Communications

The potential of statement-posing tasks

KAI-LIN YANG

A *statement-posing task* is a setting where a proof is given without its corresponding propositions and students are asked to think what these arguments can prove. Figure 1 shows a statement-posing task for undergraduates mathematics students. These arguments adopt proof by contradiction to prove that if $x > 0$, $x < y$, then $\sqrt{x} < \sqrt{y}$. **A1**, **A2** and **A3** in Figure 1 express three properties which are applied in these arguments.

If students are given a proof like this, will they be able to find the statement that the proof proves? What benefits might such a task have for understanding the proving process? This article considers this question.

What do the following arguments prove?

Assume that $x > 0$, $x < y$, and $\sqrt{x} \geqq \sqrt{y}$.

Case 1. Assume that $\sqrt{x} = \sqrt{y}$. Then $x=(\sqrt{x})^2 = (\sqrt{y})^2=y$ and hence,

by **A1**, we cannot have $x < y$.

Case 2. Assume that $\sqrt{x} > \sqrt{y}$. Since $\sqrt{x} > 0$,

by **A2**, we obtain $\sqrt{x}\,\sqrt{x} > \sqrt{x}\,\sqrt{y}$. Also, by **A2**, $\sqrt{x}\,\sqrt{y} > \sqrt{y}\,\sqrt{y}$.

Now by **A3**, $\sqrt{x}\,\sqrt{x} > \sqrt{y}\,\sqrt{y}$. Hence, by algebra, $x > y$.

But by **A1**, this is contrary to the assumption that $x < y$.

A1: *"For any real numbers x and y, exactly one of $x < y$, $y < x$, and $x = y$ is true."*

A2: *"If $x < y$ and $0 < z$, then $xz < yz$."*

A3: *"If $x < y$ and $y < z$, then $x < z$."*

Figure 1. Proof without its proposition for undergraduates of mathematics

Many studies have suggested global difficulties in understanding mathematics proof (*e.g.*, Healy & Hoyles, 2000). The need for understanding the deductive and validating method is not well supported by teaching materials for students and teachers (Lin & Yang, 2007). Formal proof is often understood as ritual by students' (Harel & Sowder, 1998). Innovative tasks for experiencing and understanding the deductive and validating method of formal proof are still called for. Problem-posing tasks provide opportunities for students to formulate problems from given situations or problems (Silver, 1994). To adapt this idea to mathematical proof, figures or properties could be substituted for situations or problems (*e.g.,* Lin & Yang, 2002). Students could make conjectures from false propositions [1] or pose statements from arguments (Yang & Wang, 2008).

According to Yang and Wang (2008), the types of statements posed by students can be classified into the five categories: premise-conclusion, only premise or conclusion, only applied property, only proof method, and description of proof. However, statements posed by students in the form premise-conclusion were not necessarily validated by these arguments. Furthermore, it is surprising that some undergraduates who had identified that these arguments adopted proof by contradiction did not distinguish assumptions from premises. Students seemed able to recognize the proof method or some of the contradictions but found it hard to deeply understand the logical function of contradiction.

Reading is a process of constructing relations among parts of the text, and between the text and readers. In language literature, there is a substantial research base suggesting that questioning while reading is one crucial strategy which can enhance reading comprehension. In Yang and Wang (2008), students used multiple reading strategies like overall review, backward inference, labeling, rewriting, and translating symbols to figures and vise versa. In addition, the reading strategy where two competing statements were posed and then validated by re-reading proof was used by some students. It is similar to conceiving of alternative hypotheses which is one of the central competencies underlying scientific thinking for coordinating between theory and evidence (Kuhn, 1991). This is one of the reasons why we thought of asking students to pose statements for given arguments. Four points are advanced as potential benefits of statement-posing tasks: (1) focusing on the logic of arguments in addition to the meaning of arguments, (2) a way to understand the logical function of proof methods, (3) facilitating dialectical reading strategies and (4) a generic example of reading tasks.

Focusing on the logic of arguments in addition to the meaning of arguments

While students read mathematics proof by recognizing the meaning of each argument separately, statement-posing tasks can compel them to identify the multiple status of arguments. According to Duval (1998) the three levels of organizing proof steps are to theoretically link a step, several steps and all steps in a proof respectively. Before organizing proof steps, one naturally understands terms or signs based on the epistemic value. While linking a proof step or several proof steps, the focus is the logical value of statements instead of the epistemic value of statements. While seeing all proof steps as a whole, both the logical and epistemic values of statements are required to correctly apply and creatively extend. Thus, readers are guided to focus on the status of arguments (*e.g.,* premise, conclusion or applied property) in addition to their meaning.

A way to understand the logical function of proof methods

Many undergraduate students still lack understanding of proof methods and show fragile knowledge (Antonini &

For the Learning of Mathematics **30**, 2 (July, 2010)
FLM Publishing Association, Edmonton, Alberta, Canada

Mariotti, 2008; Dubinsky & Lewin, 1986; Movshovitz-Hadar, 1993), even if the two proof techniques of mathematical induction and proof by contradiction have been emphasized in most university mathematics curriculums. Many students have difficulties in understanding the logical equivalence of the following two affirmations: (1) to agree with the validity of a proof, and (2) to agree that this proof guarantees the truth of a theorem (Fischbein, 1982). They also have difficulties in understanding that the following two affirmations are not logically equivalent: (1) to agree with the validity of a proof, and (2) to agree that this proof guarantees that a theorem cannot be more general (Stylianides, Stylianides & Philippou, 2007). Statement-posing tasks provide opportunities for students to overcome these difficulties by inferring one statement from these arguments and considering the extent of what these arguments can prove.

Facilitating dialectical reading strategies

A statement-posing task provides some arguments and one question, but the proof leads readers to ask more questions and to become aware of logical relations among these arguments. A recognition of logical relations leads students to move from a conception of focusing on the epistemic values of each argument to one in which logically validating is embedded in how they read these arguments. Thus, the statement-posing tasks can enhance students' decoding, semantic, and syntactic competence for understanding proof texts instead of just connecting a set of arguments of the proof for epistemically confirming the theorem. This means that students engage with the proof text in a way that goes beyond reading it as a confirmation resource and open oneself up to different ways of reading the arguments. The students must see these arguments in relation to their proposed statements as a whole. Under the tasks, understanding proof is the journey, not just the arrival.

A generic example of reading tasks

While a common weakness of doing, writing, and discoursing activities is that the logical value is not explicitly distinguished from the epistemic value (Yang & Lin, 2008), reading is suggested as a supplementing activity. However, the promising reading activities for connecting the logical and epistemic values are still few. The statement-posing task presented in this paper could be an example. Many modifications are allowed. For example, (1) to change the arguments presented in statement-posing tasks according to school levels of students, (2) to change this question into multiple-choice questions which may purposely guide students to focus on the status of some key arguments, or purposely guide students to focus on the appropriateness of the applied properties, and (3) to ask students to arrange the arguments logically and then pose a statement which is proved by the arranged arguments, which may help them understand logical implication and its converse. The statement-posing tasks are a starting point for putting reading into practice. Further research on designing reading tasks for learning mathematics proofs may create opportunities for making mathematics proofs accessible to more students.

The significance of this paper lies in revealing the potential of statement-posing tasks to facilitate students' thinking and strategies of understanding proof. In sum, such tasks can guide students to construct statements with evaluating the epistemic and logical values of the proof, to combine the instrumental and relational understanding of proof methods, and to make understanding proof dialectical. Such tasks are also a model for designing reading tasks. Future research can establish to what extent students benefit from learning the tasks.

Acknowledgments

The author wishes to thank Frank Quinn and Richard Barwell for their helpful comments on earlier drafts. This paper is part of a research project funded by the National Science Council of Taiwan (NSC 96-2521-S-018-004-MY3). The views and opinions expressed in this paper are the author and not necessarily those of the NSC.

Note

[1] See Lin, F. L. and Wu Yu, J. Y. (2005) *False proposition as a means for making conjectures in mathematics classrooms,* invited presentation at the Asian Mathematical Conference, 20–23 July, Singapore.

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