CHALLENGING KNOWN TRANSITIONS: LEARNING AND TEACHING ALGEBRA WITH TECHNOLOGY

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Technologists and educators interested in the ramifications of technology for the teaching and learning of school subjects speculate about the degree to which shifts in technological media transform the content meant to be taught with these media and the cognitive difficulties one might anticipate students might meet as they try to learn that content. There is speculation about the degree to which new technologies will lead to replacement of current curricula with new content (Papert, 1996; Schwartz, 1999; Noss, 2001). How does the use of a new curriculum that is based upon new epistemological assumptions change our capability to anticipate students' difficulties and strengths? Are phenomena like the *didactical cut* (Filloy and Rojano 1989), *cognitive gaps* or *discontinuities* (Herscovics and Linchevski, 1994; Tall, 2002; Arzarello, 2003) or *epistemological obstacles* (Brousseau, 1997) approach-independent? Or are these phenomena related to the instruction and to the cognitive tools learners have?

Critical transitions

Obviously there are difficult transitions in any sequence of learning. While understanding is often described as an incremental spiral chain of generalizations (Sierpinska, 1994), acquiring some concepts (or generalizations) requires a change in point of view. In attempting to design as smooth as possible sequences, Tall (2002) defines cognitive roots to be the kernel of continuous cognitive sequences. He argues that while cognitive roots would not always work for all students to create cognitive continuity, in many cases they offer a solution to the critical transitions. Studying cognitive obstacles, Arzarello and Paola (2003) define conceptual and structural discontinuities that demand critical transitions in the context of proof. They distinguish between a *conceptual continuity,* when the ingredients remain the same within a reference theory and a *structural continuity,* when, also, the structure does not change essentially. Arzarello and Paola offer two ways to treat the problem of the required transition: try and change the classic approach to systematically avoid the conceptual flips (that they call somersaults); design learning environments and situations centered on conceptual discontinuities – a common theme of a variety of technologically based microworlds.

Technology, when appropriately designed and used, can help to design learning environments that may change by various degrees the assumptions about previous knowledge and the order in which new concepts are introduced. In designing new-technology-supported sequences we would expect that some known discontinuities disappear but others would not and very probably new obstacles or gaps will appear.

Brousseau's (1997) didactical attempts that concentrated on epistemological obstacles that unfolded historically shed light on difficulties that are stable over time that students may experience in the classroom. Noss (2001), elaborating on the implications of rethinking the mathematics learned with new technological environments, expects that the epistemology of the mathematics learned with technology will change our ideas about cognitive hierarchies and the didactical attempts to construct them. Noss challenges what can be assumed to be a natural order of learning:

> Are three dimensions necessarily harder to understand than two? Should we just continue to take for granted that an orderly procession from 2D to 3D is somehow natural? What is natural anyway, now that it is more natural to visualize on a computer screen than in any other way? (p. 29)

Examples such as these introduce questions: Is it an interesting challenge for research to question the stability of known transitions when new computational environments are introduced into learning and teaching? How would research illuminate the nature of critical transitions? And going beyond the known transition, how might research distinguish types of transition in the context of new technologies?

In this article, *critical transition* is viewed as a learning situation that is found to involve a noticeable change of point of view. This change could become apparent as an epistemological obstacle, as a cognitive discontinuity or as a didactical gap. A transition would be identified as a necessity for entering into a different type of discourse (in terms of the language, symbols, tools and representations involved) or more broadly as changing 'lenses' used to view the concept at hand.

How might research illuminate the nature of critical transitions?

I will use two examples regarding the cognitive and epistemological changes that technology introduces: 3D representations of functions in 2 variables and the transition between recursive open rules and functional closed rules. Both examples offer ways in which the technology allows us to deal with powerful mathematical ideas that are not usually part of the traditional curriculum and in fact changes but does not eliminate known critical transitions. The examples are drawn from studies that sought a rationale for students' progress throughout analysis of curricular decisions.

The learning I will describe concentrates on students learning a functions-based, technology-intensive algebra curriculum [1]. In this function-based curricular sequence, the dominant conception of letters in introductory algebra is as representations of quantities that vary, either independently (x) or dependently (y or f(x)), and that taken together define a Cartesian plane. The explicit rule for calculating the dependent variable from the independent one can be graphed in the plane. Solving equations in one variable is conceptualized as a particular kind of comparison of two functions. This representation allows students to explore questions of equivalence as they learn to manipulate equations algebraically (Yerushalmy and Gafni, 1992; Yerushalmy and Gilead, 1997).

In general, an important goal of this curriculum is to help students develop strong algebra skills by helping them to learn to do manipulations with an understanding of the graphical and tabular meanings of these manipulations, as well as gain a sense of the purposes for which such manipulations are useful. Such proficiency involves moving across multiple views of symbols, graphs, and functions. Thus, an important and ambitious goal for instruction is to help students learn to shift their point of view. Students participate in this algebra and functions course for three years. During the last decade my colleagues and I have watched students in various classrooms and often interviewed them at different stages of the learning (Yerushalmy, 1997, 2000).

For each example I will look briefly at the curricular and technological resources students had at their disposal and describe data suggesting that students who used these resources performed in atypically powerful ways. A powerful performance might be the demonstration of a skill typically acquired at a later stage in learning or a demonstration of the non-routine use of a higher-order skill to engage in a problem usually solved with teacher-taught routine procedures. Related to this powerful performance, I will look at related data (not necessarily of the same students but that was found to be typical of the different classes we watched and students we interviewed) that tell a different story. This part of the examination of the examples identifies a critical transition. In this moment of transition, students with the technological and curricular resources that supported powerful performance act in quite a different way. We examine situations where the students confront a new problem that in standard instruction would be assumed to be simpler or similar to problems on which their performance was strong, but where these students have difficulties.

Example 1: Transition from considerations of change to an explicit functional relation

Resources: Using technology such as simulations software or other modeling tools that include dynamic forms of representations of computational processes, it is now possible to construct graphical models without first writing symbolic expressions with *x*s and *y*s. The *VisualMath* curriculum [1] is an example of an algebra curriculum where the learning of algebra is preceded by such semi-qualitative modeling. In this curriculum, an environment that allows users to construct and model motion generated by the movement of the computer's mouse supports the introduction of modeling of motion.

Relative strengths of students using these resources: Apparently, throughout the curriculum's focus on qualitative modeling, the students had developed ways of using tools to solve complex problems that concern non-constant rate of change. Graphs and staircases (a graphical depiction of differences in y value for a set change in x) emerged as models of situations, and also as models for reasoning about mathematical concepts. The software that supported this type of reasoning on temporal phenomena, the *Function Sketcher* [2], is further described in Schwartz and Yerushalmy (1995). Shternberg [3] interviewed seventh graders (12 year old students) who had studied the *VisualMath* materials on problem solving as they solved the *Ovens* problem (Figure 1) taken from a qualitative calculus unit (Taylor, 1992).

> A cook has a large portion of meat at room temperature that should be cooked as quickly as possible. At his disposal are a conventional oven and a microwave. In the microwave oven the meat temperature increases at a constant rate and in the conventional oven the meat temperature increases at a changing rate. In a cooking trial the cook found out that although the meat temperature in the conventional oven is always higher than in the microwave, cooking time is 2 hours in both ovens. Could cooking time be less than 2 hours using these 2 ovens?

Figure 1: The Ovens problem

The students started to represent the problem by exploring iconic sketches that would match the problem conditions. The following sequence of graphs (Figure 2) from one interview represents the type of work done by the students to model and solve the problem. The full description of this episode can be found in Yerushalmy and Shternberg (2001).

In their solution, the students recognized known struc-

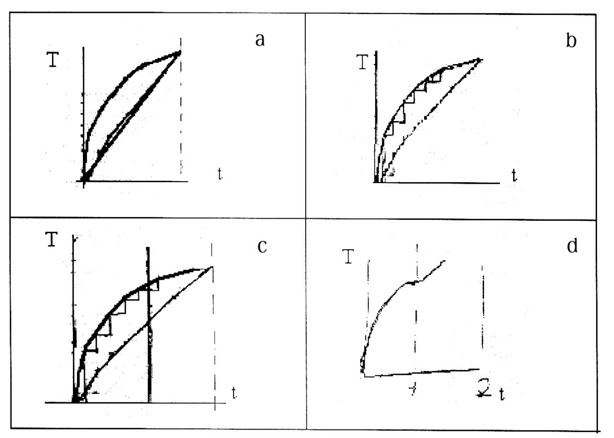

Figure 2: A modeling sequence of the Ovens problem: (a) a model that represents the two processes (b) stairs analysis of the curved process (c) an equal rate of change for both processes (d) an optimal process that would end in less than 2 hours.

tures – the two different types of processes of growth that they called *linear* and *concave down*. They compared the two processes using stairs – stairs with decreasing height on the curved one and constant height stairs for the linear process. They then used the stairs to evaluate the faster growth at each interval of time. The construction of the compound process was a totally new challenge for them. While calculus students who were trained to view graphs as representation of a function's expression had difficulties in solving this problem because they could not find a formal symbolic model, the *VisualMath* students were not looking for an expression. They were only interested in the graph as a representation of a sequence of incremental or constant differences. The solution was based on the ideas of growth and rates and on manipulations of mathematical symbols including graphical icons, descriptive verbs, and the staircases.

Transition: When Noss (2001, p. 29) asks "What is natural anyway?", he suggests that a new "digital culture" may shift our notions of pedagogical order. One of the examples given by Noss concerns the "closed form" (called explicit here) and the "recursive form" for representing a function's table of values symbolically. Recursive definitions are usually viewed as more complex and are usually studied late in school life, if at all. But, if students first become familiar with ideas of continuous change and finite differences, explicit closed forms may become less natural aspect of expressing phenomenon. Technology, which makes it natural to analyze and describe differences, either graphically like the one used by the students described above, or, numerically as structured in any spreadsheet, may suggest that a closed rule is no longer a more natural way to describe a function (Hershkowitz *et al*, 2002; Stacey and MacGregor 2000).

Here is an episode from an interview with Yael and Ronit (ninth grade students, 14 years old) *VisualMath* students who tried to solve a problem given as a table of numerical data that describes a quadratic process. Yael and Ronit are trying to answer the *Books' shipment* question (Figure 3).

They solve the problem by quickly identifying [in line 6] the result to be the squaring of the weight at the end of a given range. But, they express disappointment! The explicit closed rule gave the answer but it probably did not help them to explain the process of increasing differences that they identified at first [lines 1-4] to represent the given phenomenon in the story.

1 Yael: 1 Yael: … we (already) know that it increases by 2 each time. 5, then 7, then 9. We can continue this way as babies do because we know it increases by 2 but that's not the purpose. We can also, according to the graph.

2 Ronit: It will be a straight line.

3 Yael: It will not be a straight line…

4 Ronit: Oh, right, it has increasing constantly differences. … it would not be constant.

5 Ronit: If I'm looking at the table of values, I see... From weight 6 to 7 I have 49.

6 So I want to propose a connection between the two. I see that 49 divided by the larger number yields the larger number. I'm squaring.

7 The large number between two numbers and I have the price. It should be 54.

8 It would be 2916 the price of this package. Or it would be more. Or less.

9 Yael: We are raising weird ideas.

10 Interviewer: Why do you feel this way?

11 Yael: That's my feeling. I need something to feel sure there.

12 Ronit: That's it. I don't feel sure. I want to find the rule.

13 Interviewer: What does it mean that you need a rule? Does $Y = x^2$ tell you what the rule is? What does that describe?

14 Yael: We are so dumb. We simply have to do… what do you say: x is the weight and Y is the price, so they ask us about 54 – so let's substitute.

15 Ronit: Well, and what do you think I did right in the beginning? How did I get 2916? I simply want to translate it... I don't want to have a number for 54*54…

If the following table describes the post office rates for shipments, how much a 54 kg parcel would cost?

Weight in kg	Price in marks
From 0 to 1	1
From 1 to 2	4
From 2 to 3	9
From 3 to 4	16
From 4 to 5	25

Figure 3: The books shipment problem

Ronit and Yael were challenged by what appears to be a critical transition between the linearly increasing differences [1-4] and the output being a square of the input [6-7] describing the same phenomenon. Throughout the interview, they describe their feelings as being insecure although they found the rule that can provide the answer [9-12]. They seem to recognize that there are two ways to look at the pattern in the table but the two are described by two different set of terms – one is about terms of differences, increasing differences and linearity, while the other is about connection between input and output, trying out numbers and stating a generalization. Being unable to connect the two, they were left puzzled at their own ideas and correct solution.

Understanding the limitations: Earlier fruitful experiences of Yael and Ronit with ideas of differences (as in the *Ovens* problem above) probably complicated their use of explicit forms. This complication might not have arisen if they had not had earlier support for recursive reasoning. However, the other option, the one most traditional curricula follow of emphasizing explicit rules first and then learning to describe it as analysis of differences, is known to be problematic as well. Thus, technology that was used in this sequence helped to carry the epistemological change that makes analysis of the change of function its basic entity but left a cognitive gap between the closed and recursive views. While the difficulty is not anymore in the transition from the functional relations described as explicit rules to recursive formulations of processes, the abrupt change between views – the recursive and the explicit forms – remains the important challenge.

Example 2: Transition from an equation in a single variable to an equation in two variables

Resources: The lion's share of the early parts of the *VisualMath* curriculum focuses on functions of one variable and equations conceptualized as the comparison of two functions of one variable. With this way of thinking about equations, students acquire, alongside the algebraic procedures, alternative methods to solve equations. Prior to, and while, using symbolic manipulations, students are encouraged to use systematic guessing and intuitive numerical and graphical analysis strategies to narrow down the search interval. Students in many algebra curricula that use linked graphic, tabular, and symbolic representations similarly learn to solve equations in a variety of ways (Huntley *et al.*, 2000; Hershkowitz *et al.*, 2002).

Relative strengths of students using these resources: The *VisualMath* students who had not learned procedures beyond the linear equation worked on the problem (Figure 4):

The problem yielded a crop of answers from a group of students who had not learned to solve quadratic equations (Yerushalmy and Gilead, 1997). Most of their suggested strategies blended numerical and graphical analysis with algebraic symbols. For example, Eli sketched the two functions, marked the intersection points and expressed an objection to Jo's domain interval:

Jo was asked to determine the number of solutions for the $x^2 + 4 = x + 24$. 1. Using the table of values he decided that the equation has one solution. Explain how he arrived at his conclusion. Was Jo right? Can you suggest how he could check his answer? Explain.

x	f(x)	g(x)	f(x) – g(x)
4	20	28	-8
5	29	39	0
6	40	30	10
7	53	31	22
8	66	32	36

Figure 4: Determining solutions for a quadratic

> Jo determined the number of solutions by looking at the 0 in the difference [column]. Since 0 difference indicates an intersection point where there is no difference between the two graphs. And since the y values are the same at that x value. Jo was wrong since he did not construct the negative values part of the table. I would suggest that he check the number of solutions using graphs.

As the example illustrates, viewing an equation as a comparison of two functions, either graphically or in a tabular representation, students can solve problems for which they have not yet been taught an algorithmic solution method. Similar conclusions appear in other studies of students' learning of linear equations using computational technology (*e.g.*, Chazan, 1993; Chazan, 2000; Kieran, 2001).

Transition: Elsewhere (Yerushalmy and Chazan, 2002), we described at some length a class teacher who wondered whether his students could solve a task involving systems of equations in two variables without being taught a method for solving such systems. As a preparation he asked his students, "How would you represent the equation $x + y = 2x - y$?" We described three different groups in the class and their correct and faulty suggestions. One group of students decided to view both x and y as independent variables and to view this equation as a comparison of two functions of two variables. They then began to explore three-dimensional representations. Another group viewed y as a parameter that could take on a range of values. For every value of y, a new linear function in x would be generated. Another group began approaching the question by thinking of the equation as a comparison of two functions: $f(x,y) = x + y$ and $g(x,y) = 2x - y$. By using the difference function and symbolic manipulations, they reduced the question of finding the solution of an equation in two variables, to a question about the zeros of a single function in two variables. In the end, they changed $-x + 2y = 0$ to $y = 1/2x$.

In typical algebra instruction, solving an equation in two variables means isolating a variable: shifting from a non-explicit form ($x + y = 2x - y$) to an explicit function form: $y = 1/2x$. For our function-approach students, the equal sign of the equation represents a symmetric-comparison sign and the function-equal sign represents an asymmetric-assigning sign. Thus, the fact that simple manipulation techniques can help move from one form to the other does not seem to be useful. Students do not choose this option. They would rather keep viewing equations as comparisons of two functions. In that sense, the use of a 3D Cartesian graph to describe the comparison was the more natural representation to take. For these students, this representation directly reflects their expansion of the image of an equation in one variable to an equation in 2 variables. Although this can seem the harder one to take, it was for them more natural than the mathematically compatible 2D graphs.

Understanding the limitations: the transition from a single-variable equation to a two-variable equation in the traditional algebra curriculum is a shift in understanding of the nature of the solution: from a single definite solution to a set of solutions. For the students who take an equation to be a comparison of two functions, the nature of the solution remains the intersection values of the intersection of the two functions. However, the students have to develop new ideas about presentations of function in two variables: the 3D graphical and tabular representations of functions of two variables must be developed in order to help students see connections to the ways of representing functions of one variable with which they are familiar. An attempt to generalize to a multi-variable equation (more than 2 variables) will require departing from the graphical and tabular representation of comparisons of functions.

With this four-tier analysis: resources, strengths, transitions and limitations, I have illustrated that explanations for both strengths and complexities should be looked for when studying a curriculum that makes use of new technological tools. Apparently, the study of known transitions is an interesting perspective for studying the new approach.

The first example (of the recursive modeling) highlighted a critical transition occurring within modeling that required a shift between recursive and explicit algebraic models of the same phenomenon. The transition between the two is not new to the literature but traditionally occurred in the reverse order. The technological approach invited recursive thinking to be the more natural way to think about a phenomenon. However, the approach also challenges thinking about teaching that can support the transition to algebraic explicit rules that describe functional relations.

The second example demonstrated another known critical transition: the change in view from equations and solutions emerging when moving from an equation in a single variable to an equation in two variables. Although the function approach described here and the multiple representation technology used gave primary attention to an equation being a comparison of two functions, the transition remained critical. Reasons are of different sorts:

- it is necessary to deeply revise the graphical view where the y axis became an independent input rather than an output and the (x, y) plan describes pairs of input rather than a functional relation between x and y

- it is necessary to acknowledge that simple algebraic technique can change the mathematical objects in hand – from an equation in two variables, say, to a function in a single variable.

While a new approach and new mediators can support the smoothing of existing discontinuities, here we explored *known transitions* that remain apparent in the new setting. Thus, in looking at these two critical and traditionally known transitions, we learned about changes in cognitive hierarchies and about the different nature of the transition in this new domain.

How might research distinguish types of transition?

As Sutherland *et al.* (2004) recently suggested, identifying an approach to algebra is an important and sometimes a hard task. In previous work (Yerushalmy and Chazan, 2002), we offered distinctions to support the identification of differences in approaches to school algebra. The distinctions were made through alternative ways of conceptualizing the fundamental objects of study and their representations. We focused on the mathematical objects and their representational systems. For example, school algebra curricula differ in their emphasis on functions and equations as their cognitive root and they differ in the emphasis they put on letters, tables, graphs, and other systems of representation.

We found out that, in order to understand differences between approaches to algebra, it is insufficient to explore which objects and conceptions of representations are present in the curriculum. Additional to the question of what is included in the curriculum, there are *questions of order*. For example, what sorts of views of equal sign, letters, and graphs appear first in the curriculum? When do new views appear? There are also *questions of intensity*. How long does the curriculum stay with a particular view? When do new views get introduced and why? *How many views* of equal sign, letters, and graphs are entertained at a given time? Are students immediately introduced to a wide variety of uses of literal symbols? Or, do students work with expressions and equations for quite some time, before finding out that expressions can be thought of as correspondence rules for functions and not only names for unknown numbers? Or before learning that some equations in two variables can also be thought of as functions of one variable? *How explicit* is a curriculum about the presence of multiple views? And, how explicit are transitions of views?

It is useful to try and answer these sorts of questions in order to identify and distinguish types of transitions that might appear in new curricula sequences. For example, thinking about symbolizing and modeling, what comes first? Modeling without symbols or symbols first and modeling later? How do you make the connection? The issue of formal language first or later is a general one, and is especially connected to proposals to employ technology to support a bridging language to the algebraic symbols (*e.g.*, spreadsheet symbols or the language of graphs as symbols, Schwartz and Yerushalmy, 1995; Arcavi, 2004). These sorts of transitions are challenging when curriculum developers try to keep conceptual understanding and fluency in manipulations mutually supported.

Concluding remarks – relevance of the research

Computational technologies allow us to improve the design of mathematical learning environments. In order for research to be helpful for teaching and learning in the new context it is important to devise and use strategies and distinctions that would support systematic analysis of critical transitions.

We looked at examples that suggest a positive answer to the question: Is it an interesting challenge for research to question the stability of known transitions in the context of new computational environments? Watching a sequence of teaching and learning that involves a critical transition – a point in the learning where one has to change point of view, is valuable. Although, like Papert, I believe that obstacles should be reconsidered when new tools are involved, I suggest that often transitions between mathematical views of concepts remain complex and suspect them to be found independent of the technology used.

For a second question, How might research illuminate the nature of critical transitions?, I have demonstrated a way to study changes of cognitive hierarchies that involve learning with technology. This way is appropriate when one has a chance to follow learning and teaching for a substantial period of time, observing students' strengths, identifying the resources for these strengths, watching how the students get involved in a transition and analyzing the reasons for the discontinuity.

For a third question, How would research identify and distinguish new types of transition in the context of new environments?, a few distinctions that may support further systematic study of epistemological transformations have been offered. I believe that such distinctions could support ways to answer Noss's question, "What is natural anyway?", in a systematic manner. Indeed, the technology and the sequence of learning help to make mathematics that is assumed to be difficult for students to be "natural". However, the transitions between fundamental concepts or operations remained, in my view, a substantial challenge.

Notes

[1] For more information about *VisualMath: algebra and functions* (1995) Tel-Aviv, Israel, Centre for Educational Technology, visit http://www.cet.ac.il/math-international/first.htm.

[2] 'The F*unction Sketcher* software can be found at http://www.cet.ac.il/math-international/software1.asp.

[3] For more information about the doctoral dissertation, in Hebrew, of Shternberg, B. (2002) *Effects of the analysis of the rate of change of a function on understanding the mathematical significance of phenomena*, Haifa, Israel, Faculty of Education, University of Haifa, contact author through e-address: beba_s@cet.ac.il.

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