MIND THE GAPS: GAP-FILLING IN PROVING ACTIVITIES

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> Mold clay into a bowl.
> It is the space within that makes it useful.
> Cut out doors and windows for a room.
> It is the holes which make it useful.
> Therefore, the value comes from what is there,
> But the use comes from what is not.
>
> Lao Tzu, *Tao Te Ching,* Chapter 11

We start by inviting the readers to undertake the task shown in Figure 1 and reflect: What of your prior mathematical knowledge did it summon? And how? If you discovered a proof, a sequence of mathematical arguments establishing the proposition's truth, which parts of it appear in the figure? And which parts did you add, or create yourself? Which elements in the figure triggered you to do so?

Observe a sketch created by Yali, a tenth-grade student who worked on this task (Figure 2, left) during a lesson taught by the first author. Yali made some angle calculations, dropped the altitude of the middle triangle, added the auxiliary dashed lines to construct three squares, and after that said to his fellow students: "Look, so this *a* times this *a*, plus this *b* times this *b* equals *c* squared and this what we have here [pointing at the theorem's formula]" [1].

Yali did not just sketch at random. His notations and constructions hint at theorems and procedures regarding angle calculations, isosceles triangles, squares, and their properties. Regarding the three squares' auxiliary constructions: What in this PWW caused Yali to draw them? Is it the information the diagram presents? The information it conceals? Or an interplay between the two?

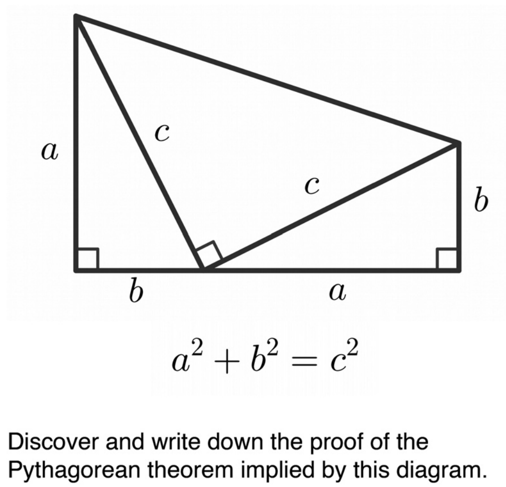

Figure 1. Garfield's Proof-Without-Words (PWW) task (adapted from Nelsen, 1993, p. 7).

In this article, a *proof-document* is an artefact given to students to help them discover, reconstruct, or understand a proof. A proof-document can be in many formats: written texts, recorded oral communications, or diagrams as in Figure 1. The information a proof-document presents is always partial. Some mathematical premises, conventions, inference rules, and readers' prior knowledge are left unspecified. So, there will always be gaps between a proof-document and the proof it represents. This article discusses the relationships between what is visible and what is absent in mathematical

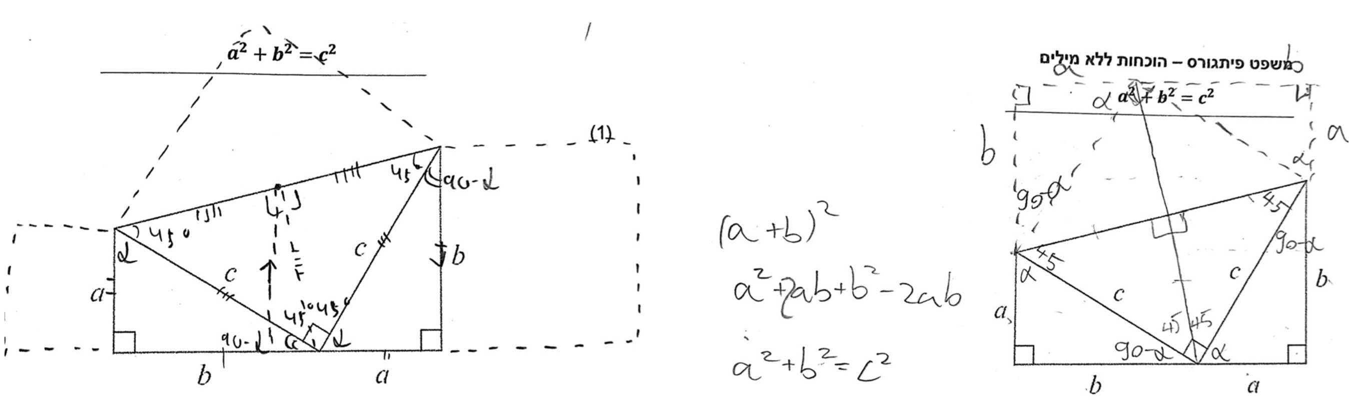

Figure 2. Yali's first attempt (left) and eventual proof (right).

proof-documents. Acknowledging the *value* of proofs in mathematics education, we will argue that *gaps* between a proof-document and its corresponding proof suggest their pedagogical *usefulness*. Gaps are pedagogically useful as they invite students' mathematical exploration, application of prior knowledge, deepening understanding, and attaining a sense of ownership over a proof.

A theory of gap-filling

Perry and Sternberg (1986) developed the idea of *gap-filling* in literary theory. They refer to any text as a 'system of gaps', which the readers need to fill in to construct meaning. The information writers can supply is restricted, and they inevitably omit some information. The mental actions of the readers alleviate this limitation. The process of gap-filling, adding supplementary information not explicitly mentioned in the text, can take place in different ways: unconsciously, consciously but tentatively, or decisively. It can range from a simple linkage of some elements in the text, which the readers do almost automatically, to "very complex systems of linkages that are constructed consciously, laboriously, hesitantly, and with constant modifications" (p. 276). Although Perry and Sternberg suggested the concept of gap-filling for understanding the reading of narrative texts, we argue that it is highly relevant to tasks based on mathematical proof-documents. Rav (1999), for instance, characterises the process of reading a proof as 'interpolation':

> In reading a paper or monograph it often happens—as everyone knows too well—that one arrives at an impasse, not seeing why a certain claim B is to follow from claim A, as its author affirms [...]. Thus, in trying to understand the author's claim, one picks up paper and pencil and tries to fill in the gaps (p. 14).

During a literary text reading, gap-filling necessitates imagination, associations, and some acquaintance with the reality depicted. In reading a mathematical proof-document, gap-filling also necessitates acquaintance with the relevant mathematical knowledge and enactment of adequate mathematical procedures. In the mathematical context, gap-filling has a universal aspect as it demands compliance with relatively scrupulous and universal mathematical standards. Yet, it is still a highly personal kind of activity as it is inextricably dependent on one's background knowledge and mathematical proficiency. In both contexts, gaps invite engagement as they create epistemic needs among the readers for a sense of certainty, conviction, and comprehension.

The term 'gap' in proofs in mathematics education has been commonly used to indicate deficiencies in proofs that render them unacceptable. However, Bender and Jahnke (1992) maintain that every proof inevitably contains gaps and that this does not necessarily prevent it being valid:

> In mathematical practice, completeness is unattainable, and undesirable. Proofs that [...] have gaps are thus not automatically considered as violating mathematical rigor. Whether a proof is judged mathematically incomplete depends on many conditions that are to some extent subject to historical change (p. 261).

To discuss the pedagogical role of gaps in a proof-document in mathematics education, we propose the following notion: *a gap consists of missing information in a proof-document, the filling of which is essential for comprehending the proof or making it more coherent for its reader.* Accordingly, gap-filling is *any action that aims at identifying and closing a gap.*

In the literature on reading proofs in mathematics education, there is some evidence for processes related to the notion of gap-filling. Selden and Selden (2003) define *proof validation* activity, which aims at determining the correctness of proofs, as follows:

> Validation [...] can include asking and answering questions, assenting to claims, constructing subproofs, remembering or finding and interpreting other theorems and definitions [...] Proof validation can also include the production of a new text [...] that might include additional calculations, expansions of definitions, or constructions of subproofs (p. 5).

This definition suggests that the actions of reading a proof and constructing a proof are interwoven. However, while proof validation addresses proofs' verification function, we see gap-filling as a more general proving activity that serves other functions, such as explanation and discovery. Gap-filling serves the explanatory function because readers may gap-fill to explain the proposition and its proof for themselves, seeking to understand the proof without ever questioning its validity (as is often the case with secondary-school students). Regarding the discovery function, if a proof-document is too vague or incomplete, one might not see how exactly it proves the proposition. In this case, the claimed proof is being rediscovered through gap-filling actions.

Let us return to the proof-document presented in Figure 1 and to Yali's gap-filling actions triggered by it. The connection between the diagram and the proposition is not explicit. In particular, the algebraic inscriptions a^2, b^2 and c^2 seem not to be represented in the diagram. Yali's sketch in Figure 2, left, suggests that he identified this as a gap and tried to fill it through the construction of the three squares whose areas are a^2, b^2 and c^2. Yali applied his prior knowledge of squares and their properties to *discover* a key idea in this proof. His gap-filling actions were aimed at finding out *how* the diagram provides a proof, not *if* it does so.

Students filling in gaps

Yali's sketches illustrate how gaps in a proof-document invite students' mathematical engagement. They create an epistemic need to complement the proof-document and stimulate the application of prior mathematical knowledge (*i.e.,* applying the concept of areas) and to interact with the proof-document in creative ways (*i.e.,* adding auxiliary constructions). It also suggests that the pedagogical examination of gaps in proof-documents may explain student mathematical behaviour by seeing their actions as aiming to fill a particular gap they identify. The next classroom episode will demonstrate these ideas further.

Lily and Mika worked on Garfield's PWW for about eight minutes. At first, they examined the diagram silently. When Mika started talking, she suggested calculating areas as a way to handle the task. However, Lily seemed to ignore her comment and immediately presented a claim of her own:

1 *Mika*	Maybe we can try *[through]* area calculation.
2 *Lily*	These two triangles are congruent. We know that.

Unlike Yali, who created new areas to calculate, Mika suggested area calculations as a general concept that may be of use. Lily's conjecture brings in the idea of congruence. They both offered mathematical concepts from their prior knowledge through which they hoped this task could be accomplished. These concepts are absent, or only implicitly represented, in the proof-document.

15 *Mika*	We need to obtain the Pythagorean theorem.
16 *Lily*	So maybe some kind of square that *[pause]*
17 *Mika*	No look here, we can prove that this area *[pointing at the whole figure]* equals this plus this *[pointing at subfigures].*
18 *Lily*	Yes!

In Turn 16, Lily mentioned a square, even though no square appears in the diagram. We cannot know if she thought of finding a square or constructing one somehow (as Yali did), but the fact that she was considering a square could be an attempt to fill the gap of the unrepresented a^2, b^2 and c^2 in the diagram. By doing so, Lily might have applied Mika's idea of area calculation (Turn 1). In Turn 17, Mika continued filling this gap by referring to areas of subfigures.

The following example shows that students can perceive a missing justification as a gap that needs to be filled. The girls tried to prove that the whole figure is a right-angled trapezoid, a property that is indicated implicitly. A discussion arose around this issue:

27 *Lily*	but this is not a trapezoid. No.You cannot prove that *[pause]*
28 *Mika*	right, I can't prove they *[interrupted by Lily]*
29 *Lily*	are parallel *[pause]*
30 *Mika*	*[whispers to herself]* or maybe yes?
31 *Lily*	I do not think there is a way to prove they are parallel.
	[30 seconds silence]
34 *Lily*	OK. Wait a minute! Look: we have $a+b$, and this side will also be $a+b$ because *[pause]*
35 *Mika*	Yes, just as in the *[formula of the]* area of a trapezoid $(a+b)(a+b)/2$
36 *Lily*	It is a trapezoid!
37 *Mika*	So, yes *[doubtfully]*
38 *Lily*	*[rotating the page at 90 degrees]* Like this, and it is a trapezoid!

The students identified that a justification that the whole figure is a trapezoid is missing and perceived this as a gap that needs to be filled (Turns 27, 28, and 31). They also knew how to prove this statement—that two of its sides are parallel (29-31). They did not recognise that the adjacent interior angles are supplementary and found two other ways to fill the gap. At first, Lily started to decompose the figure into three parts and pointed at three of its sides (34). She did not use the words 'base', 'leg' or 'height'. Instead, Lily called the bases a and b and used the word 'side' for the height. This might suggest that even though she had some insight, she had not yet fully perceived the figure as a trapezoid. Still, Mika immediately perceived her decomposition as connected to the trapezoid formula (35). Lily was finally convinced (36, 38) upon rotating the diagram until it *looked like* a prototypical trapezoid. We cannot know whether this informal justification satisfied the girls or helped them perceive the trapezoid by its formal properties. However, it seems that they felt that the gap was sufficiently filled.

The students did not justify why the middle triangle is an isosceles right-angled triangle. That raises a question: Why was the missing justification of the figure being a trapezoid perceived as a gap, while the missing justification of why the middle triangle is right-angled was not? We will later address this issue further.

The girls were delighted to discover how the theorem emerges from the area calculations in the diagram:

62 *Lily*	And it comes out to be $a^2 + b^2 = c^2$ Hey! We are so good! *[excitingly, giving high-five]*
63 *Mika*	Now we should write it down
64 *Lily*	Right. OK!
65 *Mika*	Wow, I am proud of us.

Emotions in this excerpt are high as the girls express their self-appreciation (Turns 62 and 65). These emotions suggest that the students were active and appraised their work. The identification of gaps and their filling are promising signs of progress in discovering a proof, especially when accompanied by the expression of positive emotions. Still, their enthusiasm does not mean that the purported proof they produced is rigorous. Whether or not a purported proof is accepted as a proof depends on the relevant mathematical community's expectations and standards (Bender & Jahnke, 1992). Undoubtedly, their gap-filling actions brought the students closer to a formal proof than the original diagrammatical proof-document.

This episode suggests that not all gaps have the same weight. Filling some gaps is crucial to having a sense of discovery and accomplishment (*i.e.,* the idea of calculating the area in two different ways), while others are of lesser significance (*i.e.,* proving the middle triangle is right-angled). Perhaps the 'joy of discovery', as in Turns 62 and 65, is more associated with filling the main gaps of the proof that carry its key idea(s), than with the attainment of rigorous proof.

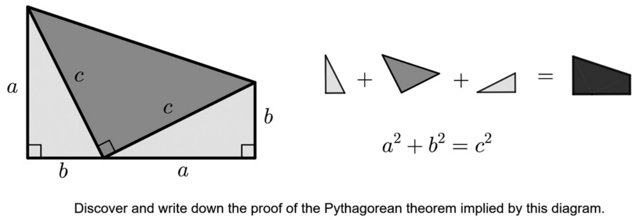

Figure 3. Given this proof-document, will students still produce sketches like Yali's?

Designing gaps for learning

The final sketch of Yali's group (Figure 2, right) shows that their proof is essentially different from Lily and Mika's. Yali's group drew a square with side $a + b$ and calculated the area of an inner square (c^2) in two different ways. Lily and Mika calculated the trapezoid area in two different ways, adding no auxiliary constructions. Imagine what would happen if the proof-document in Figure 1 was presented as in Figure 3.

Figure 3 shows a proof-document presenting more information than the proof-document in Figure 1. Its design emphasises area calculations and indicates how to create an equation that leads to the theorem's proposition. This proof-document channels students toward a proof similar to Lily and Mika's, reducing their chances to engage in probing and guessing or adding auxiliary constructions like Yali's. Lily and Mika managed to develop a proof based on Figure 1 without the additional information presented in Figure 3. Given that version, would they still have been as enthusiastic to discover the proof's key idea?

Which of these proof-document designs is preferable? That is for teachers to decide, depending on their students' mathematical competencies and the learning goals they strive to accomplish. We only indicate some general pedagogical considerations. For some students, a gap in a proof-document can be an impasse. For instance, in our example, students who miss the idea of area calculation would probably fail to develop a proof. On the other hand, a proof-document without enough gaps would leave the students with not much to do, other than following and interpreting the proof-document. Significant gaps invite exploration and creativity, but a risk of failure lurks therein. Minor gaps may ensure higher chances of success in gap-filling but are less engaging, and filling them may be less rewarding.

The changes from the representation in Figure 1, which has significant gaps, to the representation in Figure 3, that fills these gaps to some extent, illustrate the idea of tuning gaps to fulfil different pedagogical needs. Moreover, task design informed by the notion of gap-filling can help students identify and fill more subtle gaps, not just the most prominent and central ones, leading them to generate more detailed and rigorous proof-attempts.

For example, Lily and Mika's tried to explain why the whole figure is a trapezoid, in order to calculate its area. However, although they used the right-triangle formula to calculate the area of the middle triangle, they did not prove why it is right-angled. How can we design the proof-document to make students more aware of this gap? Figure 4 presents two ways of doing this visually. In Figure 4a, the angle is marked as right, while in Figure 4b, it is not. Which of these versions will increase students' chances of identifying and filling this gap?

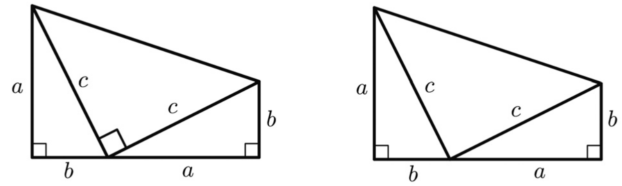

Figure 4. Marking or not the right angle in the middle triangle?

In Figure 4a, the learners should acknowledge that not every mark is a given, but rather some require verification. Students who worked on this task, like Lily and Mika, accepted the fact that the middle triangle is right-angled as a given, and did not bother to justify it. The way the information was presented prevented them from identifying and filling this gap. On the other hand, in Figure 3b, the triangle's 'right-angledness' is left as a gap. In this case, some students would not notice it, which could prevent them from continuing and filling other gaps. A third possibility, shown in Figure 5, may be useful to reconcile these two alternatives. In this version, the angle is marked as right with a dashed line, indicating a different epistemic status and forestalling taking the 'right-angledness' for granted.

This example suggests that redundant information in the proof-document can make a reader grow numb and not identify a particular statement as a gap. Sometimes, it is better to conceal information and create a gap that urges the students to fill it by themselves.

Mind the gaps

Gaps promote engagement as they create an epistemic need for a reader to interact with and add information to the proof-document, a process we refer to as gap-filling. Gap-filling can involve limited elaborations or demand expansions that include reasoning processes, inquiry procedures, and ingenuity using prior knowledge. With a proving activity based on a PWW the notions of gaps and gap-filling can be illustrated easily. However, this illustration is intended to point at some general principles that are relevant for any type of proof-document: The potential of gaps to stimulate and steer students' mathematical activity; the ped-

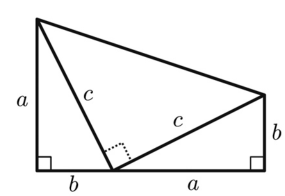

Figure 5. Why is the middle angle a right angle?

	Proof-documents with few/minor gaps	Proof-documents with many/significant gaps
Student Actions	Students try to comprehend the proof, mainly by interpreting the presented information	Students try to discover a proof by inventing/guessing missing information
Variety of proofs	Low, fewer ways to fill the gaps	High, many ways to fill the gaps
Students' chances to comprehend an intended proof	High	Low
Students' chances to generate new ideas	Low	High
Emotional effect upon completing the task	Intermediate-low sense of accomplishment	Strong sense of accomplishment

Table 1. Considerations of having substantial vs. minor gaps in a proof-document.

agogical consideration in choosing a proof-document before bringing it to class; foreseeing and designing gaps in a proof-document as a means to support students' learning.

Which information in a proof-document should designers disclose or conceal to engage students in activities that enable proof construction or discovery and foster understanding? A proof-document with too many gaps may not supply enough resources for gap-filling and limit the proof-document's usability as a pedagogical artefact. On the other hand, a too coherent proof-document can prevent students from engaging in significant gap-filling; thus, gaining only superficial understanding. This dilemma can be addressed through designing new and modifying existing proof-documents while focusing on gaps. Even a subtle change can steer students' mathematical behaviour in a proof-document activity. Thus, designers should provide the minimal information that still suffices to allude to what is absent. Designing for gaps relates to the emotional component of learning, too. By tuning the amount and nature of gaps, educators may achieve various pedagogical goals: Reducing anxiety or providing a challenge; increasing chances to reveal teachers' intentions or discover new proofs; making the intended proof more obvious or helping students develop ownership over a proof they create. Table 1 summarises these considerations.

These two types of proof-documents lead to different student activities corresponding to different proof functions. A proof-document with significant gaps would necessitate proof-construction and emphasise the discovery function of proof. A more explicit and detailed proof-document with minor gaps would be more associated with proof-comprehension corresponding to proof verification or explanation functions.

Note

[1] All quotations and transcripts of students are translated from Hebrew.

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We often present students with the solution of a problem even if they themselves are quite ready to think of a solution on their own. Just as an elevator to the third floor robs us of our chance to use our muscles, we should be alert of times when we rob our students of a chance to think.

Marion Walter (1928–2021)
from p. 16 of 'Do we rob students of a chance to learn?' in issue **1**(3).