# A Widespread Decorative Motif and the Pythagorean Theorem

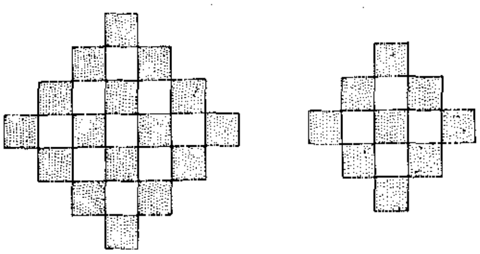
PAULUS GERDES

In their already classical study of the mathematics learning difficulties of the Kpelle (Liberia), Gay and Cole [1967, p 6] concluded that there do not exist any inherent difficulties What happened in the classroom was that the contents did not make any sense from the point of view of Kpelle-*culture* . .

Subsequent research and analyses reinforced this conclusion and recognised that in view of the "educational failure" of many children from Third World countries and from ethnic minority communities in industrialised countries like Great Britain, France and U S A , it is necessary to *(multi)culturalise* the school curriculum in order to improve the quality of mathematics education [cf. e.g. Bishop, 1988; D'Ambrosio, 1985 a, b; Eshiwani, 1979; Gerdes 1985 a, b, 1981 a, 1988 a, b; Ginsburg & Russell, 1981; Mellin-Olsen, 1986; Nebres, 1983; Njock 1985]. In other words, the mathematics curriculum has to be "inbedded" into the cultural environment of the pupils Not only its *ethnomathematics,* but also other *culture elements,* may serve as a *starting point* for doing and elaborating mathematics in the classroom [cf D'Ambrosio, 1985 a, b; Gerdes, 1986 b, 1988 a, b] In this article we explore the mathematical-educational *potential* of such a cultural element: a widespread decorative motif

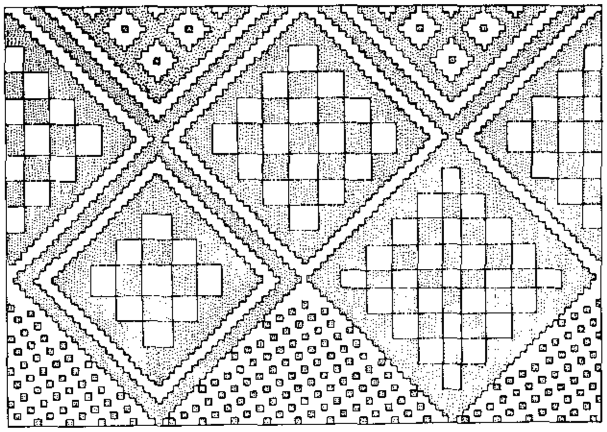
**A widespread decorative motif**

One of the best known basketry designs of the Salish Indians of British Columbia is their so-called *star* pattern [Ferrand, 1900, p 397, Plate XII See Figure 1]. The Pomo Indians of California used to name it *deer-back* or *potato-forehead* [Barrett, 1908, p 199].

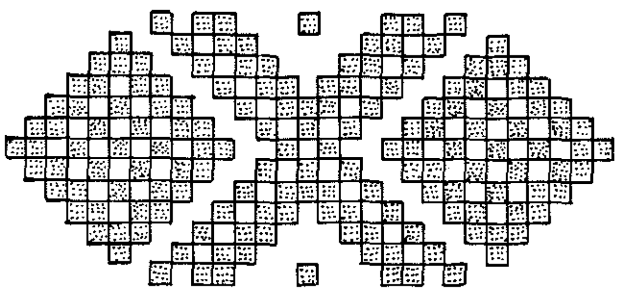


"Star" pattern
Figure 1

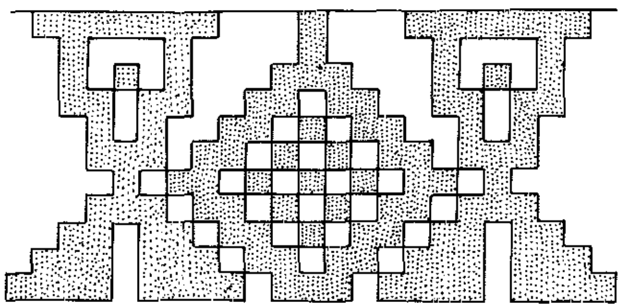
This decorative motif has a long tradition and can be encountered all over the world [1] The detail of a beautifully plaited mat from Angola, shown in Figure 2, is known as the *tortoise* design [Bastin, 1961, p 116] The same star pattern is found on tiled pavements in Arabic-speaking countries [Hutt, 1977, p 43], on textiles from Scandinavia (see Figure 3), Ancient Mexico [Weitlaner-Johnson, 1976, vol. 1, Plates 63, 64. See Figure 4], Nigeria, Algeria [Picton & Mack, 1979, p 35, 75] etc., on baskets from Lesotho, Mozambique, etc., and on a game board in Liberia [Machatscheck, 1984, p 55].



Detail of a plaited mat (Tchokwe, Angola)
Figure 2



Traditional Norwegian textile design
Figure 3



Mexican Indian design motif
Figure 4

**Discovering the Pythagorean Theorem**

Looking at the number of unit squares on each row of a "star" (see Figure 5), it is easy to see that the area of the "star" is equal to the sum of areas of the 4 × 4 shaded square and the 3 × 3 unshaded square [2].

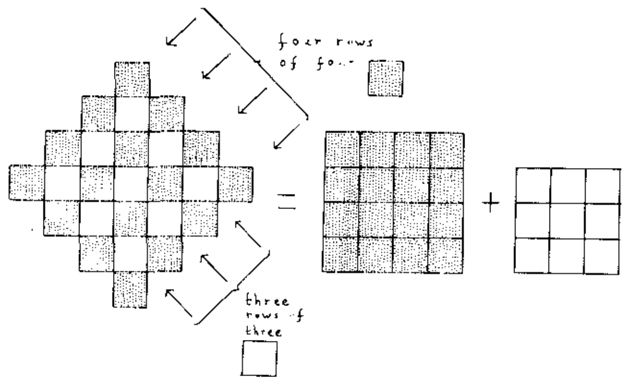


Figure 5

A "star" figure, like the one in Figure 1, may also be called a "*toothed square*". A toothed square, especially one with many teeth, looks almost like a real square. So naturally the following question arises: is it possible to transform a toothed square into a real square of the same area? By experimentation (see Figure 6), the pupils may be led to draw the conclusion that this is indeed possible.

In Figure 5, we have seen that the area of a toothed square (T) is equal to the sum of the areas of the two smaller square (A and B):

$$T = A + B$$

In Figure 6 we conclude that that area of a toothed square (T) is equal to the area of a real square (C). Since C = T, we can conclude that

$$A + B = C.$$

Do there exist other relationships between these three squares? What happens if one draws the toothed square and the two real squares (into which it is decomposed) together on square grid paper, in such a way that they

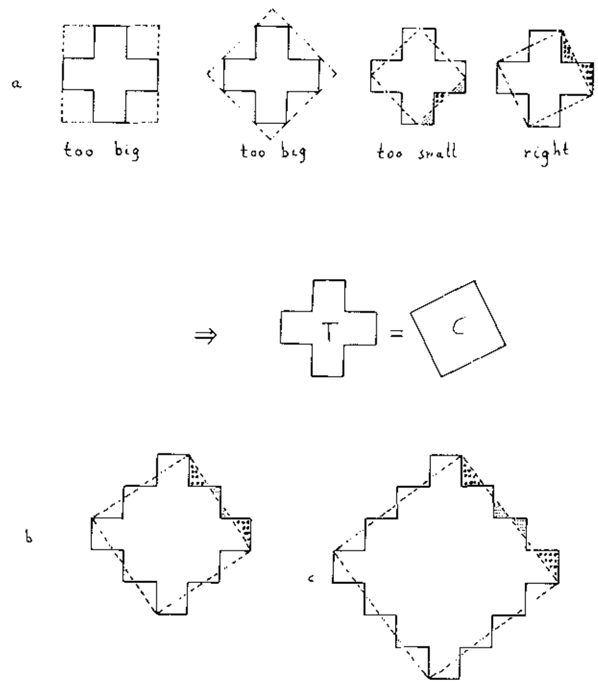



Figure 6

become "neighbours"? Figure 7 shows a possible solution. When we now draw the last real square (area C) on the same figure, we arrive at the Pythagorean Theorem for the case of (a, b, c) right triangles with $a : b = n : (n + 1)$, where the initial toothed square has $(n + 1)$ teeth on each side. Figure 8 illustrates the Pythagorean Proposition for the special case of the (3, 4, 5) right triangle. On the basis of these experiences, the pupils may be led to *conjecture* the Pythagorean Theorem in general. In this manner, toothed squares assume a *heuristic value* for the discovery of this important proposition.

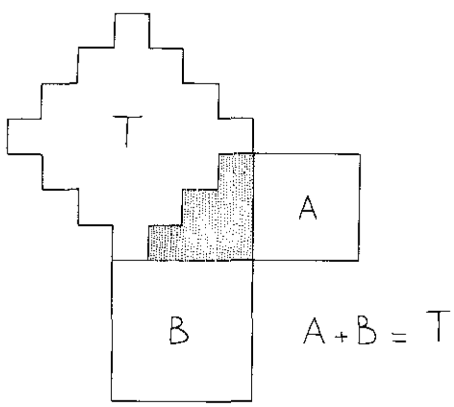



Figure 7

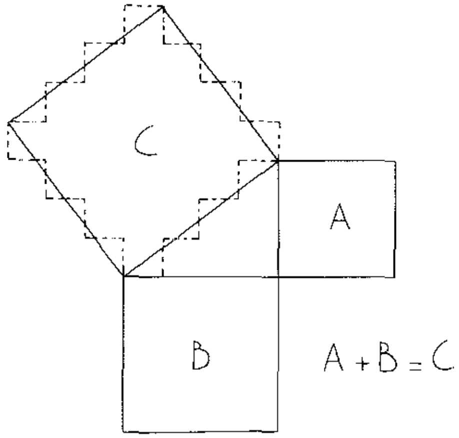



Figure 8

Does this same discovery process also suggest any (new) *demonstrations* for the Pythagorean Theorem?

What happens when one reverses the process? When one begins with two arbitrary squares and uses them to generate a toothed square?

**A first proof**

Let A′ and B′ be two arbitrary squares. We look at Figure 5 for inspiration: dissect A′ into 9 little congruent square, and B′ into 16 congruent squares, and join the 25 pieces together as in Figure 9. The obtained toothed square T′ is equal in area (T) to the sum of the real squares A′ and B′:

$$T = A + B.$$

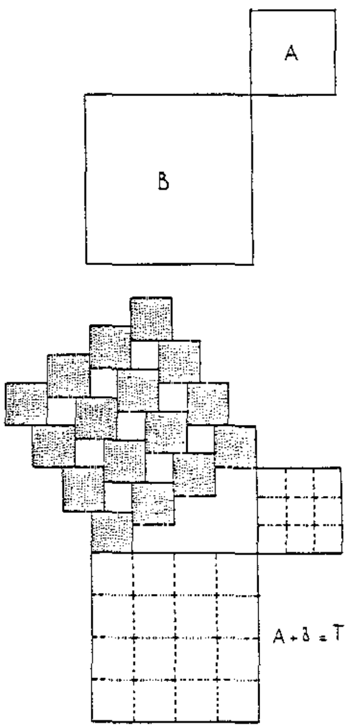



Figure 9

As, once again, the toothed square is easily transformed into a real square C′ of the same area (see Figure 10), we arrive at

$$A + B = C,$$

i.e. the Pythagorean Proposition in all its generality.

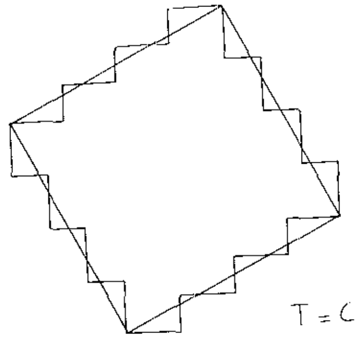



Figure 10

**An infinity of proofs**

Instead of dissecting A′ and B′ into 9 and 16 subsquares, it is possible to dissect them into $n^2$ and $(n+1)^2$ congruent subsquares for each value of $n$ ($n \in \mathbb{N}$). Figure 11 illustrates the case $n = 14$. To each value of $n$ there corresponds a *proof* of the Pythagorean Proposition [3]. In other words, there exist *infinitely* many demonstrations of this famous theorem.

For relatively high values of $n$, the truth of the Pythagorean Proposition is almost immediately visible. When we take the limit $n \to \infty$, we arrive at one more demonstration of the theorem.

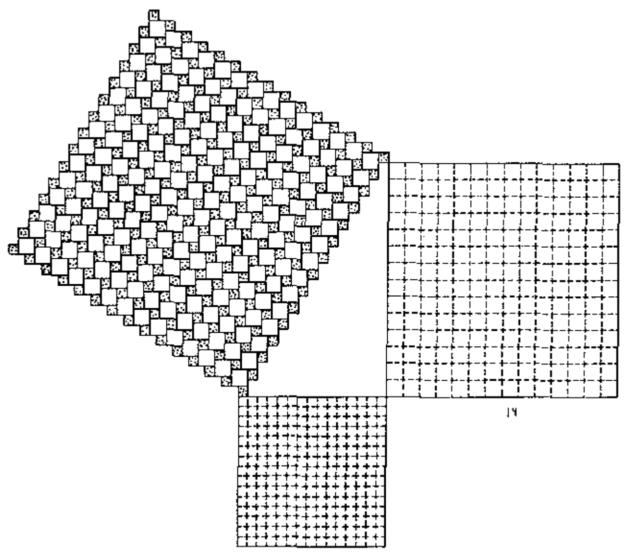


Figure 11

For $n = 1$, one obtains a very short, easily understandable proof (see Figure 12)

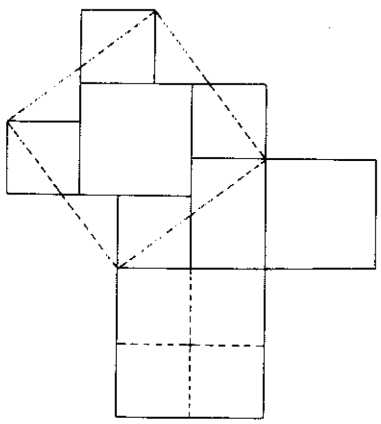


Figure 12

**Pappos' theorem**

Analogously, Pappos' generalization of the Pythagorean Proposition for parallelograms can be proved in infinitely many ways (Figure 13 illustrates the case $n = 2$).

**Example**

Loomis' well known study "*The Pythagorean Proposition*" gives " . . in all 370 different proofs, each proof calling for its specific figure" [1940; 1972, p. 269] and its author invites his audience to "Read and take your choice; or better, find a new, a different proof . . ." [p. 13] Our reflection on a widespread decorative motif led us not only to an alternative, active way to introduce the Pythagorean Proposition in the classroom but also to generate infinitely many proofs of the same theorem. May this example serve as a further stimulus to the multi-culturalisation of mathematics education.

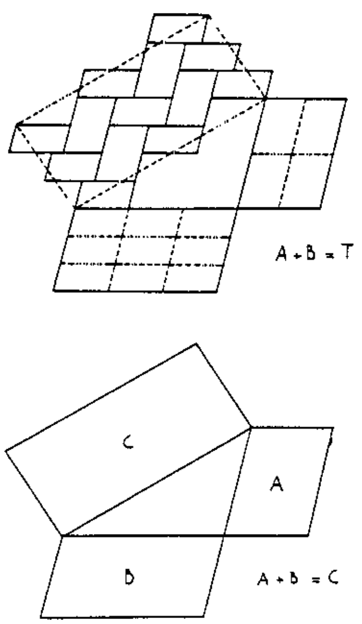



Figure 13

**Notes**

[1] Its possible technical origin in the production of flat circular baskets is analysed in Gerdes [1985, p. 47-51, 78-80]

[2] Pupils can also be led in many other ways to draw this conclusion. E.g., the teacher may ask them to transform a "star" made of loose tiles into two unicoloured similar figures. Or, one may ask them to cut off the biggest possible square from a "star" made of paper or cardboard and to analyse which figures can be laid down with the other pieces (see Figure 14)

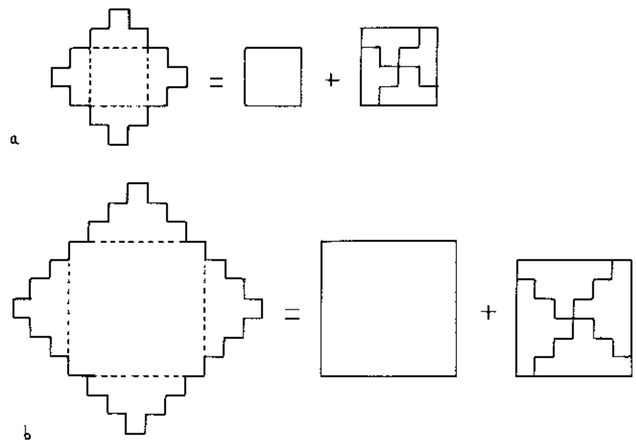



Figure 14

[3] These demonstrations have been elaborated by the author in [1986 d]. Another infinite set of possible dissection proofs of the same proposition has been outlined by Bernstein [1924]

[4] Another proof by means of limits has been given in Gerdes [1986 c]

**References**

Barrett, S. A. [1908] Pomo Indian basketry. *University of California Publications in American Archaeology and Ethnology*

Bastin, M.-L. [1961] *Art decoratif Tshokwe*. Publicaçoes Culturais da Companhia de Diamantes de Angola, Lisbon

Bernstein, F. [1924] Der Pythagoraische Lehrsatz. *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht aller Schulgattungen*. 204-207

Bishop, A. [1988] The interaction sof mathematics education with culture. *Cultural Dynamics* (forthcoming)

D'Ambrosio, U. [1985 a] Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics*. 5, 1 44-48

D'Ambrosio, U. [1985 b] *Sociocultural bases for mathematics education*. Unicamp, Campinas

Eshiwani, G. [1979] The goals of mathematics teaching in Africa: a need for re-examination. *Prospects*, 346-352

Farrand, L. [1900] Basketry designs of the Salish Indians. *Memoir of the American Museum of Natural History*.

Gay, J. & Cole, M. [1967] *The new mathematics and an old culture: a study of learning among the Kpelle of Liberia*. Holt, Rinehart and Winston, New York

Gerdes, P. [1985 a] Conditions and strategies for emancipatory mathematics education in underdeveloped countries. *For the Learning of Mathematics*, 5, 1, 15-20

Gerdes, P. [1985 b] *Zum erwachenden geometrischen Denken*. Eduardo Mondlane University, Maputo

Gerdes, P. [1986 a] On culture, mathematics and curriculum development in Mozambique. In S. Mellin-Olsen & M. Johnsen Hoines (eds.) *Mathematics and culture, a seminar report*. Radal, 15-42

Gerdes, P. [1986 b] *Un método geral para construir poligonos regulares*

*inspirado numa técnica moçambicana de entrelaçamento* TLANU Maputo
Gerdes P [1986 c] *Mais duas novas demonstrações do Teorema de Pitágoras* TLANU, Maputo
Gerdes P [1986 c] *Infinitely many new proofs of the Pythagorean Theorem* TLANU, Maputo
Gerdes, P [1988 a] On culture, geometrical thinking and mathematics education. *Educational Studies in Mathematics* (forthcoming)
Gerdes, P [1988 b] On possible uses of traditional Angolan sand drawings in the mathematics classroom *Educational Studies in Mathematics* (forthcoming)
Ginsburg, H & Russell, R [1981] Social class and racial influences on early mathematical thinking *Child Development* vol 46
Hutt, A [1977] *Islamic architecture North Africa* London
Loomis, E S [1972] *The Pythagorean Proposition* NCTM, Reston
Machatscheck, H [1984] *Stein um Stein. Exotik der Brettspiele* Berlin
Mellin-Olsen, S. [1986] Culture as a key theme for mathematics education In: S Mellin-Olsen & M. Johnsen Hoines (ed ) *Mathematics and culture, a seminar report* Radal, 99-121
Nebres B [1983] *Problems of mathematical education in and for changing societies. problems in southeast Asian countries* Tokyo
Njock, G E. [1985] Mathématique et environnement socio-culturel en Afrique Noire *Présence Africaine* 135, 3-21
Picton, J & Mack, J. [1979] *African textiles* British Museum, London
Weitlaner-Johnson I [1976] *Design motifs on Mexican Indian textiles* Graz

---

The physical sciences have succeeded largely by rejecting consciousness and intelligence from the Universe they describe Kepler started by thinking of the planets as pushed around by intelligent angels When he realized that they move in elliptical orbits, guiding intelligence did not seem necessary and was dropped. This dropping of intelligence as causal in Nature, extended from physics to Darwin's Natural Selection. This is seen as creating intelligent solutions to design problems, but not by means of a guiding conscious intelligent entity The processes are random variation and selection against competition. This generates intelligent solutions without itself being intelligent or conscious. There is indeed no "it" to have intelligence, or consciousness, in Darwin's account (As there is no valid "I" in "I think therefore I am".) If the brain was developed by Natural Selection we might well suppose that consciousness has survival value. But for this it must, surely, have causal effects. But what effects could awareness, or consciousness, have?

R L. Gregory

---