YOUNG STUDENTS' FORMS OF REASONING ABOUT MULTIPLE QUANTITIES

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Justin, a Grade 6 student, is exploring the simulation modeling the water cycle (Figure 1). It allows him to change the air temperature, mountain temperature, land temperature, lake temperature and relative humidity, and to observe the effects on quantities such as the rate of evaporation from the lake, the amount of precipitation (rainfall) and the amount of runoff.

Justin moved the lake temperature slider and argued that "the lake temperature if it's like very high it can evaporate, and the rain that falls could be like more" (Figure 2). Justin's statement shows that he is coordinating the change in three quantities: lake temperature → evaporation → precipitation.

In this article, we define forms of students' reasoning around variation, especially in cases like Justin's in which multiple quantities vary together, and we show how students as young as in Grade 6 can engage in complex forms of variational reasoning in suitable contexts.

Figure 1. The Water Cycle simulation.

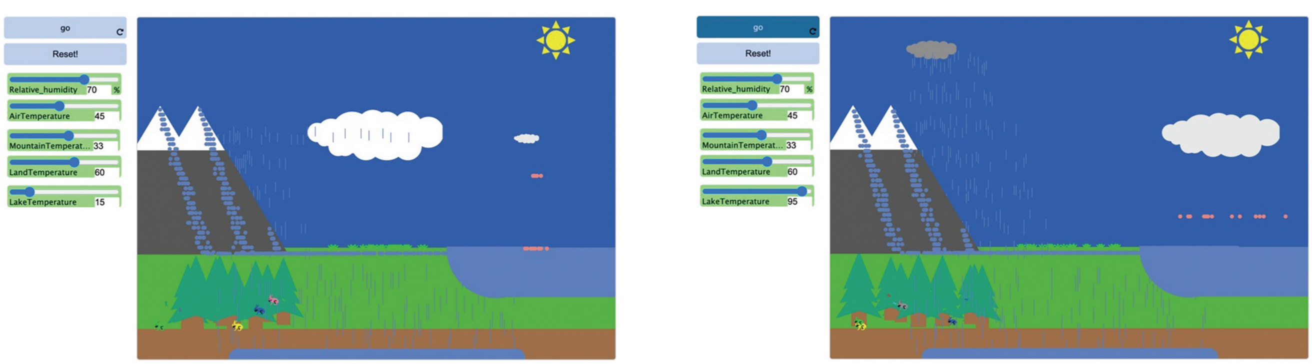

Figure 2. When the temperature increases, there is more evaporation (dots) and consequently there is more rain and more runoff.

Variation, covariation and multivariation

Quantities, as described by Thompson (1994), are measurable conceptual attributes constructed by an individual when conceiving a quality of an object. For instance, during the exploration of the Water Cycle simulation students identified quantities such as 'humidity', 'air temperature' and 'mountain temperature'. In constructing relationships between quantities, Thompson and Carlson (2017) distinguished between *asynchronous variation* which involves envisioning one variable changing, then the second, then the third, and *synchronous variation* which involves reasoning about the quantities changing simultaneously. Envisioning changes in one variable's value as happening simultaneously with changes in another variable's value is what Thompson and Carlson refer to as covariational reasoning. For instance, while exploring the Water Cycle simulation, Kelly dragged the relative humidity slider to change the value from 0% to 100% and observed that "the higher the relative humidity, the lower the rate of evaporation". Reasoning about variation and covariation of quantities has been studied extensively by the mathematics education community as a way of supporting students' learning of rate of change (*e.g.,* Johnson, 2012) and functions (*e.g.,* Paoletti & Moore, 2018). However, Thompson and Carlson argue that most of the studies examining students' variational and covariational reasoning use those constructs to frame their investigations but do not contribute explicitly to the development of the definitions of these constructs.

In analyzing the data from our study (*e.g.,* Basu & Panorkou, 2019; Panorkou & Germia, 2020), we noticed that students not only reasoned covariationally but also reasoned about more than two quantities changing simultaneously, similar to Justin's reasoning above. This finding opened the opportunity to make a significant contribution to the field since the only prior research study characterizing students' multivariational reasoning focuses exclusively on undergraduate mathematics education (Jones, 2018). Indeed, although the mathematics of change and variation are important for students in order to understand various environmental, economic and social trends of the twenty-first century, most of these concepts are introduced to students at higher grade levels, by which 90% of students are already filtered out by prerequisite classes (Roschelle, Kaput & Stroup, 2000). As a result, a large number of students do not have opportunities to engage with these concepts, in spite of the importance of this type of mathematics.

Sixth-grade students' forms of reasoning about multiple quantities

In this article, we discuss our characterizations of students' multivariational reasoning by presenting examples from data gathered from a series of whole-class design experiments (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003) conducted in different sixth-grade classrooms (11–12 years old) in a large suburban school district with a heterogenous population in the Northeast of the United States. In the district of the participating schools, 46% of the students are classified as economically disadvantaged and have an average mathematics proficiency of 29.3%. Three design experiments focused on examining scientific phenomena using interactive simulations: two on the water cycle and one on the rock cycle.

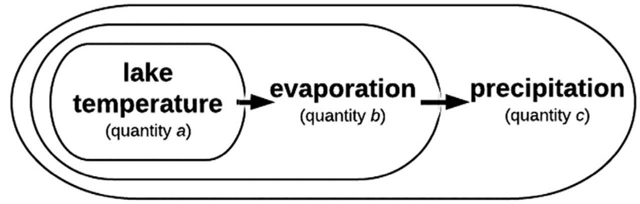

Figure 3. Illustration of nested multivariational reasoning expressed by Justin.

Similar to Jones' (2018) characterizations of multivariational reasoning by undergraduate students, in analyzing data from our sixth-grade students we identified forms of nested, independent and a form of partial dependent multivariational reasoning. In addition to demonstrating examples of these forms from our data, we also present some new multivariational forms that were not identified by Jones but were present in our data, such as nested and transitive, integrated and connective multivariational reasoning. We also discuss how the science context played a key factor in students' conceptions of synchronous versus asynchronous variation.

Nested multivariational reasoning

Jones (2018) defined *nested multivariation* as involving a chain of related dependencies. For instance, Justin's statement that "the lake temperature if it's like very high it can evaporate, and the rain that falls could be like more" reflects nested multivariation reasoning. We interpret his reasoning to show a sequential image of change illustrated in Figure 3: that the change in lake temperature (quantity *a*) impacts the rate of evaporation (quantity *b*), and that the change in evaporation (quantity *b*) affects a change in precipitation (quantity *c*).

Nested and transitive multivariational reasoning

In their nested multivariational reasoning, students sometimes reasoned transitively, in a similar manner as recognizing a transitive property. For example, when students were asked whether a higher rate of evaporation would lead to more runoff, Ian stated that more evaporation would "more likely" lead to more runoff because "if there is higher evaporation, there is more rain. If there's more rain, there is more runoff". We interpret Ian's reasoning to illustrate a form of nested and transitive multivariational reasoning, because he used the relationship between evaporation and rain, and the relationship between rain and runoff, to reason explicitly about how a change in evaporation (quantity *a*) causes a change in runoff (quantity *c*) (Figure 4).

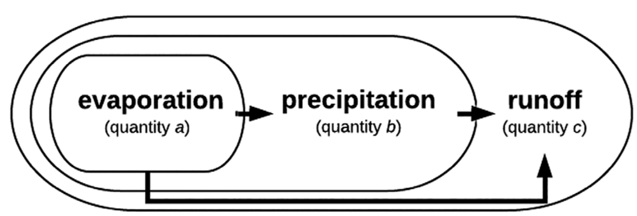

Figure 4. Illustration of nested and transitive multivariational reasoning expressed by Ian.

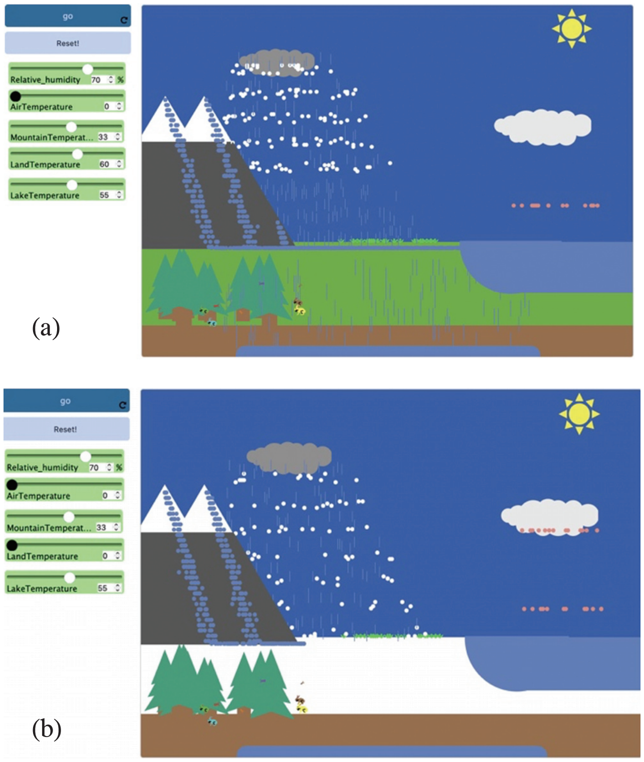

Figure 5. Amount of snow when (a) only the air temperature is decreased and (b) both the air temperature and the land temperature are decreased.

Independent multivariational reasoning

Students also exhibited reasoning similar to Jones' (2018) definition of *independent multivariation* as involving at least two quantities that are independent from each other and affect the change in another quantity. For example, while exploring the Water Cycle simulation, we prompted the students to make the precipitation in the form of snow by manipulating only the air temperature and the land temperature, and then asked them, "What conditions will release more snow?" Chloe and Justin worked as a pair to explore the conditions of snow and reasoned that "We need both of them to be cold" (Figure 5). To explain her reasoning, Chloe decreased the value for the air temperature by dragging the slider to the left, showing that the cloud was releasing a mixture of rain and snow, stating, "if you just move for air temperature, it only snows a little bit". Next, she decreased the value for the land temperature by dragging the slider to the left and showed on the simulation that the cloud released more snow which accumulated on the ground. She then explained, "But if you put it with a land temperature, it starts to accumulate in the ground and it produces more".

We interpret Chloe's reasoning to show independent multivariational reasoning because she coordinated the change of land temperature (quantity *a*) and air temperature (quantity *b*) as unrelated independent quantities with the change in snow (quantity *c*) as the dependent quantity (Figure 6). Her statements also show that it was her interaction with the simulation and our questioning that helped her to notice the two changing quantities at first, air temperature and amount of snow, and then to notice the change in snow when a third quantity, the land temperature, was added to the existing covariational relationship.

Integrated multivariational reasoning

We also noted instances where the students merged more than one form of multivariational reasoning in their statements. For instance, after students explored the Water Cycle simulation, we asked them to draw and explain their model of the water cycle. Lorna then reasoned about five quantities, namely relative humidity, air temperature, rain (precipitation), runoff and amount of water going into aquifers (infiltration):

> *Lorna* So, this is when the relative humidity and the air temperature are above 32 degrees. So then, there's gonna be a lot of rain and a lot of runoff. And the more runoff there is, like the more rain there is, there's more runoff. And the more runoff, the more water is going to go into the aquifers [*pointing at the aquifer of her model*].

We interpret Lorna's reasoning to illustrate integrated multivariational reasoning. Specifically, her first two sentences illustrate independent multivariational reasoning because she coordinates the change in two independent quantities *a* and *b* (relative humidity and air temperature) with the change in precipitation and runoff. Her last two sentences illustrate nested multivariational reasoning because she is coordinating the amount of precipitation as quantity *c*, the amount of runoff as quantity *d* and the amount of infiltration as quantity *e* (Figure 7).

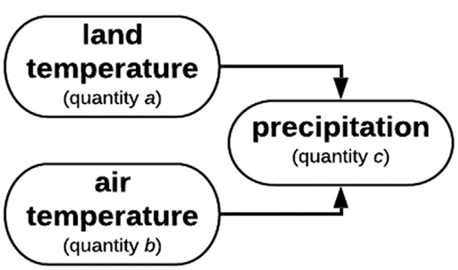

Figure 6. Illustration of independent multivariational reasoning expressed by Chloe.

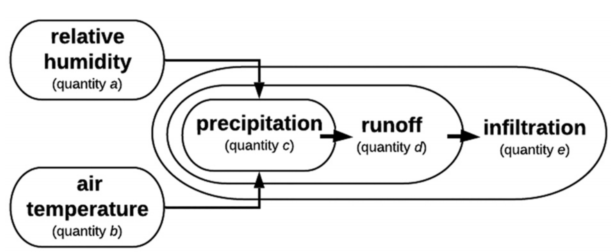

Figure 7. Illustration of Lorna's integrated multivariational reasoning.

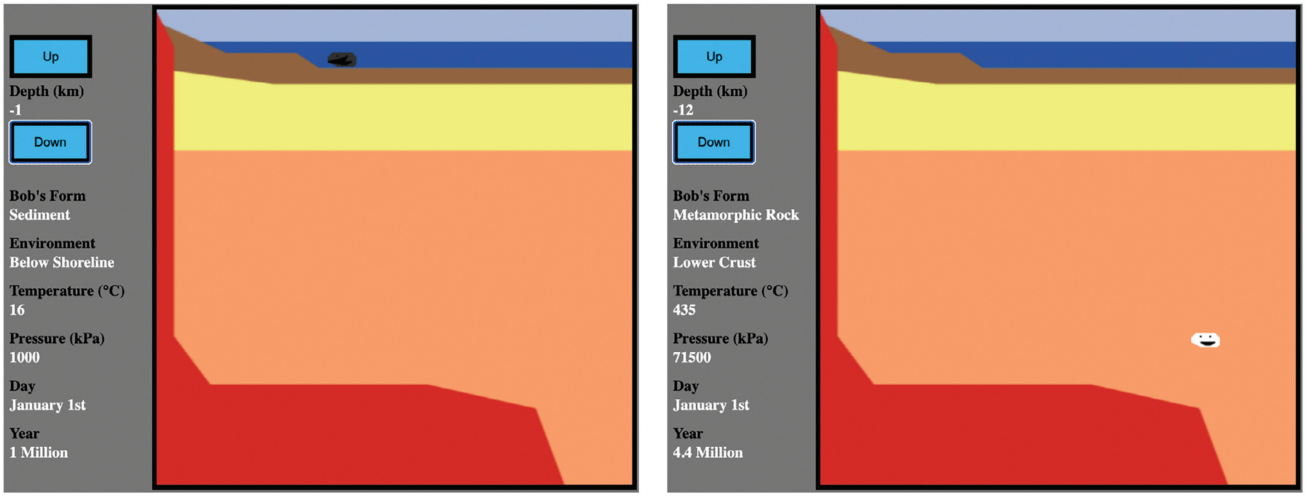

Figure 8. Increasing Bob's depth in the Bob's Life simulation and observing the change in temperature and pressure.

Partial dependent multivariational reasoning

Students also illustrated reasoning that we would characterize as a subset of Jones' (2018) *dependent multivariational reasoning*. Here we present an example from Michael's work on the Bob's Life simulation (Figure 8), which shows one of the many possible travel paths of a rock named Bob through the rock cycle. At the start of the simulation, Bob is located on the top of a volcano. Michael used the up (decreasing the depth) or down (increasing the depth) buttons to change Bob's depth in kilometers. As Bob changes his depth, Michael observed the changes in his color, form, environment, temperature, pressure and the date.

When we asked him to describe what he had noticed in Bob's Life to someone who has never seen this simulation before, Michael clicked the down button on the simulation multiple times to move Bob deeper into the ground and stated, "I would say that, the deeper, the deeper you get, the higher the temperature is, and the higher the pressure is". In contrast to Jones' (2018) definition in which all three quantities involved are interdependent, Michael's reasoning emphasizes the simultaneous change of the two dependent quantities, temperature (quantity *b*) and pressure (quantity *c*), as influenced by the independent quantity of Bob's depth (quantity *a*), while quantities *b* and *c* are not related to each other (Figure 9).

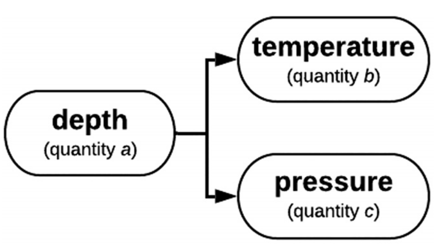

Figure 9. Illustration of Michael's partial dependent multivariational reasoning.

Connective multivariational reasoning

We also noticed instances where students related their explorations with quantities that were not part of the specific study. We refer to *connective multivariational reasoning* as the form of reasoning that connects the relationship of two or more quantities with another quantity that students bring in from their prior experiences (what we refer to as a connective quantity). Connective multivariational reasoning is always expressed together with another form, such as nested multivariational reasoning. For example, when Jared was asked to state the relation between Bob's depth and the temperature, he argued that "The farther Bob goes down, the closer he gets to the magma so the hotter the temperature gets". Jared connected the existing covariational relationship between Bob's depth and temperature to the 'distance from magma', a quantity we define as *connective* because it was not part of the quantities explored in the specific lesson.

In the above example, Jared added the connective quantity in a nested relationship as quantity *b* (Figure 10a). However, we observed that students could add the connective quantity in any place of the multivariational relationship. For instance, in discussing the relationship between lake temperature and evaporation during the Water Cycle simulation,

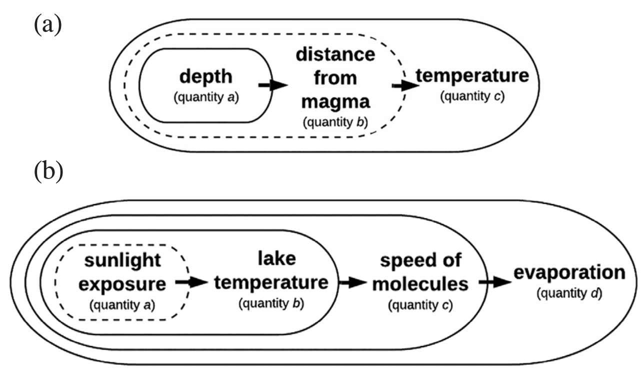

Figure 10. Illustrations of connective multivariational reasoning expressed by (a) Jared and (b) Lorna.

Lorna brought in the connective quantity of sun rays from her prior experiences at the beginning of her multivariational relationship (quantity *a*) (Figure 10b), stating "the sun rays are heating the water making the molecules go faster, and then the water turn it into gas".

Synchronous versus asynchronous multivariational reasoning

In our data we found traces of both synchronous and asynchronous variation and our analysis showed that the way the phenomenon is modeled has an impact on students' expression of synchronous or asynchronous reasoning. In science phenomena, while sometimes the change in one quantity is synchronous with the change in the other quantities, at other times this change is sequential and illustrates an asynchronous relationship. For example, in the Bob's Life simulation (Figure 8), the change in depth causes a simultaneous change in both temperature and pressure. Thus, the statements that students made during this exploration illustrated more synchronous reasoning, such as Michael's statement above "I would say that, the deeper, the deeper you get, the higher the temperature is, and the higher the pressure is".

However, in the Water Cycle simulation, the change in lake temperature causes an asynchronous and sequential change to evaporation, precipitation, and then runoff. As a result, the multivariation statements that students expressed during this exploration showed an asynchronous variation, such as Chloe's and Justin's statements above "But if you put it with a land temperature, it starts to accumulate in the ground and it produces more" or "If there is higher evaporation, there is more rain. If there's more rain, there is more runoff".

Although Thompson and Carlson explicitly consider the construction of covariational reasoning to be indicative of a conception of synchronous variation, students' statements above show that it is possible to reason multivariationally about three quantities but still think asynchronously. For instance, Justin's statement couples three quantities together (evaporation, precipitation, runoff) but he still thinks of this coupling as being asynchronous by envisioning the evaporation changing first, then the precipitation and then the runoff. These results can initiate a discussion around the possibility of interpreting such statements as illustrating both multivariational reasoning and asynchronous change.

Conclusions

Thompson and Carlson call for more research contributing directly to defining students' constructs of variation. Our investigation of how students may reason about more than two quantities makes a contribution to this call. However, more research is needed to examine young students' multivariational reasoning. To begin with, our initial goal in the study was to engineer opportunities for students to reason variationally and covariationally, therefore our tasks and questioning were restricted to only a few prompts to connect multiple quantities. However, these few prompts illustrated the importance of targeted questioning for prompting students to study the variation in multiple quantities and reason multivariationally. In the next iteration of our design, we plan to engineer more opportunities for this type of reasoning. Furthermore, future research can explore further the role of technology, and specifically simulations, as a context for this kind of reasoning. It would be also worth investigating how students may utilize the multivariational reasoning they develop through their interactions with the simulations to non-digital tasks.

Additionally, future studies may also examine further the effect that the modeling of the science phenomenon has on students' multivariational reasoning, especially for bringing in connective quantities, as well as their expression of synchronous or asynchronous reasoning. Our study has set the groundwork for discussing whether envisioning an asynchronous coupling of quantities is partly dependent on the context and not solely on the student's developmental ability. All this work can shed light into understanding better how we may democratize access to the mathematics of change and variation for our young students.

Acknowledgments

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References

Basu, D. & Panorkou, N. (2019) Integrating covariational reasoning and technology into the teaching and learning of the greenhouse effect. *Journal of Mathematics Education* **12**(1), 6–23.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R. & Schauble, L. (2003) Design experiments in educational research. *Educational Researcher* **32**(1), 9–13.

Johnson, H. L. (2012) Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior* **31**(3), 313–330.

Jones, S. (2018) Building on covariation: making explicit four types of "multivariation." In Weinberg, A., Rasmussen, C., Rabin, J. & Wawro, M. (Eds.), *Proceedings of the 21st annual Conference on Research in Undergraduate Mathematics Education.* SIGMAA on RUME.

Panorkou, N. & Germia, E. (2021) Integrating math and science content through covariational reasoning: the case of gravity. *Mathematical Thinking and Learning* **23**(4), 318–343.

Paoletti T. & Moore K. C. (2018) A covariational understanding of function: putting a horse before the cart. *For the Learning of Mathematics*, **38**(3), 37–43.

Roschelle, J., Kaput J. & Stroup W. (2000) Simcalc: Accelerating Students' Engagement with the Mathematics of Change. In Jacobsen, M., J. and Kozma, R. B. (Eds.) *Innovations in Science and Mathematics Education: Advanced Designs for Technologies of Learning,* 47−76. Lawrence Erlbaum Associates.

Thompson, P. W. (1994) The development of the concept of speed and its relationship to concepts of rate. In Harel, G. & Confrey, J. (Eds.) T*he Development of Multiplicative Reasoning in the Learning of Mathematics,* 181–234. SUNY Press.

Thompson, P. W. & Carlson, M. P. (2017) Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.) *Compendium for Research in Mathematics Education,* 421–456. National Council of Teachers of Mathematics.