A Research Programme for Mathematics Education (I)

In March 1981 I sent a letter, excerpted below, to a round dozen mathematics educators. Three replies are printed following the letter. The (I) in the heading is optimistically inserted to say that other statements will be printed when they arrive. Any reader may now read the letter as addressed to himself or herself and make a contribution to the symposium. I hope that the variety of the projects outlined, together with the implied visions and aspirations of the writers, will help all of us become more clear about what mathematics education research could yield.

"I suggest publishing in FLM a number of short articles (statements, communications, proposals . .) each under the title *A research programme for mathematics education*. Depending on the number received and when I receive them, I will put 5 or 6 in a single issue, or 3 or 4 in each of two consecutive issues, or . .

The words "research programme" have a deliberate echo of Kuhn, Lakatos, etc., but I don't insist on the meaning these writers give. You would be free to write a "research programme" for some particular problem within mathematics education or, in a larger sense, outline a programme which research should be concerned with. In either case you will be giving readers a picture, your picture. Out of the juxtaposition of a variety of pictures may come sufficient material for a vigorous and productive discussion. That, at least, is my hope.

Each programme should be presented as definitely as possible. Indicate what it could be expected to deliver, and its controlling "heuristic". I suggest you do not use much space, if any, for justification of your plan, but get on with describing it. Do not feel constrained by current research paradigms, but be realistic in the sense that you believe your programme could be successfully pursued if enough people were committed to it.

I think 1,000 — 1,500 words will be enough. Will you help? I very much hope so. David Wheeler"

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From Pearla Nesher

A research programme, in my terms, will be an articulation of a general problem which directs many other more specific research questions, each of which will give a partial answer to the general problem. In a framework of such a programme different people from different disciplines may work on quite different questions yet they are all aimed at progressing in the understanding of the general problem. This implies that efforts are made to share concepts, terminology and findings in a way that makes the efforts susceptible to continuity and accumulation. Characteristic of such a programme is that it tackles a problem which seems to be broad enough to justify a long-range endeavour, and that people who suggest it believe that significant progress can be achieved in working in this direction.

My positive attitude towards a research programme is obvious from the way I have just described it. It should not be understood, however, that I think that there is room for just *one* research programme in math education. On the contrary, I believe that there is a room for several research programmes. The only point I would like to emphasize is that there is an urgent need for a genuine research programme which will be accepted by many researchers, and that efforts put into research in math education should be accumulated so as to gain continuity and progress.

The last research programme we had was the "New Math" programme, which was mainly in the realm of curriculum changes and development. We now witness a movement calling for "back to basics" which can be characterized by its negative attitude towards "New Math" rather than by having its own answers to the needs in math education. It is based more on a mood than on a research programme. This movement, however, expresses the deep dissatisfaction of the public with the present situation in math education.

The failure in learning mathematics is well documented and is in sharp contrast to the growing need in our technological society for adults who can handle mathematics fluently. The failure in mastering mathematics and some other formal languages is also in sharp contrast to the situation in many other learned disciplines. We still do not know, for example, what is inherently different in the learning of mathematics to the learning of natural languages, which seem to be mastered by everyone.

The central problem of math education is, in my view, our almost total ignorance about the cognitive processes involved in the acquisition of mathematics. I think that this is the reason that the enormous effort put in the past into curriculum innovations did not fulfill the hopes that accompanied their introduction into schools. We do not know the relationship between the modes of instruction and the internal representation of mathematical objects and concepts. One can improve methods of teaching and advance learning, systematically, only if one knows what are the mechanisms of learning. I propose, therefore, *to concentrate on understanding the processes involved in the acquisition of mathematics as a language system*. I would like to clarify that when I speak about "language" I mean not merely the symbolic notation and its syntax but also the sense and the reference of that language: its semantic interpretation and pragmatic operation as well. That is, I include what some call "conceptual frameworks". My claim is that articulating the general problem of the acquisition of mathematics as an acquisition of a language system will enable us to compare it with the acquisition of a natural language and other human skills which seem to be examples of a successful learning, and we might gain deep insights by drawing these fruitful analogies.

In order to study this general problem of the processes involved in the acquisition of mathematics, an interdisciplinary effort should be made. The explication of what is characteristics of mathematics vs. ordinary language is a philosophical question that should be explored by philosophers, mathematicians and linguists, who will view their research question as subordinated to the educational problem that underlies the entire research programme. Linguists should participate in the detailed analysis of mathematics as a language. How children acquire knowledge in general, and language in particular and what are the developmental constraints — are typical psychological, or psycholinguistic questions There are probably sociological and sociolinguistic aspects in the leaching and learning mathematics; and, of course, research on what can be implemented in schools, in a systematic manner, calls for an educational expertise. It is true that separate studies in the areas mentioned above may already exist, but in the lack of a research programme they are not directly contributing to our understanding of why we fail in teaching mathematics.

As I see it, the general problem suggested above will have to be elaborated into more detailed and workable questions. I will mention some of them here to give the flavor of what is included in such a research programme. For example: What does it mean to "understand" in ordinary language, and what is the parallel to it in mathematics? Is there a semantic component in the language of mathematics, or is it a symbolic system that has only syntax? How does a child acquire knowledge about abstract objects in ordinary language, and what could be the parallel to it in mathematics? Is it possible to use the mechanisms observed in learning natural language for the learning of mathematics? How does one reason in ordinary language, and what parts of this reasoning are transferable to mathematics? How can the child use mathematical models in contexts described in ordinary language? The above questions should be answered in a rigorous manner by examining events of learning, possibly with the aid of computational models.

In recent years many researchers in the fields of cognitive science and artificial intelligence have attended to similar questions. The research question, however, in each case was different from the one I suggest The artificial intelligence studies are interested in the question: how can we produce a computer that will mimic a more intelligent human behavior than it does now? The cognitive scientists are mainly interested in the study of the memory and, therefore, in the structure of the internal organization and representation of any knowledge structures Although both disciplines are committed to formal descriptions of their theories, they are not committed to the study of what is needed for the acquisition of the competence to describe things formally by means of mathematics They may occasionally and in a sporadic manner study some processes involved in learning a mathematical concept, but this is only because such concepts are usually easy to describe and easy to control Therefore their answers do not always serve the questions I have suggested for a research programme in math education

I think it is in the interest of math educators to pose the question concerning the failure in learning mathematics and I believe they have to enlist other disciplines for this research programme

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From Alan Bell

We know that most of what is taught is not learned. For example, we know that although most pupils are taught to understand and operate with numbers containing several decimal places, most of them are in the end only successful with one place and with a limited range of operations. The problem is therefore how to design teaching to take account of this important fact; how, that is, to look at a proposed lesson and to recognise what aspects of the material and the experience constitute the likely residual acquisition, after the normal process of forgetting has taken its toll. To attack this problem, I propose a programme of teaching experiments of which the aim is to achieve a discrete permanent change in the level of understanding of individuals in a key conceptual task. Some such experiments have been successfully performed in relation to the Piagetian conservations; Inhelder, Sinclair and Bovet [1974] taught conservation of quantity and length, and class inclusion, and obtained in some cases both permanent gains and transfer to untaught concepts, and Gelman [1969] taught number and length conservation with spectacular success by simply presenting situations requiring the relevant discriminations and giving immediate feedback of correctness.

Nothing comparable has yet been achieved in secondary school mathematics But the time is ripe for such attempts, since there is now a rapidly increasing volume of research data describing pupils' conceptions in many mathematical topics [e.g. Hart, 1981]. The results and the test material from such research can be used to define levels of understanding on which to base the teaching experiments. We have begun some experiments on these lines; I will describe the present state of one of these.

Our work with directed number shows that a considerable number of 14-15 year olds, probably the majority, add such numbers correctly but cannot reliably subtract them. They have an understanding of such numbers which includes the addition of them by the use of the number line in the mode "point + move = point". The identification of positive or negative *points*, and positive or negative (to the right or the left) *movements* exists. This corresponds to the teaching they have experienced But other untaught notions exist, apparently as spontaneous generalisations from their experience; some of these are correct and some not Adding two negatives is seen as adding two quantities of the same kind and obtaining a total quantity also of the same kind. Subtracting a smaller negative from a larger one (e.g. $^-10 - {}^-3$) is seen in the same way. For a number of pupils, this completes the set of correct conceptualisations. Subtracting a larger negative from a smaller (e.g. $^-3 - {}^-10$) cannot be dealt with correctly in such a system, and may be assimilated to the "like quantity" notion, by regressing to the commuting of subtraction ($^-3$ from $^-10$), or to the number line addition notion, by trying to use "point + move = point"; in the latter case the minus signs signal a leftward move but the significance of there being *two* minus signs is lost These conclusions arise from videotaped interviews with a number of pupils. In subsequent attempts to teach a few pupils, we have so far failed to bring them to replacing these subtraction procedures by correct ones, though we have succeeded in enabling them to see correctly how to interpret states of credit or overdrawing and transactions in a bank account, and in particular to interpret the formula "final state − starting state = increase" In this latter case, the initial misconception was the inability to reverse the temporal order of events (start → change → final state) to fit their order of appearance in the given relation.

The theory on which the teaching is based is that (i) conceptualisation of subtraction, a meaning for it, is what is lacking and is needed; (ii) this may be provided by (a) exposing some relevant aspect of existing knowledge (money) or (b) sufficient work with a suitable concrete representation (e.g. black and white cubes) to enable this to remain as a model to which future reference can be made; (iii) an understood means of immediately checking correctness has to become habitual, so that misconceptions can be exposed and extinguished rather than being reinforced by being performed without correction (correction "tomorrow" is rarely successful); (iv) the correct conceptualisation and procedure must link directly with the pupil's thought process as he faces the question (e.g. $^-3 - ^-10$); his existing train of thought must be diverted into a correct path

So far our teaching has explored only (i) and (iia) It may be that incorporating the remaining principles will result in successful long term learning of subtraction. The intention is to pursue the experiment until this is reliably achieved.

The videotapes of the pre and post test interviews and of the teaching have been transcribed and closely studied, and we feel that at last we are beginning to be able to relate aspects of the teaching to changes in the pupils' conceptions as shown in the interviews. I say "at last" because we began this research project with whole-class teaching method comparisons which, while they yielded quantitative "results", gave us little insight into the relation between what was and was not learned and its relation to the teaching offered The present method looks much more promising

References

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From Caleb Gattegno

Since mathematicians have found only three "mother structures" — order, algebraic and topological — and have managed to produce *most* of classical mathematics by considering the effects of each on each other and the resulting structures on each other (cf. *Traité de mathématique* by Nicolas Bourbaki), it seems of paramount importance for mathematics education to manage to spell out the mental structures and the mental activities which go to form these mathematical structures.

When Piaget tried to find out how order, inclusion of sets, one-to-one correspondence and the definition of a unit, as mental percepts, merged to produce the natural numbers he seemed close to undertaking this work. But it did not affect mathematics education Piaget "theorized" in a skilled and simple way but remained at a distance from the children (i e. very young children) who must actually take the steps that will one day give them entry to the dynamic set of integers. Only if we find out what children actually do in order to own, consciously, the various mental structures that they will, also consciously, merge to produce each of the elementary mathematical structures, shall we be in a position to influence education in a manner that will be significant.

"Stressing" and "ignoring" are the two components of the act of abstraction. The order structure only becomes mathematically meaningful when algebra merges with it to give the transitive property of order Is order, then, a true "mother structure" or is it only a mathematical abstraction? Is not order the awareness of the presence of "irreversible time" in all mental activities, forcing any conscious person to know that events in awareness either follow or precede each other?

The awareness of co-presence is concomitant with the phenomenon of seeing. It is the awareness of how sight operates on the infinitudes of simultaneous impressions to generate a kind of orderliness that forces on the reflecting mind the awareness of topological structures

Awareness of change, of transformation of the content of the mind, leads to awareness of the mental dynamics that are found to be present in all mental activities. Is it not then legitimate to consider all "algebras" to be present in dynamic human minds before some people took the trouble to formalise singular forms as they met them in special experiences? Algebraic operations must be seen as virtual mental actions, and operations upon operations appear as soon as different operations merge in consciousness. This kind of mental functioning is required in order to speak the language of our environment, and we do that before we are three years old. Hence algebra exists in the form of the dynamics behind mental structures long before it can become a "mother structure" in the mathematical sense

A collective research programme that started from the recognition that the mental structures required by mathematics are the endowments of seeing, hearing, perceptive people becoming aware of their own functionings would make a decisive difference in the way we construct a mathematical curriculum and present its contents to the students. At this time the technology of microcomputers with graphic components could make research findings immediately available as syntheses of the temporal, visual and dynamic components of all human minds, yielding mathematical structures owned by their users.