. "Not until we consciously set out to observe learning processes can we create the means to organize, describe and evaluate them." [Freudenthal, 1980, p. 164]

Many of the more recent studies have relied not only on observational and correlational techniques, but also on think-aloud, stimulated recall, and other introspective procedures such as metacognition. "Metacognition refers to knowledge about one's own thought processes. This includes the monitoring and control of these processes" [Romberg & Carpenter, 1986, p. 858] As pointed out by Berliner [1986], the effectiveness of the latter approaches is frequently marred by the experts' inability to articulate clearly, meaningfully and succinctly to others the constituent components of their performance That the problem of communication is magnified when the participants are non-expert learners is illustrated by transcripts such as those provided by Span and Overtoom-Corsmit [1986]. The protocols of their average pupils were considerably less lucid and informative than those of their gifted pupils. Less than 10% of the former compared with just over 50% of the latter were able to provide a comprehensive discussion of the strategy they used to solve a difficult mathematics problem.

In our culture mathematics is generally considered to be sufficiently important to warrant being a compulsory subject for study until well into secondary school

> There can be no doubt that there is general agreement that every child study mathematics at school. Indeed the study of mathematics, toghether with that of English, is regarded by most people as being essential [Cockcroft, 1982, p 1]

Yet references to the difficulty and painfulness of doing mathematics abound. In the introduction to his recently published book Hughes [1986] asserted:

> I am aware that many people who read this book will have strong negative feelings aobut numbers: for them, terms such as "mathematics" and "arithmetic" will bring back unpleasant memories of tedious and pointless exercises carried out at school. [p x]

The tension between the declared importance of mathematics and the perceived difficulty, dislike, criticism of the method of teaching and consequent avoidance of mathematics by many of those who recognize its importance prompted the present study.

A number of priorities shaped the methodology employed:

1. The study should increase our understanding of processes and techniques which appear to facilitate or inhibit the learning of mathematics;
2. Information obtained should rely heavily on the subjects' own experiences in learning mathematics;
3. Participants should react to a set of shared, and non-trivial experiences;
4. The sample chosen should be articulate and able to communicate their reactions to the learning process;
5. The learning climate should resemble, as far as possible, a regular classroom setting.

The strategy that best seemed to fulfill these requirements was one advocated by Tobias [1986]. She argued that identifying and describing the difficulties experienced during learning require maturity, confidence, sophistication, and an appropriate vocabulary — qualities rarely present in young learners. Using peer evaluation of teaching should yield much useful information not readily available or tapped by other means. Specifically,

> a teacher/learner laboratory [should be established] where the only difference between the teacher and the learner would be naiveté about the subject. In all other respects — age, intelligence, confidence, maturity, and self-image — the students would be peers of the instructors. [Tobias, 1986, p. 36]

The study described here details the establishment of such a teacher/learner laboratory and the information it yielded about the learning and teaching of mathematics.

Method

The sample comprised 18 of the author's colleagues and three academics from other tertiary institutions visiting Monash University in Melbourne, Australia.

All were members of an Education faculty and taught pre-service and post-graduate courses in education in which they typically sought to increase their students' appreciation and understanding of various aspects of learning and teaching processes. Of the 15 males and six females who participated, eight (seven males and one female), had studied mathematics at tertiary level, though only one was currently involved in teaching mathematics or a related course. Participation was on a volunteer basis and involved attendance at all four teaching sessions spread over two consecutive mornings.

Two mathematicians, experienced tertiary teachers, and members of an applied and pure mathematics faculty respectively, were recruited to act as instructors. Each was asked to select a topic that could be taught in two one-hour sessions and would be suitable for mature, educationally sophisticated but mathematically relatively naive learners. Both accepted the challenge enthusiastically.

All participants — learners as well as instructors — understood that the purpose of the experiment was to increase our understanding of the difficulties that might be encountered in learning mathematics and to identify teaching strategies that might exacerbate or minimize these difficulties. Thus the instructors were encouraged to present their material as meaningfully as possible. As well as attending to the material being taught, e.g., by taking notes, the learners were encouraged to respond in a margin left on each page for this purpose to the twin questions: "What is making this subject difficult for me?" "What could I, or the instructor, do to make it come clear?" These prompt questions had also been used by Tobias [1986]. To maximize the effectiveness of the teaching sessions the students were encouraged to interrupt the instructor if they needed another explanation, further clarification, or if they wanted to ask any pertinent question. Finally, the participants were asked to reflect in writing, on the totality of their experiences at the conclusions of the teaching experiment. The focus of this study is on this last set of data in particular.

Pythagoras' theorem and Pythagorean triples, and matrix applications to digraphs were the topics taught. The former occupied the first two one-hour sessions, while the latter filled the second morning. Selection of material was guided by the following principles. The work had to be

1. Capable of being understood from first principles (or at least from well known mathematics)
2. Capable of yielding interesting results in a short time
3. Evergreen
4. Many-faceted and
5. Open-ended

[J. S. Stillwell, Personal communications, December, 1986]

The students' comments on the teaching sessions were examined for recurring themes. Four broad groupings emerged: comments on the nature and structure of the "experiment," reactions to classmates' behaviour, reflections on the learning process, and reflections on teaching strategies. For ease of presentation, each category is presented separately.

The size and composition of the sample did not allow separate analyses by sex or mathematics background to be carried out. However, for maximum information, these details are appended to every quotation included.

The nature and structure of the experiment

The initial difficulties experienced by some of the participants in separating role playing from genuine involvement in the activities can be gauged from the following excerpts:

> Early in the first session I became quite angry with my colleagues. I sensed hostility and an "I don't want to learn approach." Surely they were role-playing — they could not be so thick. Later there seemed a more genuine desire to learn.
> (Male, tertiary mathematics background (TMB))

> Early on, on the first day there were a few people trying to be smart.
> (Male, no tertiary mathematics background (NM))

> The atmosphere of the class was relaxed — I think. That's not a bad thing, but it removes the sense of purpose if it goes too far. For example, for some it becomes an exercise in picking up witty points, nit-picking, outplaying the teacher, rather than a genuine attempt to learn. . Of course, some students in school get to the flippant stage because they've failed already or don't want to be in school at all. So there's a point of relevance there.
> (Male, TMB)

> Even though I volunteered to be part of the experiment, my mind kept on drifting — I felt pressured, unable to apply myself fully to mathematics. There are so many other things I should have been doing! A class of children would not be pressured in this way.
> (Female, NM)

At the same time, there were many comments which indicated that the participants soon settled down and became absorbed in the learning activities. To quote another student:

> The exercise, despite its moments of unreality, was a useful reminder (and more) of the associations which go with teaching and learning. . Some interesting ideas are buzzing around my head!
> (Male, TMB)

Reference to the difficulty of learning and simultaneously reflecting on that learning also testifies to the students' genuine application

> At various times I got so absorbed in the subject matter, I forgot to write anything down in that extra margin.
> (Male, TMB)

> I found it really difficult to concentrate on the content of the lesson, and remember to write extra comments.
> (Female, NM)

> Note taking and understanding aren't easy processes; Note taking, understanding and monitoring reactions are very difficult.
> (Male, TMB)

> The *task* you set us was a particularly difficult one, one in which it was almost impossible to do justice to everything: (1) learn, (2) observe and comment on the *teaching* (3) reflect on and write about our *learning* I just could not do all three *and write* about them in real time
> (Male, NM)

The final set of reflections to be summarized in this category refer to a different structural facet: the increasing cohesiveness of the group, and the attendant advantages of this, as the exercise progressed

> By the third session the "class" had developed sufficient solidarity and sense of identity . to act in concert and to rein back the lecturer and get him to conform to our speed of working rather than his
> (Male, NM)

> By the second day the class had gelled. Things went much more smoothly. The audience had become much more receptive
> (Male, TMB)

Reactions to classmates' behaviour
There were substantial individual differences in the ways interruptions by classmates to the teaching flow were perceived. Some, as can be seen from the following excerpts, clearly found them helpful.

> I probably did my most valuable/interesting "learning" as a result of questions which might have been (and once or twice *were*), classed as "distractions"..
>
> But people become self-conscious if they think their questions are holding up other people.
> (Male, NM)

> I like the *simple* questions that clarified the maths. I found the "digression" excellent.
> (Female, NM)

> I needed the support from my colleagues. Wouldn't it have been dreadful if the "teacher" had said "Be quiet — listen — and you'll learn, etc " — which some teachers do. But I panicked when a comment showed that a classmate who earlier did not understand did, before me.
> (Female, NM)

Others, too, noted the debilitating, confusing or irritating effect, of certain questions:

> The presence of, and remarks by, the more knowledgeable (mathematics) colleagues was probably an intrusive and even unwelcome part of the process It *did* highlight the feelings experienced by learners of "low" ability in a mixed class context
> (Male, NM)

> For me, the first day was bitsy and disjointed The fact that the teacher was trying to please everyone by answering questions was a real problem.
> (Male, NM)

> Some people were almost petty in their questions/issues, e g., direction of arrows That point was really clear
> (Female, NM)

Several of the mathematically more sophisticated participants also commented adversely on the questions or comments from others Their reactions mirror those of the previous excerpt

> The first teacher allowed the students to distract him from the main arguments for too long. Although many points were cleared up through students' discussions, I feel many of them would have become clear anyway had the teacher continued . While (many) discussions were not uninteresting, they did hinder the flow of the lesson and obviously shortened the time available to talk aobut the main issues.
> (Female, TMB)

> I was very irritated by some of the attention-seeking behavious of one or two of my colleagues who had to do their thinking aloud. As time went on I became increasingly annoyed and frustrated. The sort of behaviour I'm referring to is actually voicing the steps in the exercises in a loud voice apparently showing "how quick and good I am"
> (Male, TMB)

The extracts cited so far have all referred to behaviour that affected the whole class. To conclude this section, the only two reactions to less intrusive interruptions are cited. In both cases, they reflect appreciation of help offered by another class member.

> I hated being the first to expose a difficulty in understanding. It was really comforting to sit next to Paul on the second day. I felt as though I was sitting next to the brightest kid who let me copy.
> (Female, NM)

> I liked the way the first teacher allowed students to interact with one another to help clear up points.
> (Female, TMB)

A number of themes more appropriately included under learning and teaching have already been foreshadowed in the comments quoted so far. In the next two sections such themes are expanded and categorized more methodically. To minimize the inevitable overlap between the learning and teaching topics, comments through which participants clearly focussed on self are highlighted in the former, while reflections that emphasized the instructor's role are left for the latter.

The learning process

A striking feature of the participants' reflections about their own learning was the frequent reference to anxiety and lack of confidence about the subject matter. These comments, it should be recalled, were made about lessons carefully planned and prepared by two experienced and enthusiastic teachers aware of the strengths and weaknesses of their students.

> Why was there no attempt to allay my anxieties right at the beginning? Why does mathematics have to be more fun for the teacher than the students?
> (Female, NM)

> I found the classes difficult. I like the simple calculations. They gave me a sense of success. I like rote laws (I know that they are frowned on but I'm *very* prepared to accept rote rules especially in the subject area where I'm uncertain). The more uncertainty I have, the more I need structure.
> (Female, NM)

> I was very disappointed with my attitude to mathematics. I felt I had not gained any confidence with the subject.... I still lacked confidence to admit my difficulties and admired those who could. I still found that I could not concentrate fully when I began to feel out my depth. My latter-day tenacity seemed to dessert me.
> (Female, NM)

> I felt "out" in very marked ways yesterday — and I felt very much in the position of whole groups of classes of working-class kids of all colours.., who just simply "turn off," laze around, laugh, joke and have a good time. That laughing and joking often come about in an effort to cover one's own embarrassment at not being able to cope.. they are a way of reinforcing peer pressure *against* maths.
> (Male, NM)

> I became totally confused on the first day. I stopped being able to concentrate. I was constantly confused, lost the thread. My notes are bitsy and disjointed. I loved the handouts on the second day. I needed the visual and verbal clues... At various times I experienced the same difficulties and anxieties that I suffered 30 years ago.
> (Male, NM)

> But why was I having difficulty: well, the above makes me realize that all my old fear of and hostility to maths, engendered by bad teaching and all the other personal factors long ago, were coming up again and encouraging me to re-apply the old stereotype of the subject I had first constructed all those years ago and which had otherwise lain so long dormant.... What made me upset was the recapitulation of experiences I had some 36 years ago... when I gave up maths at school...at the age of 14.
> (Male, NM)

The difficulty experienced by some class members did not go unnoticed.

> It came as a revelation to see mature colleagues having the same levels of difficulties as children. I had to speak severely to myself not to feel superior. Basic questions were being asked by people for whom I have a high regard in other aspects of professional life.
> (Male, TMB)

> I had sympathy with the reaction "I can follow the steps, but I can't see how you know where to start." This is a very familiar reaction...treating symbols with freedom — i.e., without worrying too much what they mean — is like taking your feet off the bottom of the swimming pool. You've just got to try it!
> (Male, TMB)

The wisdom of encouraging students to expect understanding to come quickly was canvassed by at least one other participant.

> The class seemed to ask "why" in an unconstructive way. Non-mathematicians seemed unwilling to accept procedures if there was no ready explanation. They were not prepared to follow and see where it led.
> (Male, TMB)

Several other sets of reactions are worth citing. Willingness to do addtional out-of-class work to facilitate learning is reflected in the first set of quotations.

> I felt ..some sense of satisfaction and mastery (which later revision confirmed)
> (Female, NM)

> With help from another participant, I solved the problem after the session.
> (Male, NM)

> I was satisfied with my own understanding, although one point slotted home as I was looking through my

notes this evening!
(Male, TMB)

Other positive comments referred to the enjoyment derived from exposure to mathematics

> The sessions rekindled an *interest in mathematics* though perhaps not to the point of wanting to take a course
> (Male, NM)

> The topic was developed in a way which had made me curious at the tea break how it would continue (How were the digraphs going to be transformed into matrices?)
> (Female, TMB)

> I found the second topic really interesting; the material was new to me
> (Male, TMB)

The next two quotations, below, not only reveal a tendency by some learners to shift the blame for the difficulties they experienced from self to the instructor, but also serve as a useful transition to the category to be considered: reflections on teaching procedures

> There is a tendency to believe that mathematicians think differently Rather, they are careless in their vocabulary They don't break things down
> (Male, NM)

> The teacher bored on . . Was he really trying to teach something, or was he just being provocative?
> (Male, NM)

Teaching strategies

All the participants reflected on the teaching styles and procedures they had just experienced — though there was considerable variation in the degree of detail that characterized the responses Several strategies received warm commendation.

1 Objectives of the lesson must be stated at the outset

While participants agreed about the importance of being given a brief outline of the material to be covered, some noted that a particular instructor had done this while others, sitting in the same sessions, accused the instructor of failing to do so

> The printed material distributed at the outset by [the instructor] was very useful in giving a sense of direction.
> (Male, TMB]

> It was helpful to have the objectives of the lesson stated at the start
> (Female, TMB)

> I needed to be given an "advance organizer" to know what and where I was going.
> (Female, NM)

2 Technical terminology should be explained

> Teachers must always explain the technical terms of the subject.
> (Male, NM)

> [The instructor] tended to assume more familiarity with technical terminology etc , than was appropriate
> (Female, NM)

The wisdom of using technical terms prematurely was also questioned

> Show how to do it first Only then let slip the technical terminology
> (Male, TMB)

3 Students must be given opportunities to work on examples during the lesson.

> Some time should have been given to students to engage in some kind of activity and to discover simple facts on their own
> (Male, TMB)

> Maths could perhaps best be learned by individuals working through an interactive programme and having a tutor available to the class
> (Male, NM)

A somewhat different plea for more active student involvement was voiced by another class member

> The deadening familiarity of an individual verbally annotating interminable lines of symbols, figures and diagrams made me remember school maths from my own school days . . Has maths teaching not advanced in the last two decades?.. Why are students mere observers and patient scribes?
> (Female, NM)

4 The work discussed should be linked to practical examples.

> This was good teaching — there were good links with real examples
> (Male, TMB)

> The power of analogy is very important in teaching I need a teacher who does this well
> (Female, NM)

> Mixing in the application illustrations with the theoretical discussion made the topic look relevant straight away.
> (Female, TMB)

But not all the students recognized the relevance of the examples given, however:

> The session lacked a contextual framework and hence it was difficult . . to find a reference point
> (Female, NM)

Ambiguity about the value of following up questions from individual class members has already been discussed in an earlier section. While some participants reflected that teachers should be particularly receptive to such questions, others considered that teachers should be less tolerant and more discerning It is instructive to compare the following two excerpts which refer to the same incident

> Teachers must go slowly, slowly . and must explain and demonstrate . as many times as it takes for everyone to go forward with the confidence that comes from deep understanding.
> (Male, NM)

> I can see how a class can become very polarised by something like this. Those who want to pursue one point whereas others want to go for the main target
> (Male, TMB)

The importance of careful preparation was also noted

> The teacher needs to analyze the content of the lesson and set it out in logical fashion. The psychological and pedagogical order must also be decided. And this is not necessarily the logical order .. the teacher should check whether students know and remember key terms and concepts.
> (Male, TMB)

> In future I will make sure that I have plenty of visual aids and concrete materials when I teach (my course) I will begin with a clear practical problem and use it to illustrate identifiable competing solutions.
> (Male, NM)

While the extensive use of quotations thoughout the results section most fully and effectively illustrated the flavour and scope of the reflections drawn from the sample of educators, some final interpretive comments are appropriate.

Concluding comments

In many ways the present study was successful in meeting the aims outlined. Much useful information was gained through the teacher/learner laboratory in which both teachers and learners were experienced and competent professionals, shared many characteristics, but differed in depth of knowledge of mathematics in general, and the subject matter taught in particular. Despite several participants' early unease and misgivings, role playing soon ceased and was replaced by genuine involvement in learning mathematics Class members' references to the similarity between the lessons they attended and the ones they remembered from school days suggest that the learning climate established duplicated many features of regular classrooms. Parallels drawn between aspects of the learning environment created during the experiment and those experienced by younger students lead to the same conclusion.

As indicated earlier, recent research on teaching has often incorporated, or relied on, introspective procedures, with learners being encouraged to be aware of, reflect on, and control their own thought processes Yet comments from the educationally experienced and sophisticated subjects in this study point to the difficulty of doing this in a normal teaching setting in which material is presented at a rather rapid pace — too fast to allow time to attend to the actual ideas being discussed as well as the ways in which they are being processes, organized, integrated, or assimilated. These reflections suggest that metacognitive strategies that have been shown to be effective in improving the quality of learning in a one-to-one or small group setting may not be readily translated to the regular classroom without special training of both teachers and students. Monitoring one's thought processes imposes substantial additional demands on students.

Participants' preference for a clearly recognizable statement of lesson aims, their need for the linking of unfamiliar material to known concepts, their expectation that knowledge of technical terms not be taken for granted, that the presentation rate not be excessive, and that they should be more than passive listeners to the instructor and be given opportunities to work on practice examples, mirror many of the features of effective lesson highlighted by Brophy and Good [1986] and Doyle [1986]. Yet interpretations of what constituted a recognizable statement of objectives, an acceptable explanation of technical terms or mathematical process, a reasonable lesson pace, or a helpful practice example showed considerable variation, with previous background in and attitudes to mathematics being crucial factors. While some participants were satisfied that the teaching techniques used ranged from largely adequate to most satisfactory, fellow students found them confusing, anxiety provoking, unhelpful and in need of considerable modifications. The reflections exemplify the dilemma faced by regular mathematics classroom teachers anxious to facilitate the learning of all students. Meeting the needs of some is likely to alienate others; extending one group of students may curtail the learning environment of another. Once established, negative feelings about mathematics are not changed easily

While some participants commented that the lessons reinforced for them previously held negative beliefs about mathematics, other indicated that they were unexpectedly interested and challenged by the new material being presented. Feeling more prepared to admit to early difficulties and to seek help were noted as important elements.

Even in this group of educators, sympathy for fellow students experiencing difficulties was limited Appreciation of the depth and extent of the anxiety, uncertainty, feelings of hopelessness and frustration (and attendant rejection of mathematics) suffered by some was rare. Those with a strong mathematics background tended to express impatience with the frequency and quality of questions being asked and focussed primarily on organizational and structural aspects of teaching. These reactions are instructive and illustrate the need for greater attention in mathematics classes to the affective components of learning For it is worth recalling that typical high school mathematics teachers resemble those with a tertiary mathematics background in the present study.

In focussing on the reflections of individual participants, one important feature of the teacher/learner laboratory exercise has been omitted. It is difficult to quantify accurately the impact of the informal discussions and arguments about various aspects of teaching and learning generated both among fellow "students" and with faculty members who had not been part of the experiment Being a learner in a setting that was unfamiliar, challenging and perhaps threatening was a sobering experience, one that shook complacency and provoked much re-analysis by individuals of teaching practices they themselves used The likely benefits of sharing these deliberations with their own students should not be underestimated.

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Outsiders see mathematics as a cold, formal, logical, mechanical monolithic process of sheer intellection; we have argued that insofar as it is successful, mathematics is a social, informal, intuitive, organic, human process, a community project Within the mathematical community, the view of mathematics as logical and formal was elaborated by Bertrand Russell and David Hilbert in the first years of this century They saw mathematics as proceeding in principle from axioms or hypotheses to theorems by steps, each step easily justifiable from its predecessors by a strict rule of transformation, the rules of transformation being few and fixed. The *Principia Mathematica* was the crowning achievement of the formalists. It was also the deathblow for the formalist view. There is no contradiction here: Russell did succeed in showing that ordinary working proofs can be reduced to formal, symbolic deductions. But he failed, in three enormous, taxing volumes, to get beyond the elementary facts of arithmetic. He showed what can be done in principle and what cannot be done in practice. If the mathematical process were really one of strict, logical progression, we would still be counting on our fingers.

De Millo, Lipton and Perlis
