THE METHOD OF VARYING INQUIRY FOR STIMULATING LEARNING

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We present the Method of Varying Inquiry (MVI), aimed at fostering students' involvement in discerning mathematical ideas through an inquiry-based approach to mathematics. Conceived within a post-Vygotskyan perspective towards teaching and learning (*e.g.,* Stetsenko & Arievitch, 2002), MVI also helps teachers in becoming aware of how students discern mathematical objects through inquiry processes and in designing activities that support both students' inquiry and teachers' didactical goals (see, *e.g.,* how to lead classroom discussions to foster these processes, together with students' reflections on them).

The MVI-model is a theoretical construct that specifies the role of the teacher in supporting students' inquiry processes and suggesting a flexible didactic method to frame them. In this article, we present the MVI-model by introducing the two main theoretical components upon which the MVI-model is built: the *inquiry-based approach* to teaching mathematics and the *variation theory* of learning, its novel elements, and an example to illustrate how it can guide the design of tasks for students and of inquiry oriented classroom interactions.

Inquiry-based mathematics education

Inquiry-based mathematics education has roots years ago in many scholars' ideas on the need to focus on the experimental dimension of mathematics. Artigue and Blomhøj (2013) proposed a conceptualization of inquiry-based approaches, covering a range of factors: (a) the epistemological relevance of the questions, (b) the modeling dimension of the inquiry process and the experimental dimension of mathematics, (c) the development of problem-posing and problem-solving abilities and inquiry habits of mind, (d) the collaborative dimension of the inquiry process, (e) the autonomy and responsibility given to students, the guiding role of the teacher, and teacher-student dialogic interactions.

The guiding role of the teacher in supporting students is essential to the effectiveness of inquiry-based learning (Lazonder & Harmsen, 2016). In fact, if inquiry-based learning is interpreted as an approach in which the "learner is not provided with the target information or conceptual understanding and must find it independently and with only the provided materials" (Alfieri, Brooks, Aldrich & Tenenbaum, 2011, p. 4), neglecting the role of the teacher, it has limited educational value (Scott, Smith, Chu & Friesen, 2018).

Given the importance of the teacher in inquiry-based learning it is important to design educational activities that might promote several layers of the inquiry processes and, as the implementation of these activities in the classrooms is a complex task for many teachers, to develop frameworks to support their work. Our research aims to generate an explicit model of how teachers can actually lead inquiry processes in their classrooms, and how such processes can be concretely fostered, beyond the inquiry approach itself. This model combines the inquiry-based framework with another educational perspective that could support the design of effective educational activities aimed at creating experiential spaces where students can have opportunities for understanding, seeing, and acting in the world: variation theory.

Variation theory

Variation theory (Marton & Tsui, 2004) defines learning as a change in the way an object of learning is discerned: how it is seen, experienced, understood. According to variation theory, an object of learning can be formulated in three different ways of increasing precision, in terms of: content (*e.g.,* linear function), educational objectives (e.g., generalizing patterns), and critical aspects, which the learners should simultaneously discern to make the object of learning their own. To help students notice the critical aspects, the teacher must search for them in advance, and this is challenging, because they are relative to the object of learning and learners as well.

According to Marton (2015), the creation of meaning occurs through *contrasting* the object of learning with other objects from the same dimension of variation, which refers to the aspect to focus on (*e.g.,* numbers, function, shapes, *etc.*), and its *values* that are features of the aspect (*e.g.,* 1, 2, 3, …; linear function, cubic, trigonometric, *etc.*; triangle, square, kite, *etc.*). To illustrate contrast, this example is useful: to discern the color 'green', we should contrast it with different colors, keeping some objects invariant (green ball, red ball, blue ball, *etc*.). Once the meaning is found through the contrast pattern, the *generalization* of the object of learning is necessary: not only the ball that is green but also other things (*e.g.,* green ball, green window, green bottle, *etc*.). The learning object (color green) is invariant in such cases, while the other objects vary. After separating the objects by contrast and generalization, the whole must be put together again, to simultaneously experience certain aspects of the object of learning: this pattern is called *fusion*. The order of these patterns is important for learning, according to Marton.

To help the learners to discern the critical aspect while preserving the students' inquiry process, the teacher should design a set of tasks to achieve this aim. This can be done by separating the aspects first and then fusing: "seeing a certain class of phenomena in terms of a set of aspects that are analytically separated but simultaneously experienced provides a more effective basis for powerful action than a global, undifferentiated way of seeing the same class of phenomena" (Marton & Tsui, 2004, pp. 16–17).

In this article, we employ key principles of variation theory to support teachers' design of activities aimed at scaffolding students' inquiry processes.

Method of Varying Inquiry

MVI consists of designing challenging tasks for students (as in the inquiry approach), in meaningful contexts (real-world or mathematical), by varying some variables of phenomena while keeping the others invariant (as in variation theory) to let students discern the object of learning embedded in the phenomena. Moreover, in order to create a model that could operationally support teachers in fostering and leading inquiry processes in their classes, we add two elements useful to a meticulous design of classroom activities and discussions that foster students' inquiry processes.

The first element is didactical: the *mathematics laboratory*, elaborated in the institutional context of the Italian Ministry of Education (Anichini, Arzarello, Ciarrapico & Robutti, 2003) and representing a teaching approach based on group and peer work, sharing and comparing ideas, classroom discussions led by the teacher, and 'acting' instead of 'listening' through problem posing and problem solving. It is aimed at fostering the construction of meanings of mathematical objects through the use of different tools and through social interaction.

The second element is theoretical: the *virtuous cycle* (see Figure 1), introduced by Swidan, Arzarello & Beltramino (2017) as a process that supports students in making sense of mathematics by enabling them to connect different pieces of theoretical knowledge (not only mathematical). The cycle draws its origins from similar and more complex cycles for using formal mathematics to interpret real-world situations (Schoenfeld, 1991).

The *virtuous cycle* consists of four intertwined processes: (I) representation of aspects of a situation (related to a phenomenon in a real-world or mathematical context) into a formal system; (II) treatment of the representation within a formal system or conversions between systems towards a generalization in a class of formal systems [2]; (III) interpretation of the generalisation in relation to a family of situations; (IV) interpretation of the initial situation within a

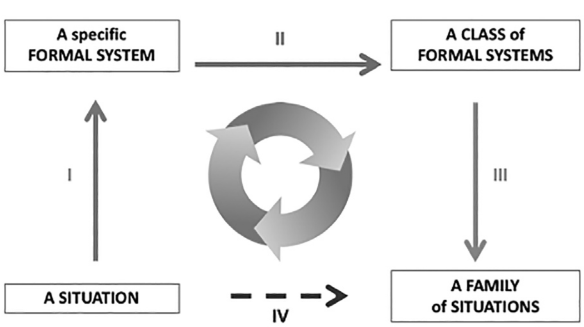

Figure 1. Virtuous cycle (adapted from Swidan, Arzarello & Beltramino, 2017).

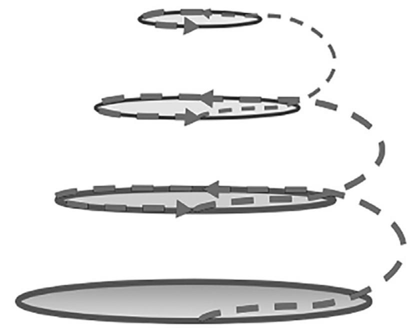

Figure 2. MVI: a spiral-shaped and multi-layered model.

family of situations. It is important to stress that (IV) is, therefore, the outcome of (I), (II), and (III). In tune with Godino (1996), we take the notion of situation (or problem-situation) as a primitive idea. A situation could refer both to real-world or mathematical phenomena and it represents for us a macro-object of inquiry. This object becomes a problem when questions are posed, making the students focus on specific variables that emerge when phenomena are analyzed and on the relationships between these variables.

When we use the term 'formal system' we mean not only the register that is chosen to construct representations, but also all the theoretical frames (properties used, typical procedures, …) within which the situation is represented, interpreted, and studied. Of course, the cycle is productive if Arrow IV (in Figure 1) represents a genuine 'epistemic gain' in how the students interpret the initial situation after the cycle, that is, if they consider the explored situation as particular case of a family of situations.

The MVI-model interprets learning as a layered inquiry process (see Figure 2) promoted—in the context of the mathematics laboratory—through a task-design that may support students in making sense of mathematical concepts

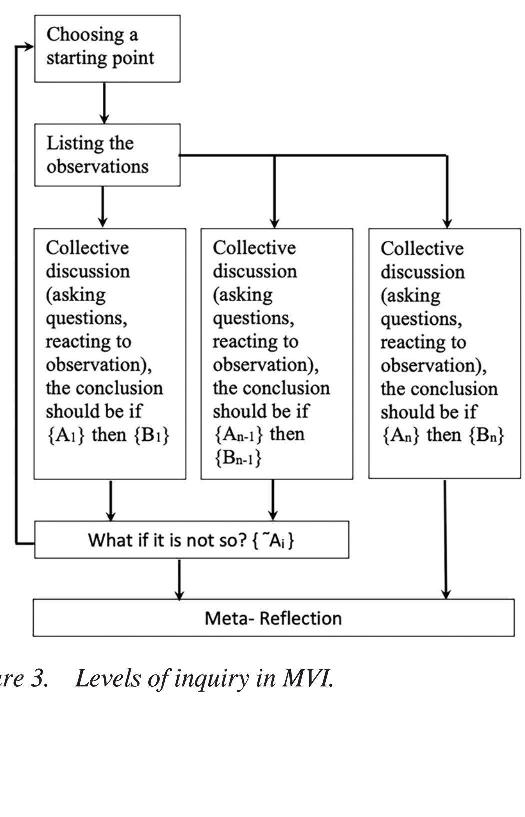

Figure 3. Levels of inquiry in MVI.

through the processes that characterize the virtuous cycle.

This layered process to the building of meanings is a novel element introduced by the MVI-model with respect to the inquiry approach and to variation theory. The several levels of inquiry that characterize MVI are represented in Figure 3.

Implementing MVI: a representative example

MVI is not only aimed at introducing a layered process of students' learning, but also at supporting teachers in both the design of activities aimed at fostering inquiry processes and the planning of how to lead classroom discussions. In this section, we focus on both aspects by means of an example, contextualized in Grade 8. The initial level of the inquiry starts with providing the students with a task that introduces a situation to be investigated and draws their attention to specific aspects of the object of learning under investigation (*Choosing a starting point,* in Figure 3). In our example, the educational objective that characterizes the object of learning is to enable students to generalize patterns of numbers algebraically. Each possible task supports students' focusing on different dynamic aspects of the situation, that is, on different dimensions of variation (Marton, 2015). In our example, students are provided with a number table (see Figure 4a, inspired by Brown & Walter, 2005, pp. 80–81), and asked to observe it with a 'mathematical eye' to identify regularities. Henceforth, we will refer to this activity as 'patterns of numbers'.

At the '*Choosing a starting point*' level, the teacher introduces the problem to be faced and describes how the lesson will be conducted (that is, the students' tasks and the role they have to play during the class). Students are introduced to '*a situation*' (the first box in the virtuous cycle, Figure 1) and start their observation of variation of three interrelated variables, corresponding to the numbers within the three columns of the table in Figure 4a. The first step is *contrasting* the variables within the columns with each other, perceiving their different values, and identifying the possible patterns in the table. A further dimension of variation intervenes: the different ways in which students could explore the table. They could, in fact, focus on each column separately (*generalization*) to highlight the way in which each variable varies inside the column, or they could look at two or three columns simultaneously to highlight possible relationships within the table (fusion).

At the following level (*Listing the observations,* in Figure 3), students are required to work in groups and to list their observations and conjectures, while the teacher monitors their work and encourages them to discuss and share their observations. The teacher may also pose and promote questions if the students are blocked, suggesting they focus on specific variables/relations that can be generalized. These teacher interventions are directed at scaffolding the students' inquiry process by drawing their attention to different ways of exploring the table, by varying single variables or by observing the relationships between them. This scaffolding phase is crucial, since the teacher guides the students' inquiry, aiming at rendering them autonomous in developing it. Moreover, the teacher can also identify some student observations that should be discussed (with a certain order) in the following level of the activity (*Collective discussion,* in Figure 3).

Afterwards, the groups are asked to share their observations publicly and the teacher lists them on the whiteboard. At the *collective discussion* level, the teacher manages the class discussion in order to share the conjectures produced by the different groups. The teacher asks for clarifications, posing questions such as 'What does this word mean?', 'How can we better define this idea to present our observation in a more precise way?' The aim is to support students in sharing their ideas and to gather together the produced conjectures in the form: 'If A_i, then B_i'.

Referring to the 'patterns of numbers' activity, we assume that Grade 8 students can observe, for example, that—in each line—the difference between two numbers in the first and second columns (Figure 4a), is 2, while the number in the third column is the product of them.

Another regularity that can be observed is that each number in the third column is the difference between the square of a number (the mean between the two numbers in the first and second column). Suppose that the students have difficulties in discerning this regularity. In this case, the teacher can provide them with the table in Figure 4b, to direct their attention on the generalization pattern as the numbers vary, while keeping something invariant (*e.g.,* the constant –1). Students can become aware of the new pattern 'square of the mean minus 1' by *contrasting* square numbers (4, 9, 16, 25…) of the mean (2, 3, 4, 5, …) with the numbers in the third column (3, 8, 15, 24, …). To help the students being aware of the squared numbers, the teacher asks them to write the numbers in the third column in Figure 4b in different ways. Then, the teacher poses questions aimed at making students evaluate the correctness (or not) of the proposed observations through the analysis of the data or the development of experiments ('Do you agree with this?', 'Is it always true?') and at stimulating students' constructions of argumentations to support their conjectures and justify the

(a)

1	3	3
2	4	8
3	5	15
4	6	24
5	7	35

(b)

1	3	3=4-1
2	4	8=9-1
3	5	15=16-1
4	6	24=25-1
5	7	35=36-1

(c)

1	5	5=9-4
2	6	12=16-4
3	7	21=25-4
4	8	32=36-4
5	9	45=49-4

Figure 4. The pattern of numbers activity: (a) the situation (b) possible relations (c) further variation.

statements ('How can we justify this statement?'). At this point, the students' role changes from observer to discussant. The students should also be able to refer to theoretical knowledge to justify their observations, constructing statements in the form of 'this is/works/happens because…'. For example, in the 'patterns of numbers' activity, teacher/students can introduce symbolic representations to highlight that each product could be written as $x(x+2) = x^2 + 2x = (x^2 + 2x + 1) - 1 = (x+1)^2 - 1$, justifying, through generalization, the regularities they noticed. Therefore, aspects of a mathematical situation are represented in '*a specific formal system*', through a conversion from oral to symbolic register (Arrow I, Figure 1). Then treatments within a formal system are developed to prove the observed regularities (Arrow II, Figure 1).

After students come to an agreement and construct their justifications, the teacher (and ideally the students) should pose other questions in the form 'What happens if not A_i?' to make the students move forward in their inquiry (*What if it is not so,* in Figure 3). The formulation of such questions, which are aimed at making the students shift from one layer of inquiry to another one, is inspired by the *fusion* pattern of variation theory. At this level, the students' role gradually shifts from discussant to problem poser (Brown & Walter, 2005). Initially, the teacher could suggest possible new conjectures, asking 'What happens if A^*_i instead of A_i?'. In the case of the 'patterns of numbers' activity, the teacher, or better the students, can introduce a further dimension of variation and create additional inquiry by varying one of the invariant variables in the preceding situation. For instance, students can be asked about what happens if the difference between the numbers in the second and first column is 4 (or 6, 8, …) instead of 2 (Figure 4c). This question could lead them to observe that, when the difference is 4, the numbers in the third column can be written as the difference between the square of a number (the mean between the two numbers in the first and second column) and 4. Students, therefore, could experience generalization, since introducing this further dimension of variation corresponds to introducing new tables to be explored, characterized by a structure similar to the one of the initial table. The following exploration would lead them to conceptualize the difference between the numbers in the second and first column as a parameter, representing the general situation with the expression $x(x + 2n)$ and transforming it to gain a general result:

$$x(x+2n) = x^2 + 2nx = (x^2 + 2nx + n^2) - n^2 = (x+n)^2 - n^2$$

The construction of the expression $x(x + 2n)$ and its consequent transformation represents again the Arrow II (Figure 1), since it enables the students to interpret the expression $x(x + 2)$ as a representative of a broader class of expressions in the form $x(x + 2n)$. The interpretation of the expression $(x + n)x - n^2$ in relation to the problem (Arrow III, Figure 1) enables the students not only to prove a general property of all the possible tables constructed according to the rules identified during the initial exploration, but also to interpret the table in Figure 4a as a representative of a family of tables (with different values of parameters). In this way, the initial situation is interpreted as a member of a family of situations (Arrow IV, Figure 1).

Subsequently, the teacher could introduce a further dimension of variation and ask students to investigate possible new hypotheses. For example, within the 'patterns of numbers' activity, students could consider couples of numbers whose difference is not even. Through questions in the form 'What happens if …?' and 'What happens if not …?', it would be possible to progress to a new layer of MVI, which has a new situation to be explored. The diagram in Figure 3 can therefore be interpreted as a multi-dimensional diagram: every time a question in the form 'What if…?' or 'What if not…?' is posed, a shift to another layer is activated. This enables students to enlarge the family of problems/situations to which the initial problem/situation belongs.

At each step of students' new explorations, the teacher guides a classroom discussion (*collective discussion*, in Figure 3), during which new conjectures are formulated ('If A_{i+1} then B_{i+1}') and new arguments are constructed to justify them.

After some cycles of this inquiry activity, the discussion could be brought to the *meta-reflection level* (Figure 3), a new element that the MVI-model brings within the framework of the inquiry-based approach: students are asked to connect the real/mathematical situation with theoretical knowledge. The teacher stimulates the students to identify connections between the different conjectures produced, making them highlight the mathematics behind the examples. In reference to the 'patterns of numbers' activity, the teacher could make students reflect on the comparison between the regularities observed if the difference between the numbers in the first two columns is even and if it is odd, reflecting on the structure of the proof based on the algebraic expressions. Moreover, an analysis of the different roles (variable, parameter) played by the variables (x and n) could be developed.

The *meta-reflection level* (Figure 2) could represent a fundamental moment of the MVI-model within which the *virtuous cycle* can be effectively completed, since it is aimed at involving students in reflecting on connections between theoretical knowledge and the family of situations (Figure 1) under inquiry.

As stressed in the example above, MVI is designed in such a way that at each layer the students are supposed to construct specific knowledge regarding the specific task, and to connect different pieces of theoretical knowledge (not only mathematical) in the transition through the layers. The example also shows that an essential aspect of this method is that students are provided with resources (tables, algebraic expressions, graphs, dynamic diagrams…) to experience inquiry, to formulate conjectures and verify/refute them.

Conclusion

The MVI-model has been implemented in some teaching experiments, and the first results show that students did indeed develop competencies such as abstract representation, geometric thinking, functional thinking, explaining phenomena, building models and describing them, formulating questions, formulating claims and justifying them. Other, equally important competencies, the development of which we anticipated in the model's design, have yet to be empirically investigated.

For example, based on the idea that mathematics is a social activity, the instruction according to the MVI-model is built around group and class discussions. We assume that such teaching, under the proper guidance of the teacher, could introduce students to the discourse of the field of mathematics and provide an opportunity to engage in discussion according to the conventions of that discourse. Moreover, since the model asks students to formulate and justify their research conclusions in the form of 'If … then' and 'If not … then', we hypothesize that this requirement could improve the students' argumentation—their ability to present information in a fluid and convincing manner. Working in small groups (inspired by the idea of a mathematics laboratory) to solve mathematical problems should, we believe, develop students' ability to identify thought processes and make group decisions, and their ability to make decisions and carry out tasks in a cooperative manner (Anichini, Arzarello, Ciarrapico & Robutti, 2003).

As mentioned above, the tasks the teacher should prepare are not limited only to pure mathematical tasks, but also tasks and questions that include real-life aspects, which the students are asked to explain by mathematical means. Tasks of this type can develop the ability to: use mathematical knowledge to describe and explain phenomena and events; identify phenomena, and construct mathematical models to describe, explain and predict these phenomena. We assume that developing the ability to ask 'what if' questions should help students ask questions and explore new and unexpected directions, look for solutions in unusual places, and use existing knowledge in new contexts (Brown & Walter, 2005).

Further empirical investigations are needed also to face theoretical challenges related to the novel elements that the MVI-model introduces within the inquiry-based approach. In particular, empirical studies are needed to identify tools and constructs to develop a theoretically based analysis of the roles played by the teacher in fostering students' engagement within the virtuous cycle and to deepen the investigation of how teachers' interventions during both students working group activities and collective discussions affect students' inquiry processes.

The results of our case studies to date have also revealed possible challenges in implementing the model. These include teachers' difficulty in translating the model's principles into concrete task designs, in deciding when and how to intervene in the students' inquiry process, and in matching that intervention to the model's goals. Last—and perhaps most prominently—our results have revealed teachers' difficulty in relinquishing old instructional practices and adopting new ones. These results have raised questions such as: What are the features of effective professional development programs we might use to educate teachers to design tasks that match MVI's goals, and to critically reflect on their ongoing instructional practices? Is the MVI-model applicable to all mathematical topics? If not, to which topics is it most usefully applied? What kind of teacher intervention promotes inquiry processes and what might instead be hindering them? Investigating some of these questions will be the target of the next stages of our research.

Finally, the implementation of the MVI-model will need to be examined in a wider variety of schools and populations. We know that the MVI-model's components, as described above, constitute an overall 'big picture'. Additional studies are still needed to further develop and elaborate upon the MVI-model by breaking it down into smaller, more distinct components that will overcome the concrete challenges involved in its implementation.

Note

[1] The terminology employed here is from Duval (2006). Treatment refers to transformations within the same register, *e.g.,* when $(x - 1)^2 + (y - 1)^2 = 2$ is developed into $x^2 + y^2 - 2x - 2y = 0$. Conversion refers to transformations between different registers, *e.g.,* when the previous formula is interpreted as a circle in the Cartesian plane.

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