NUMBER WORK: TEACHERS AS EXPERTS WHO CAN THINK LIKE NOVICES

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> "A Math 30-1 (Grade 12) student asked me why we can't imagine imaginary numbers."

These brief descriptions of 'pedagogical impasses' were offered by teachers in response to the prompt, "Tell us about a time you were teaching mathematics in which you found yourself stymied by something a student said or did."

Those teachers, all from the same school, were taking part in a longitudinal project aimed at improving the mathematics learning experiences of their students in Grades 1 to 8. Then in its seventh year, the project had multiple strands. The main focus was on enacting a more tentative and attentive mode of mathematics teaching, and that's what drove the request for impasses. Collectively, we were wondering about what students might be saying that just was not being heard. Yet.

The notion of 'pedagogical impasse' is not entirely new. There have been rich examinations in the research literature of moments in learning and teaching when momentum is lost. In the 1990s, the phrase 'epistemological obstacles' rose to some prominence as a means to account for many such events. As Sierpinska (1994) defined them, epistemological obstacles are

> ways of understanding based on some unconscious, culturally acquired schemes of thought and unquestioned beliefs about the nature of mathematics and fundamental categories such as number, space, cause, infinity […] (p. xi)

As developed below, I am confident the above impasses were rooted in epistemological obstacles. But, that said, I believe there to be an important difference between obstacles and impasses. An epistemological obstacle is a cultural pothole for learners—an identifiable aspect that can be anticipated by teachers. A pedagogical impasse is more amorphous. It arrives as a sensation of not being able to comprehend a sentence even while understanding every word. Phrased differently, a pedagogical impasse typically raises questions for which the answers tend to be epistemological obstacles.

Certainly, that is how most of the few dozen teachers who participated in the project experienced the impasses they related. The telling of each impasse drew nods of familiarity and smiles (or sighs) of sympathy from the other teachers. But at no time did these narratives trigger discussions of likely epistemological obstacles—which, in fact, prompted me to grow more and more uneasy as the session unfolded. Indeed, I could not resist asserting something that I thought should be obvious to all: across multiple concepts and multiple grades, every incident had to do with the same obstacle: they all revolved around constrained understandings of *number*.

Experts who can think like novices

How teachers listen to their students is something I have been studying for a long time (*e.g.,* Davis, 1996). A decade ago, my attentions shifted more specifically to the relationship between teachers' mathematics knowledge and their inclinations to attend to what students say.

Teachers' disciplinary knowledge is a notoriously difficult topic to study, and partly for that reason, my working definition of the phenomenon is that it is "what an expert needs to know to think like a novice" (*cf.* Davis & Renert, 2014). Rooted in the expert–novice literature, this characterization highlights a defining feature of expert knowledge across domains—namely, the expert's ability to recognize when a concept is appropriate, immediately, without conscious mediation, no matter the situation. Novice understanding, however, tends to be much more piecemeal, deliberate, and context-bound (Ericsson *et al.,* 2006). Experts have had time and opportunity to integrate diverse encounters into consolidated, coherent wholes. For novices, concepts often lack such coherence, and so different instantiations of the same concept can be experienced as unconnected. Consequently, situations sometimes arise in which experts cannot distinguish among elements that novices cannot reconcile. I interpret the pedagogical impasses introduced at the start of this writing in precisely these such terms—that is, as moments in which teachers were unable to parse their expert knowledge of 'number' in ways needed to make sense of learners' quandaries.

That mathematical concepts are regular sites of pedagogical impasses is unsurprising. As has been argued and researched by phenomenologists and cognitive constructivists for more than a century, concepts are not static forms or unified wholes that can be shared among knowers. Rather, they evolve across experiences and interpretations that are specific to individuals. Hence, pedagogical impasses should not be met as mis-takes, but as divergent construals.

As mentioned, I experienced my own pedagogical impasse as a session leader. For me, it seemed obvious that everyone was talking about number. However, when I artlessly said so, the shaking heads and hasty objections made it clear that few, if any, participants were able to put together what I had. Recognizing that I had occasioned a pedagogical impasse in a session devoted to making sense of pedagogical impasses, I floated the suggestion that the group might use 'number' as a focus for the year's shared inquiry.

Over our years of working together, 'shared inquiry' had come to involve deconstructing and reconstructing a concept, attending to the many threads of experience and noticing that are braided together in mathematical ideas. In terms of experts and novices, within these inquiries we worked together to analyze our now-consolidated (expert) understandings, endeavoring to recover some of their constitutive elements—instantiations, applications, and other encounters. In that spirit, we began our year-long commitment to interrogating number by reading Lakoff and Núñez's (2000) explication of 'four grounding metaphors of arithmetic'—namely, object collection, object construction, using a measuring stick, and motion along a path. These active, body-based notions, they argued, provide sufficient ground to derive and deploy increasingly complex mathematical constructs, ultimately rendering even the most abstract formulations comprehensible. So oriented, participants reviewed classroom resources such as textbooks, teachers' guides, manipulatives, and games, aiming to identify the metaphor(s) that are foregrounded for learners. This work was accomplished in grade-based group settings.

Once the teachers had sufficient time to generate preliminary analyses of the tools and resources in their classrooms, they provided grade-by-grade reports. Unsurprisingly, every group noted inconsistencies and slippages—that is, instances in which inappropriate metaphors were invoked through images or vocabularies, and thus opening possibilities for pedagogical impasses.

This work occupied most of the professional learning time set aside for mathematics. Through the year, we revisited the pedagogical impasses that the teachers brought to the start-of-term session, looking to answer the question of just how useful more nuanced understandings of number might be for teachers.

Resolving some of the impasses

Some more fine-grained detail on Lakoff and Núñez's four grounding metaphors of arithmetic would be useful before getting into the teachers' follow-up discussions about their reported teaching impasses. Owing to my focus here on the concept of number within school mathematics, I limit the analysis to entailments for conceptions of number afforded by the grounding metaphors. Some additional entailments for topics beyond number are presented later, but it is important to note that the webs of association and the mathematical power that arises in these webs vastly exceed what is offered here.

In Figure 1, the four grounding metaphors are interpreted in terms of the sorts of number-related questions that learners might ask or be asked. Each is phenomenologically distinct—that is, each invokes a specific cluster of experiences, gestures, and associations. In turn, each calls forward a distinct sense of number.

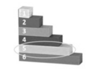
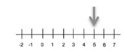
Lakoff and Núñez's Grounding metaphor	Associated metaphor of number	Matter addressed (situation modeled)	An instantiation of '5'
OBJECT COLLECTION	NUMBER AS COUNT	How many?	
OBJECT CONSTRUCTION	NUMBER AS SIZE	How big?	
USING A MEASURING STICK	NUMBER AS LENGTH	How long?	
MOTION ALONG A PATH	NUMBER AS LOCATION	Where?	

Figure 1. Four instantiations of number, associated with Lakoff and Núñez's four grounding metaphors of arithmetic.

As I develop below, these metaphors proved sufficient for making sense of the pedagogical impasses presented at the start of the writing (in addition to many others presented in the session). However, they are not sufficient to span every encounter with number in school mathematics.

Grade 2, making smaller

> "When I was brainstorming different ways of saying 'subtract' with my Grade 2s, one of the children got angry that 'make smaller' was on the list. He argued that '*making smaller* can't be *subtracting* since five will still be five no matter how small you make them'."

From the vantage point of different metaphors for number, there is a fairly obvious and highly likely interpretation of this learner's quandary. It would appear that the child was thinking about number strictly in terms of cardinality—number as count. For him, the actual count of things could not be subjected to the physical process of making things smaller, but the things that were counted could be. In terms of literal meanings, the child was using the entwined notions of *size* and *make smaller* consistently, and the teacher was not. That does not mean that the teacher should have immediately perceived the inconsistency, however; rather, as with most expert knowers, she was likely locked in what Rorty (1991) called a 'dead metaphor'—one that has lost its original figurative power by being subsumed into the grander web of associations. It was an instance of coherent expert knowledge that smoothed the rough inconsistencies of its roots.

Grade 5, doubling π

> "I teach Grade 5. Last year, when we were looking at the formula for circumference of a circle, $C = 2\pi r$, one student knocked everything sideways when she asked, 'If π goes on forever, how can you times it by 2?'"

At first hearing, this child's observation that "π goes on forever" might seem indicative of the metaphor, number as length. It is not an instance of this metaphor, however, because π in this interpretation would be the length of the interval between 0 and 3.14159… on a number line. As for doubling the value, it is easy to imagine making two hops of that length.

What the student was likely referring to, then, was not the number, but the symbolic representation of the number. While I cannot be certain, I would guess that she had applied a number-as-count metaphor to the digits in that representation and was troubled by the logical impossibility of applying a digit-by-digit algorithm for multiplication to a number with infinite digits. If correct, then this impasse underscores an issue that came up in the previous impasse. There seems a strong disposition among young learners to treat ‘number’ and ‘numeral’ as synonyms—a conceptual move that, I worry, renders number a meaningless operator far too soon in learners’ mathematical experiences. I return to this issue later, when I look at interpretations of number beyond those considered by Lakoff and Núñez.

Grade 12, imagining imaginaries

> “A Math 30-1 [Grade 12] student asked me why we can’t imagine imaginary numbers.”

I will defer to Lakoff and Núñez’s more nuanced explication on this one, sufficing here to highlight that the metaphor number as location offers a way through this common impasse. Briefly, one must first invoke the commonplace interpretation of ‘multiplication by -1’ (*i.e.,* by $(-1)^1$) as a 180°-anticlockwise rotation of the number line about 0 (mapping a onto $-a$). So framed, multiplying twice by -1 (*i.e.,* by $(-1)^2$) is interpreted as two 180° turns, and the resulting 360° rotation of a number line about 0 will map each number onto itself. Accordingly, multiplying by the square root of -1 (*i.e.,* $(-1)^{1/2}$) is interpreted in this frame as half a 180° turn, or a 90° anti-clockwise rotation. That rotation generates the complex plane, the horizontal axis of which comprises the Real Numbers and the vertical axis of which comprises the Imaginary Numbers. In other words, any imaginary number can be imagined as its location on the vertical axis of the complex plane.

Minding the gaps

As mentioned earlier, one aspect of our inquiry was a grade-by-grade inventory of the metaphors invoked within the classroom resources. In groups, according to the grades they taught, teachers looked across vocabulary, images, and applications to generate a rough mapping of how numbers were framed for learners at different levels. Their initial impressions are presented in Figure 2—which, unsurprisingly, suggests an almost-exclusive emphasis on number as count in the first years of school mathematics, giving some way to a much-more-varied (and, arguably, conflicted) landscape dominated by number as count and number as length by the end of the middle grades.

The group was unsatisfied with this table, however. In fact, they started to express frustration almost immediately when the inventory was undertaken. By the time they were ready to give reports, every person in the room was convinced that the four metaphors we had listed in the chart,

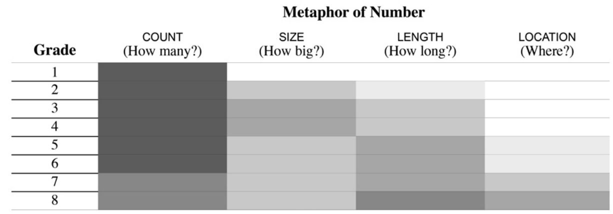


Figure 2. Teachers’ initial impressions of relative emphases of varied interpretations of number in classroom resources. (The clear-to-dark shadings indicate absent-to-heavy emphases.)

based on Lakoff and Núñez’s four grounding metaphors of arithmetic, were insufficient. Other interpretations of number seemed to be at play in school mathematics. In particular, they noted, four frequent encounters with number did not seem to be included in Figure 1’s categories. Two of these instances address matters of ‘Which?’ and ‘How much?’, and the group settled on notions of number as rank and number as amount for these situations. Another common encounter with number was identified to be associated with ‘What?’ questions, which linked to notions of number as reification or number as numeral.

Which?—number as rank

In our jurisdiction, students are formally introduced to the distinction between cardinal numbers and ordinal numbers early on. Our group had initially overlooked the ordinal numbers, owing to the naïve assumption of a one-to-one correspondence between our identified metaphors of number and Lakoff and Núñez’s grounding metaphors of arithmetic. However, that omission was immediately evident when we delved into curriculum materials. Many lessons in the first few grades rely on the distinction, and one of its primary markers is a shift from questions asking ‘How many?’ to questions asking ‘Which?’, associated with tasks requiring learners to attend to, for example, positions in groups and locations in sequences.

We settled on the notion of number as rank to refer to this instantiation. Other options included place and position, but we worried these were too similar in everyday meaning to the location. As well, we felt that the notion of rank better served to underscore the discrete character of ordinal numbers.

The notion of rank can also be used to highlight an interesting difference in the way that cardinal and ordinal numbers are encountered by learners. Mathematically, cardinals and ordinals can be defined in terms of one another for finite numbers. Experientially, however, they are not the same. Cardinals are used to describe whole sets; ordinals are used to refer to single members of sets. However, in naming a part (or a rank), an ordinal number implies a whole. Following Coles and Sinclair (2017), it would seem that number as rank might be better considered a metonym than a metaphor. Thus, in the case of ordinal numbers, a statement about ‘item n’ implies a set with a cardinality of at least n.

How much?—number as amount

In the early stages of our inquiry, the discrete–continuous distinction emerged as a useful and frequently invoked idea.

While all participants would have encountered it somewhere in their histories with formal mathematics, it was received as new by most—and, in fact, was a site of struggle for many.

In an effort to render the distinction accessible, I sidestepped formal definitions and suggested two rules of thumb:

> If the situation involves counting, it's discrete; if it involves measuring, it's continuous.
>
> If it's grammatically correct to say 'fewer' in the situation, it's discrete; otherwise, it's continuous.

While imperfect, those guidelines served us well across the analyses of number behind Figures 1 and 2. However, they are inadequate around a few applications that are frequently encountered in grade school mathematics, especially ones involving money. As a 4th-grade teacher expressed the issue, during our grade-by-grade inventory of interpretations:

> They're [*i.e.,* situations involving quantities of money] discrete, right? We *count* money. But we don't say, "How *many* does this cost?" We say, "How *much*?" And we never use 'fewer' when we're talking about money.

Her colleague added:

> We noticed kind of the same thing with the way fractions are introduced in the Grade-4 book. Most of the exercises are based on counting—like [holding up the exercise book] this picture where five out of six balls are colored in, that asks, "What fraction is shaded?" That's not a 'How many?' or a 'How big?' question, it's a 'How much?' question.

Much more time was given to mulling over the matter, but these teachers' remarks seem to sum up an important experiential truth: in many contexts and occasions where questions of 'How much?' are asked, number can be deployed as discrete but sensed as continuous. In our analyses, the most common of these situations involve large quantities and/or discrete fractions (including terminating decimals).

Since we were immersed in a discussion of orienting metaphors, it was no surprise that interests turned to identifying an analogy fitted to this situation. Suggestions immediately gravitated to notions of 'piling up' and 'clustering', at which point some hasty googling of original word meanings prompted us to suspect others had long ago grappled with a similar issue. It turns out that English has several terms that invoke precisely the same images that the teachers had suggested to address matters of 'How much?', such as *amassing* (*e.g.,* a fortune), *amounts* (*e.g.,* owed), and *accumulating* (*e.g.,* parts into a whole). According to the Online Etymology Dictionary (etymonline.com), all three of these have to do with mounding bits into larger unities:

> *amass* derives from the Old French *à* 'to' + *masse* 'lump, heap, pile';
>
> *amount* derives from the Latin *ad* 'to' + *monten* 'mountain';
>
> *accumulate* derives from the Latin *ad* 'to' + *cumulare* 'heap up'.

The group thus settled on the metaphor number as amount for this new category.

Importantly, there is no suggestion here that number as amount has the same epistemic status as the metaphors for number presented in Figures 1 and 2. Rather, as hinted by the ambiguity experienced around the discrete–continuous distinction, this metaphor is better seen in phenomenological terms than mathematical terms. It is something encountered in day-to-day applications (mainly situations involving discrete fractions, such as money)—and so while it does not appear to be integral to concepts in pure mathematics, it is certainly important in school mathematics. We also concluded that it likely plays an important conceptual role in bridging discrete, quantity-focused (number as count) and continuous, magnitude-focused (esp. number as size) conceptions of number.

Indeed, a distinct quality of the instances of number as amount that we examined was that they always involved two components: a quantity and a magnitude. Three such examples are '5 dollars', '5 million', and '5 ninths'. In each case, the leading value is likely to be experienced as a count and the latter as something more like a measure. Perhaps then, like number as rank, number as amount may be more appropriately understood as a metonym than a metaphor—given that the leading information in amounts (*i.e.,* the '5' in each of the examples just given) focuses attentions on the part while deflecting considerations away from the much-more-significant whole.

What? number as numeral and number as reification

A somewhat more surprising observation for the teachers was how early and how often classroom resources invoked numbers and posed questions in complete absence of interpretive referents—that is, asked 'What?' questions in which numbers were presented as naked operators. Practice exercises devoid of metaphorical anchors were already evident in first grade, and they represented the most common variety by the middle school years. Indeed, the steady increase in proportion of 'What?' questions was taken by the teachers as evidence of a systematic process to wean learners from specific, meaningful-but-necessarily-limiting interpretations of number.

In some regards, this progression should have been expected, especially given that the group's analyses of several pedagogical impasses homed in on children's habits of equating 'number' and 'numeral' in even the lowest grades. Clearly, something is pressing learners toward seeing numbers and things in and of themselves. Nonetheless, we experienced the presence and the press of number as numeral as disconcerting. As was evidenced multiple times in our concept study, a premature compulsion to treat numbers as only symbolic operators—or, worse, as symbols—can debilitate efforts to interpret and extend mathematical concepts. Conversely, failure to elaborate number into a symbolic operator might be similarly debilitating at higher grades, as illustrated by other pedagogical impasses in which learners' interpretations of number did not keep pace with the increasingly abstract nature of the concepts under study.

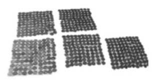
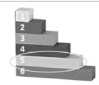
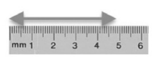
Metaphor, Metonym, or Metaform of Number		Matter addressed (situation modeled)	Associated Grounding Metaphor(s) of Arithmetic	An instantiation of '5'	How 'less' and 'greater' tend to be expressed: less	greater	How addition tends to be seen	Some encounters/ contexts/ uses	Numbers made available
COUNT	quantity	*How many?* (discrete)	OBJECT COLLECTION		fewer	more	combining sets	counting; sorting; clustering	Whole Nºs; Natural Nºs; cardinals
RANK	quantity	*Which?* (discrete)	OBJECT COLLECTION		ahead	behind	changing rank	sequencing; ranking; grading	ordinals
AMOUNT	quantity	*How much?* (discrete, but experienced as continuous)	OBJECT COLLECTION & CONSTRUCTION		less	more	pooling amounts; amassing	pricing; accounting; apportioning	large numbers; discrete fractions
SIZE	magnitude	*How big?* (continuous object)	OBJECT CONSTRUCTION		smaller	larger	growing; joining pieces	assembling; sharing; ratios	continuous fractions
LENGTH	magnitude	*How long?* (continuous dimension)	USING A MEASURING STICK		shorter	longer	extending; moving farther	scale-based measuring; traveling	Rational Nºs; Irrational Nºs; Integers
LOCATION	entity	*Where?* (discrete site in continuous space)	MOTION ALONG A PATH		left of (or, lower)	right of (or, higher)	shift in location	locating; scheduling; reading time	Real Nºs; Imaginary Nºs; Complex Nºs
REIFICATION	entity	*What?* (disentangled from physical instantiations)	All of the above	5	<	>	binary operation	symbolic manipulation; computing	Any/all of the above
NUMERAL		Any of the above	None of the above	5	<	>	procedure	repetitious practice	N/A

Figure 3. Eight instantiations of number in school mathematics, along with some illustrative entailments.

There is no quandary here. Humans' understandings of number are both embodied (*i.e.,* rooted in bodily based experiences) and embedded (*i.e.,* called forward in culturally meaningful situations). It is entirely reasonable to expect school mathematics to be structured in a way that draws on and nurtures the former while anticipating and enabling the latter. The issue is not whether school mathematics should channel learners toward a consolidated concept of number, but how and when it should happen. Such matters, in turn, can only be settled through nuanced appreciations of how integrated concepts emerge and what the integrated concept is expected to do.

Two notions drawn from cognitive science proved helpful to the group while engaging with these matters. Firstly, one of the participants called attention to Fauconnier and Turner's (2003) research into conceptual blends, which they characterized as combinations and integrations of existing concepts to create new or more powerful concepts. In this regard, we found Danesi's (2014) notion of *metaform* useful. Contrasted with a metaphor, a metaform is abstract distillation—a fusing that foregrounds common functional elements while suppressing idiosyncratic and potentially dysfunctional elements. In the process, the metaform can be experienced as non-spatial and acausal—as an idea that is unencumbered by the interpretive specificities of a metaphor or a cluster of metaphors. In this sense, a metaform of number would be what Hilbert (1928) dubbed an 'ideal object', which "in themselves mean nothing but are merely things that are governed by our rules" (p. 470).

"So, basically we should be drawing attention to metaphors to teach a metaform so that students don't have to rely on the metaphors," one participant summed it up, to the general approval of the group. His thought prompted the suggestion from another participant that we should name the metaform under discussion, in order to distinguish it from the clutter of meanings for number that we had encountered. Ultimately, we settled on number as reification—as an abstraction that is treated as something real.

Three other choices figured prominently in our protracted discussion of what to call the metaform of number: numeral, object, and operator. The first two were rejected because, although we aimed to flag the 'thing-ness' of the metaform, we also wanted to signal its emergent character. The third, number as operator, was initially compelling because of its current prominence in efforts to incorporate computational thinking into school curriculum. In computer-coding contexts, an *operator* is a logical symbol. That is, an operator is not simply a numeral; it represents an action or process. While that particular meaning seemed fitting, we decided to set it aside because of the explicit and deliberate meaninglessness (in phenomenological terms) of computer- and computation-based number usage.

Revising the maps

With these four additional interpretations of number distinguished and named, the group undertook to elaborate Figure 1 into Figure 3, the contents of which hint at considerably more discussion and debate than I have reported here. (Note that number as numeral is set off in grey at the bottom of Figure 3. Because this instantiation lacks interpretive anchors,

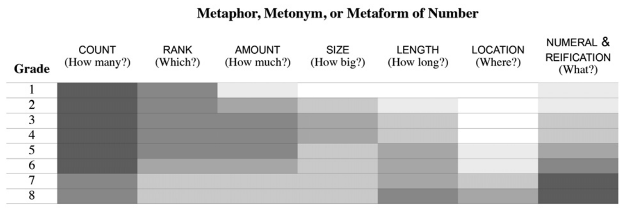


Figure 4. Teachers' elaborated impressions of relative emphases of varied interpretations of number in classroom resources. (The clear-to-dark shadings indicate absent-to-heavy emphases.)

participants felt it important to distinguish it from the others, even while they argued its persistent presence in their teaching experiences necessitated its inclusion in the table.)

The group also redid their analysis of relative emphases of varied interpretations of number in classroom resources (Figure 2) to include the three additional meanings (Figure 4). Adding of the 'rank' and 'amount' columns were uneventful, but compiling the 'numeral & reification' column was fraught. As participants analyzed 'What?' questions in classroom resources, they frequently struggled with the imagined intentions of textbook authors. Most often, they concluded, 'What?' questions were not framed in a way that enabled and compelled learners to consolidate their evolving conceptions of number. Rather, they most often seemed to be presented as attempts to wean learners off physical referents by ignoring (rather than inviting) processes of differentiation, bridging, and consolidation of varied instantiations. Consequently, for participants, the final column of Figure 4 points more to 'opportunities to develop number as reification' than as actual attempts to prompt learning in that direction.

And so ...?

It goes without saying that one might expect a strong emphasis on pragmatics when engaging with educators. It was thus no surprise that the teacher participants pushed our discussions toward recommendations for action, ultimately articulated as a series of principles to guide mathematics teaching. Their advice to themselves, when dealing with number-related topics, revolved around being attentive about which instantiation(s) are being invoked and to possible needs for interpretive bridging when more than one instantiation is in play. There was general agreement that the ultimate goal is a nimble, consolidated-but-flexible metaform of number that is enabled by the richness of diverse instantiations, but unencumbered by the limitations of any singular interpretation.

With those principles as a backdrop, teachers were primed to look for incongruent or inappropriate interpretations of number when new pedagogical impasses were reported at the most recent start-of-term session, which took place one year after the one described at the start of this article. Here are two impasses that came up:

> "I found it difficult to get my seventh-graders to measure angles. They can't seem to figure out how to use their protractors properly, no matter what I do. They don't place it right, or they read the wrong scale, or they can't figure out if they need to go higher or lower if the angle doesn't land exactly on the markings."

> "My students in Grade 8 struggle with subtracting integers. They can follow the rule, but no one seems to get why 'adding the opposite' makes sense."

I will spare the details, but these and other instances invited rich discussion and provocative suggestions. Regarding use of protractors, for example, it was suggested that students should be alerted to think about number as length—that is, to view the scales as curved number lines, thus channeling attentions to what those lines are curved around, to which scale should be used, and so on. Regarding integers, it was quickly evident that this teacher's lesson on integer addition was grounded in a number-as-count metaphor, prompting the suggestion that a blend of number as length and number as position might support more fluid understandings of operation.

Of course, these discussions afforded no direct insights into participants' classroom practices. I have no idea whether any of them, when encountering an actual pedagogical impasse, would be inclined to set aside frustration and think about whether muddled interpretations of number might be at play. But I can say that these start-of-term discussions were about paying attention to what learners might bring to the situation. That is, the main concern was not with what students do not know, but with how their current habits of interpretation might condition what they are able to perceive. In terms of the inquiry's theme of 'what an expert needs to know to think like a novice', and with regard to my personal obsession with the relationship between teachers' mathematics knowledge and their inclinations to attend to what students say, I am confident that participants had come to understand aspects of mathematics in ways that, ultimately, will contribute to their students' learning.

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