Mathematical Thinking and Intellectual Technologies: the Visual and the Algebraic

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Introduction: media, cognitive ecology and the reorganization of thought

" ... may I think on paper?"

The above question was asked by a student as she attempted to solve a mathematical question I had posed Accustomed to the presence of paper and pencil in mathematical practice, in its teaching and its learning, it could be considered trivial if one did not stop to analyze the situation with greater attention This student's comment attracted my attention: paper as a place where thought can develop; paper as an interface that allows thought to be expressed Student, pencil and paper: a *thinking collective*

The student's comment illustrates the idea developed by Levy (1993) that:

> our thinking is deeply molded by material devices and socio-technical collectives (p. 10, *my translation*)

The concept of *cognitive ecology* that Levy developed defending the "idea of a thinking collective *human beings–things*" (p. 11, *my emphasis*) is central to having a different understanding, a new vision of the work that students can develop in a computer environment

The above student's comment points toward the importance of the medium in her thought processes; in this case, paper and pencil - a medium that is generally so evident and incorporated into daily life that its use appears natural and unimportant Nevertheless, the medium of paper and pencil reveals itself as an object that is mediating human thought Similarly, the computer constitutes a new tool that transforms and, at the same time, is part of human thought, also integrating a thinking collective *human beings–things*.

Levy's concept of *cognitive ecology,* which marks the emergence of new cognitive styles in the time of the computer, appears compatible with Tikhomirov's (1981) *theory of reorganization* This latter author states that the mediating nature of the computer in human intellectual activity produces a reorganization of the processes of creation, search and storage of information and of human relations Tikhomirov refers to the constitution of *human being–computer systems* as the basis of this reorganization. In talking of a thinking collective human beings-things, Levy goes beyond this system and creates the concept of cognitive ecology which contemplates the technical and collective dimensions of cognition, broadening and complementing in this way the vision of the Russian author.

However, neither author offers a characterization of cognitive ecology or reorganization in the processes of teaching and learning of mathematics. Filling this gap has been one of the objectives of the Research Group on Technology in Mathematics Education (UNESP at Rio Claro) during the last three years. Studies such as the two described below offer examples of this reorganization

Borba (1997) shows how using graphing calculators to approach mathematical questions posed by the teacher or generated by the students themselves leads to a production of knowledge characterized by independent investigation and the generation of hypotheses A change in the dynamic of the discussion in the classroom is also characteristic of this reorganization.

The work of Borba and Villarreal (1998) shows that, even without the actual presence of the graphing calculator at the time of a mathematical discussion among a group of biology students enrolled in a math course, the *students–graphing calculator system* was in action New forms of relation between the teacher and the students and new modes of legitimizing and justifying mathematical questions emerged The authors also indicate that the use of the graphing calculator does not preclude the use of the spoken language channel nor paper and pencil.

These media form part of a thinking collective and are related to the intellectual technologies described by Levy: orality, writing and computers. Clearly, speaking and writing continue to be the most common intellectual technologies in students' mathematical work The algebraic form of solving mathematical questions is characteristic of a culture of writing, which provides the fundamental support for such solutions. This aspect has already been discussed in various articles by Borba (1993, 1995b), who points out the influence of traditional media (paper and pencil) in the style of mathematical production which emphasizes:

> knowing a given phenomenon primordially through algebra (1995b, p. 71)

It is in this sense that he suggests that the medium permeates the mathematics and the thinking of whomever does and learns mathematics

Without the paper and pencil medium, it would be difficult to approach mathematical questions algebraically It is for this reason that the student quoted at the beginning of this article needed to think on paper and, perhaps, it is also because of this that some students do not feel comfortable in front of a computer that demands other ways of thinking and new approaches. Levy describes this tendency with clarity:

The temptation to condemn or ignore that which is strange to us is great. It is even possible that we do not perceive the existence of new ways of knowing, simply because they do not correspond to the criteria and definitions that constituted us and that we inherited from tradition. (p. 117, *my translation*)

For example, in a cognitive ecology dominated by writing, a mathematical proof based on a graph or generated with the help of the computer is not considered rigorous. Horgan (1993) discusses the transformations that the use of the computer brings to the discovery, proof and communication of mathematical ideas. The author refers to the emergence of what has been termed *experimental mathematics*, where the computer carries out the fundamental role of experimentation, including in the processes of proof.

Nonetheless, many mathematicians still consider, along with Mumford [1], that:

the pure mathematical community by and large still regards computers as invaders, despoilers of the sacred ground. (cited in Horgan, p. 76)

Although the focus of this article is not directly on the influence of media in the production of research in mathematics, it is important to point out that opinions like this are common in the mathematical community, and that this affects teaching, especially in higher education.

The ways of knowing characteristic of the culture of information technology can be rejected, purposefully ignored or not perceived because they do not satisfy the traditional criteria and definitions that came out of the 'civilization of writing'. But the image stands out as a fundamental support in a new intellectual technology where the computer is central. As part of a new cognitive ecology, the visual approach to a mathematical question could be recognized as characteristic of new ways of knowing.

The thinking collective of which the computer is part appears to privilege what can be called 'visual thought', without implying a rejection of the algebraic. Perhaps the computer has come to restore the value of the process of visualization in mathematics education. An emphasis on visual aspects in mathematics education linked to the use of technology has been developed by several authors where mathematical topics related to the study of functions are discussed.

Borba and Confrey (1996) and Borba (1995a, 1994) studied the understanding and the construction of mathematical ideas of students who worked with transformations of functions with a multiple representational software in an approach which begins with the visual and the graphic.

In a study in which individual interviews were conducted with students who were working with topics of quadratic functions using graphing calculators, Souza (1997) also emphasized the importance of visualization and experimentation as notable aspects of the strategies used by the students in their mathematical investigations, even indicating that a focus which emphasizes the visual:

opens new options in the study of mathematics for those who are blocked with respect to algebra. (p. 121)

In the following section, I will present the study that gave rise to the characterization of what I call *algebraic* and *visual approaches* in the mathematical thinking processes of calculus students in computer environments. In this way, the present study is intended to contribute to the clarification of the characteristics that distinguish these approaches and that do not clearly appear in the studies summarized so far.

The study

My objective in this study is to describe and understand the thinking processes of students in a computer environment while undertaking mathematical tasks related to the differentiation of functions defined on the real numbers. To achieve this objective, it was necessary to observe and record in detail the work of students in computer environments.

Working with three pairs of volunteer students, constructivist teaching experiments (Cobb and Steffe, 1983) were conducted and videotaped. We used the software *Derive*, version 3.14. In addition to the computer, paper and pencil were always available for the students to use as they felt the need, since they occasionally wrote down their conclusions which were used in later sessions, either to apply to new situations or to question or improve them.

The participants were first-year biology students in the Institute of Biosciences of the State University of São Paulo. At the time of the study, they were taking an applied mathematics course in the first semester of 1997. All participants in the teaching experiments were women and unfamiliar with the software.

When the teaching experiments began, the students knew:

- how to determine tangent lines to a curve using secant lines;
- the derivative of a function at a point as the value of the slope of the tangent line to the curve at that point;
- the derivative as a function and some rules of differentiation.

Each teaching experiment was organized in four work sessions where the students performed previously assigned tasks. The one-and-a-half-hour sessions were structured around broad questions aimed at generating mathematical discussions between students and eventually involving my participation as well. There was no linear sequence: thus, each pair followed a different path.

After each session of the teaching experiments, the corresponding videotapes were watched, searching for elements that might shed light on the research questions. I made notes of frequently recurring situations that attracted my attention. With them, I identified regularities and patterns of behavior and/or of problem solving, as these could lead me to establish invariants in the students' actions and to generate conjectures.

The students' performance as well as the interviewer's interventions were recorded and analyzed. This allowed me to re-structure upcoming sessions, to establish 'working hypotheses', and to re-examine previously observed situations. At the end of the teaching experiments, the videotapes were totally transcribed in order to perform an in-depth analysis

I chose an inductive/constructive analysis approach (Lincoln and Guba, 1985) Accordingly, I did not begin with previously established hypotheses: rather, I constructed 'working hypotheses' from the data collected, by generating propositions, conjectures and relations between them The validity of the working hypotheses was tested in the course of the teaching experiments

Twelve episodes - that is, relevant passages with useful elements to characterize the students' thinking processes and strategies - were selected and analyzed. I looked at these episodes through 'bibliographic glasses', that is, contrasting them with the pertinent literature, with the intention of constructing new understanding From the analysis, two approaches in students' thinking processes were identified and characterized: the visual and the algebraic These approaches are influenced and conditioned, although not determined, by the technologies of intelligence which belong to the thinking collective of the students

The visual approach and the algebraic approach in the computer environment

There is a tendency in mathematics education to recognize the relevance of visualization in this area (e.g Tall, 1991; Zimmermann, 1991; Zimmermann and Cunningham, 1991) However, its *status* does not reach the same level of legitimacy as the analytic.

In the literature, for example, studies can be found that focus on the relation between visualization and the mathematical performance of those students that have a visual preference in the processing of mathematical information (Presmeg, 1986a, 1986b, 1992) These studies classify students as 'visualizers' and 'non-visualizers', and the first asserts that the latter have better performance in mathematics than the former

The work of Presmeg made valuable contributions in the area of studies about visualization. However, it is important to point out that, with reference to the relation between visualization and mathematical talent, the students who were considered 'stars', who generally were 'non-visualizers', were evaluated using the same tests from the mathematics curricula that favor non-visual forms of thinking. It seems that the advantage of the non-visualizers over the visualizers is not recognized, as though it were natural, considering that the environments of traditional learning generally do not favor nor encourage the development of visual strategies to approach and solve mathematical problems

The emphasis on the non-analytic is related to the way that mathematics is presented and communicated by teachers and researchers, generally in oral or written form Thus, in this process of communication, the media condition the type of thinking However, as Levy points out, it is not determined by them.

The different episodes that were analyzed in this study reveal different styles of thinking and approaches to dealing with mathematical questions, even though the computer was always present. The thinking collective (formed by each pair of students, the surroundings, the media and the interviewer) constituted a particular cognitive ecology with a thinking practice, which was also particular to that thinking collective

Two styles or approaches can be observed in the present study: the algebraic and the visual, which should not be considered mutually exclusive but rather as complementary, each with their own characteristics.

I characterize here what I consider to be an *algebraic approach* and a *visual approach* in the thinking processes of the students. This characterization is inductive, since it was elaborated based on the activities they carried out in the particular context of the research study itself. Examples extracted from the different episodes complete the descriptions of each approach.

An *algebraic approach* in the process of mathematical thinking could be characterized by:

- a preference for analytic solutions when graphic solutions are also possible;
- difficulty in establishing graphic interpretations of analytic solutions;
- when a graphic solution is requested, a brief run through the analytic one is needed;
- facility with formulating conjectures and refutations or generating explanations based on formulas or equations.

In this case, the computer is not used very much, and the calculations that could be carried out on the computer are done with paper and pencil or just mentally. Rules are remembered to justify diverse mathematical questions

For example, faced with the graph of the function $y = x^3 - 3x$ (see Figure 1), studied in one of the episodes with a pair made up of Karine and Maristela, Karine stated that the graph has this form:

> K: because if it is to the third power, it has to . there will be three roots

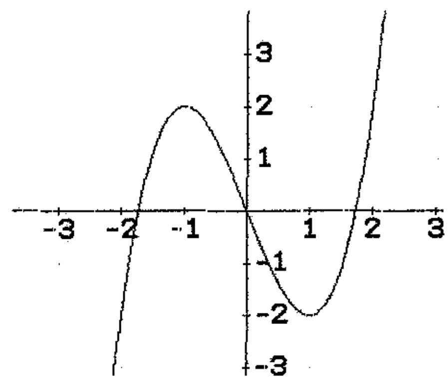

Figure 1

When asked by the interviewer why this did not happen with the graph of $y = x^3$, the student recalled a rule:

> K: I don't know, but in the equations, when we say, like, graphs of the third degree have three roots, in this case [referring to $y = x^3$], I don't know.

In another episode of the teaching experiment conducted with Karine and Maristela, after the students had traced the graphs of $y = 2x^2 + 7x - 4$ and its derivative $y = 4x + 7$ on the computer (see Figure 2), the interviewer asked:

I: Does that parabola have a point that . . that the derivative is equal to zero?

The response of Karine shows her algebraic preference in the interpretation and later solution to the question posed:

K: There isn't any value that you can substitute there [referring to the expression $4x + 7$] to get zero

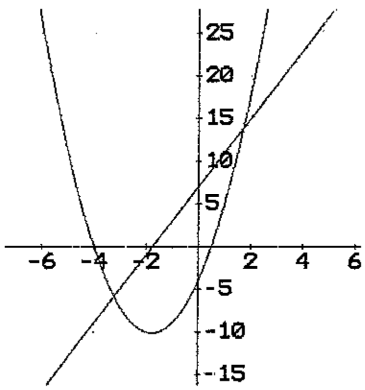

Figure 2

It is important to point out that it was not possible at this time for the students to make a graphic interpretation of the question posed, not even of the correct numerical response ($x = -7/4$), later given by Karine.

In another episode with the same pair, the interviewer proposed that they trace graphs of the derivatives of the quadratic functions without looking at the algebraic expressions. The graphs of $y = x^2$ and its derivative $y = 2x$ were traced on the computer to be used as models (see Figure 3)

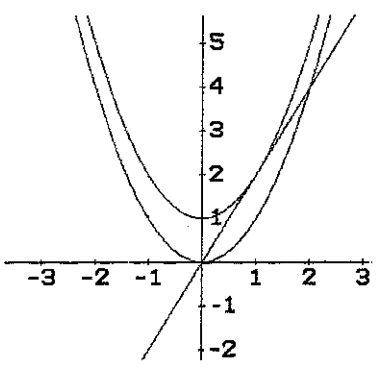

Figure 3

When asked about the graph of the derivative of a parabola such as $y = x^2$, although vertically translated upward by one unit, i e. $y = x^2 + 1$, Karine responded that the derivative would be the same as for the function $y = x^2$ and recalled a rule to justify the response:

K: when we do the rule of the derivative, when we've done all that, $2x$, ... In the case of, of numbers [referring to the independent term 1], we don't find the value of the derivative, we don't have a term to be able to .. that thing about decreasing the power, you know, that little rule . in the case of numbers, we don't have a value . . because the derivative will be in function of the x .... and in the independent term, there is no x, there isn't a variable.

In this same episode, explaining the relation between derivative and tangent lines, a preference for the algebraic can also be observed in the following statements made by Karine:

K: From the derivative, we find a slope of the tangent line

K: Because we have an expression, it is the derivative, and if you substitute a value of x there, of the variable, we're going to have the slope.

For her, the tangent line comes from the calculus of the slope through the expression of the derivative While this statement is correct, it also demonstrates an algebraic preference in the understanding of the relation between derivative and tangent lines.

Now, I can specify the meaning of what I call a visual approach. A *visual approach* in the mathematical thinking process would be characterized by:

- use of graphic information to solve mathematical questions that could also be approached algebraically;
- difficulty in establishing algebraic interpretations of graphic solutions;
- when graphic solutions are requested, there is no need to run through the algebra first;
- facility in formulating conjectures and refutations or giving explanations using graphic information

In this case, the computer is used to verify conjectures, to calculate and to decide questions such as: what is the point where a particular graph crosses the y-axis?

Let us examine other examples of the apparent intertwining of both an algebraic and a visual approach. In one episode, the students (Mayra and Carolina) explained how they decided whether the line $y = 4x - 4$, tangent to the parabola $y = x^2$ at the point $x = 2$, touched the parabola at more than one point (see Figure 4). This tangent line was determined algebraically using the computer, which was in turn used to trace its graph.

The conflict arose with the graph shown on the computer, where the tangent line appeared to be touching the parabola at more that one point This provoked the following question from Carolina:

C: Can a tangent line touch at various points?

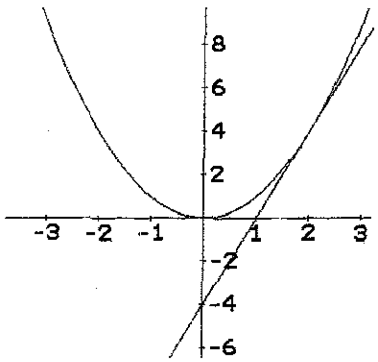

Figure 4

The students' first strategy was to use the *Zoom* command to get a closer view of the graph and a better view of the region where the tangent line and the parabola appeared to be "touch[ing] at various points". However, this failed to resolve the problem, since the parabola and the tangent line tend to get confused in the neighborhood of the tangent point (see Figure 5).

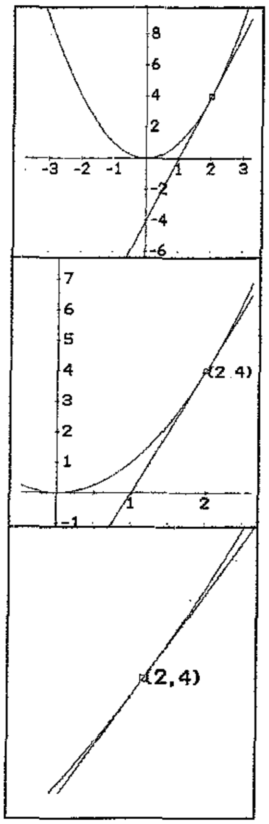

Figure 5

Thus, the need for an algebraic approach to solve the question arose: set the equations of the tangent line and the parabola equal to each other, to verify if they touch at more than one point besides the tangent point.

To summarize, there is algebraic information provided by the computer when it is used to determine the equation of the tangent line. This information says that $y = 4x - 4$ is a tangent line to the function $y = x^2$ at the point (2, 4). On the other hand, there is also graphic information provided by the computer when it is used to trace a tangent line, showing that this line appears to touch the parabola at more than one point. It is from the confrontation between these two types of information that a conflict and two strategies to resolve it arose: first visual, and later algebraic. This process is depicted in the following diagram.

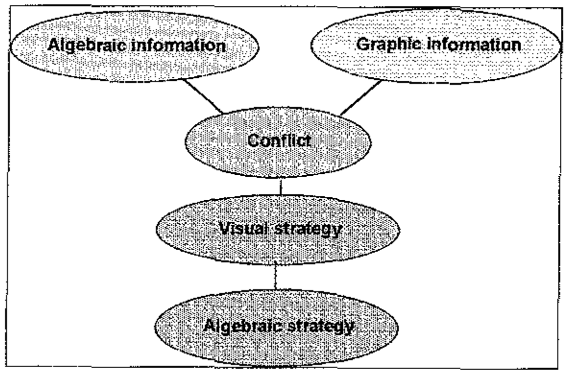

Figure 6

Another situation that illustrated the need to complement the visual and the algebraic was the following: faced with the graphs of the functions of $y = 3^{x^2}$ or $y = 3^{(x^2 + x)}$ (Figure 7), Mayra and Carolina associated them with quadratic functions, as shown by Carolina's comment:

C: An exponential [function], when it is squared, becomes a parabola?

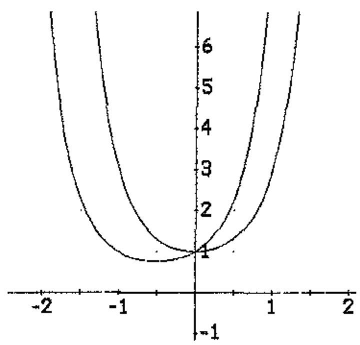

Figure 7

Thus, to find the minimum of the function $y = 3^{(x^2 + x)}$, they first attempted to use the formula $x_v = -b/2a$, for the vertex of the parabola $y = ax^2 + bx + c$. Although the minimum of the function $y = 3^{(x^2 + x)}$ coincides with the minimum of the quadratic function $y = x^2 + x$, which is the exponent, the reasons the students were inclined to use this formula were associated with the parabolic character attributed visually to the graph of the exponential.

Conclusions

In the previous section, I described an *algebraic approach* and a *visual approach* in the thinking processes of calculus students. Such descriptions were elaborated from the work the students developed in a computational environment.

The examples presented in that section show how the media could influence the students' mathematical thinking and provide evidence that algebraic and visual approaches are neither exclusive nor disjoint. The same person can work with an algebraic approach or a visual one, depending on the situation. However, what seems clear is that students who prefer algebraic approaches can experience difficulties or discomfort in the computer environment, preferring a medium that has formed part of their cognitive ecology for some time: paper and pencil

Some examples show the importance of working with multiple representations and the relations that link them, to be able to connect domains that would otherwise remain separate; if connected, they generate more ample and complete mathematical understandings. In addition to the need for co-ordination between multiple representations, the introduction of the computer into the cognitive ecology of students also suggests the need for inter-media co-ordination, which allows moving from one medium to another, taking into consideration the individual characteristics of each one. The inter-media co-ordination is as indispensable as the work with multiple representations in the thinking collective of which the computer forms a part

The reflections and elaborations presented in this article are intended to look for equilibrium between the visual and the algebraic in mathematics education. The emphasis given to rules and techniques in the teaching of calculus, for example, that is an emphasis on the algorithmic and algebraic manipulations, can be challenged precisely with the use of the computer. Paradoxically, the computer that is regarded the *prototype of the algorithmic* [2] has come to help in order that this aspect is not so emphasized in mathematics education

The emphasis given to the analytic in the teaching of mathematics in general, and that of calculus in particular, the consequent need for exact and single solutions and the limited value attributed to the visual and the experimental in mathematics education could be considered, in the terminology of Levy, as ecological effects. These effects are related to the predominant technologies in mathematics classes and, also, in the very process of production of mathematics, that is, the oral and the written. According to the remarks previously presented, a re-thinking of mathematics education is called for, in order to take into consideration the presence of new intellectual technologies in academic cognitive ecologies

Notes

[1] David Mumford was granted the Fields Medal in 1974 for his research in pure mathematics.

[2] This expression was used by Dr Nilson Machado, Faculty of Education, University of São Paulo, in a personal communication.

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