MATHEMATICS TEACHING—JOY IN TIMES OF TROUBLE?

TIM ROWLAND

"When it's dark enough, you can see the stars" [1]

As I write, we live in strange and troubled times, with self-isolation a requirement for most residents in the UK, and in many other parts of the world. Only recently have I learned that the word 'quarantine' derives from a dialect of 14th century Italian, designating the period of 40 days during the Black Death when ships were required to be isolated in port before disembarkation. It denoted a state of enforced isolation, intended to prevent the spread of the disease. In 2020, these 40 days have stretched to 50, 60, … who knows? With no opportunity to go out for coffee, to the cinema, theatre—indeed, to most shops, but that makes space for a lot of thinking time.

Commentators, in the newspapers and other media around the world, are now alerting us to the near-certainty that things will not be 'as they were' when the shops, offices, schools and universities, cafes and restaurants—and the rest—reopen. But what of teaching and learning? Early in 2020, my own university became the first in the UK to announce that there would be no face-to-face lecture-hall teaching for the whole of the next academic year, because of the coronavirus pandemic. Not long afterwards, other UK universities were planning forms of 'hybrid higher education' in order to avoid large gatherings in confined spaces and to enable 'physical distancing'. What were once 'lectures' would be presented to students in a format that could be accessed online.

Now, these 'lectures' are a mode of instruction—what I prefer to call 'teaching'. In an account of a mathematics lesson with 5-year-old students, Marie Therese Farrugia (2017) reports having "experienced the joy of teaching very young children, who are quick to show appreciation and affection" (p. 13). In this article I describe such an experience of joy, but with much older, university students. For teaching is a joyous profession; predictably, perhaps, I feel compelled to add that the teaching of mathematics is—or can be—especially joyous. Which brings me to what now follows.

Some years ago, I emerged from a mathematics session with a class of young adults, overflowing with that sense of joy, and amazed that my occupation made it possible. Over the next few days, when I could, I reflected upon that session, and recorded what had happened in it. After a while, I realised that this might be something that I could share with others, in an article.

About 'joy'

Here, I pause to consider what it might mean to experience 'joy', if only because use of the word must be unusual in the mathematics (education) literature. Comments from the 'readers' of my first submission of this article included a request that I "develop a deeper conceptualisation of 'joy'". In fact I had begun to do so already, and I en*joy*ed (!) giving it further thought and effort.

The Greek philosopher Epicurus distinguished between two kinds of enjoyment: the rewards of activity experienced whilst immersed in the activity, and those experienced after engaging in it. His 'Vatican sayings' include "In other pursuits, the hard-won fruit comes at the end. But in philosophy, delight keeps pace with knowledge. It is not after the lesson that enjoyment comes; learning and enjoyment happen at the same time" (Saying 27). So, how might joy, or delight, or pleasure, be related to activity? I might compare beholding a loaf of bread that I've already made with the 'live' pleasure experienced when I'm dancing or singing. Wherein is the joy in each case? Epicurus tells us that the joy derived from engaging in philosophy is of the second kind; that it is inherent in the activity itself. In this article, I shall describe an experience of this second kind, in the activity of teaching mathematics, in interaction with a class of students. At least, that joy-in-the-moment is what I recorded in my account of one hour with a small class of undergraduate students. And yet, as I recall it, even now I experience something of Epicurus' second kind of joy, in the outcome.

Attempting a stipulative definition, we might say that joy is an extreme kind of happiness. But my account here, of a classroom episode, comes closer to an ostensive exposition of 'joy'. Wittgenstein (2009) wrote that "an ostensive definition explains the use—the meaning—of a word if the role the word is supposed to play in the language is already clear" (¶ 30). An ostensive definition consists of an example of the object under consideration. Like 'love', perhaps, maybe we only come to know what 'joy' is by experiencing it.

One final comment, before I move on. The philosopher Bertrand Russell (1919) captured the (or his) experience of joy in engaging with mathematics as follows, "Mathematics, rightly viewed, possesses not only truth, but supreme beauty […] The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry." (p. 60). But for the moment, in what follows, I am not so much reflecting on joy in doing or learning mathematics, but in assisting others to do so.

An article?

For some time, something prevented me from writing-to-share. Eventually I realised what was inhibiting me. It was this: I had recorded something personal, and I found it difficult to distance myself from what I had written. Worse, I feared that the style might be too self-congratulatory; I had done some teaching, and felt good about what had happened. I put the 'article' to one side, somewhere on my hard drive, probably in good company. But that sense that there was something worth sharing persisted.

One influence that encouraged me to return to what I had written was a chapter by Bob Burn (2002), on the possibility of the genesis of 'new' mathematics in university mathematics classrooms. Burn refers to Hans Freudenthal's distinction between three different phenomena in mathematics education:

> 1. The articulation of formal structures, their logical connections and their applications are the substance of the phenomenon of *mathematics*.
> 2. The interactions between teacher and student, and the activities engineered by the teacher for the student are a *didactical* phenomenon.
> 3. The experiences that the student goes through in learning some mathematics are a *genetic* phenomenon of a psychological nature. (Burn, 2002, p. 21)

Freudenthal writes, "No mathematical idea has ever been published in the way it was discovered" (1983, p. ix).

Burn comments:

> Freudenthal claimed that mathematical practice often led to a didactical inversion in which the genetic sequence was reversed in exposition. […] The best pedagogy is informed by knowledge of both genetic and formal structure and the difference between them. (p. 21)

What follows is an account of teaching which was underpinned by a desire to resist that didactical inversion, despite its elegance and temporal economy—a *conscious* effort to respect the genetic sequence, and to observe and describe the didactical phenomenon.

First, I should explain that I was enjoying the luxury of teaching mathematics to small classes of about 25 students. 'My' students were learning to be teachers of young(ish) children within the age-range 5 to 13. They had already experienced success as mathematicians at school, so as to achieve their university places. Now they were learning to be mathematicians in a university context. Hopefully, they were also learning to be learners themselves. Arguably, the distinctions between the different kinds of learning are false. How can one teach mathematics without knowing (some) mathematics? How can one know what is required of a teacher without awareness of what it is to learn? How can one become a mathematician in the absence of knowledge about how to learn it? These questions are intended to be rhetorical, yet, in each case, it is clearly possible to function in the first domain in the absence of sensitivity to the second. I return to these issues later.

The mathematics lesson

That day, in the classroom, I experienced something very good. It felt good at the time, and I write about it now because the sense of its being special persists.

My 'lesson' was with a class of first-year undergraduate students who intend to become teachers in primary or middle schools. Many (or most) of them were not long out of school; I smiled inwardly when, on occasion, one of them spoke of my 'lessons', our times together which the system dignifies with the name of 'lecture'. In fact, both 'lesson' and 'lecture' derive from the same Latin root *legere*, 'to read'. Since I had no intention of reading notes at them, both names seem inappropriate. We might speak of a 'session'. This time the root is *sedere*, to sit. Yes, they did sit most of the time, and so did I whenever an opportunity presented itself.

The mathematics belongs to the Theory of Numbers—accessible, but not trivial. It is pertinent to remark that this is an area of mathematics that fascinates me, to which I have significant personal commitment. To teach it is not a chore, but a privilege, and in saying so I am not indulging in hyperbole. For me, the Theory of Numbers has been an inner laboratory, more like a playground, in which I experimented and theorised as an adolescent. In my late teens I first discovered the existence of books on primes, divisibility and the like, and I then studied number theory as an undergraduate, and eventually returned to it as a teacher.

The topic under scrutiny with this first year class is the Euler function, usually denoted φ. The function can be approached and subsequently defined in a number of ways, but the following needs least introduction: for any natural number n, $\varphi(n)$ is the number of integers in the range 1 to n which share no factor (apart from 1) with n. Thus, $\varphi(6) = 2$, because 1 and 5 are the only positive integers less than and 'coprime' with 6. In principle, $\varphi(n)$ can always be evaluated by inspection of a list of the integers from 1 to n, but this can be a tedious task if n is large. The purpose of the session was to arrive at a more 'efficient' way of finding $\varphi(n)$ in such cases. The previous session had concluded with agreement that, when p is a prime number, $\varphi(p) = p - 1$. It was a straightforward inductive inference from a couple of examples, and an explanation was articulated without undue difficulty by the students whom I questioned.

Following a short review of where we'd got to with $\varphi(p) = p - 1$, I suggested that we went on to consider $\varphi(p^2)$. I asked them to find $\varphi(9)$ and $\varphi(25)$, and to make and write down a conjecture about $\varphi(p^2)$, to discuss it with a colleague—as one does. (You might like to try it yourself, before proceeding.)

In discussion with groups of students, and then with the whole class, they proposed that $\varphi(p^2) = p^2 - p$, as an inductive inference (Rowland, 1999) from the data $\varphi(9) = 6$, $\varphi(25) = 20$. I had prepared to 'give' them a 'generic' proof (of which more later) but one of the students, Hadley, had already offered me one earlier (he called it an 'explanation') when I had spoken to his group. I had congratulated him on his insight, and asked whether he would be prepared to share it with the class. He had groaned in mock dismay. Nevertheless, I asked him to say why it was clear to him that $\varphi(p^2) = p^2 - p$ for any prime p. Hadley rose from his chair and walked to the whiteboard. From my chair at one of the tables, I passed him a pen.

Hadley hesitated, then wrote 1 2 3 4 5 6 7 8 9, and went on to say that the numbers in this list that are not coprime with 9 are the multiples of 3, and there are 3 of those because 9 divided by 3 is 3. That leaves 9 – 3 numbers that are coprime, so $\varphi(9) = 9 - 3$. And *it would be the same* for 25, added

Hadley. In the list of numbers from 1 to 25, only the 5 multiples of 5 are not coprime with 25.

Hadley's proof is by 'generic example'—a confirming instance of a proposition, carefully presented so as to provide insight as to *why* the proposition holds true for that single instance. The class was familiar with the term, and the intention behind embedding a general argument within a particular case.

> The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a *characteristic representative* of the class. (Balacheff, 1988, p. 219, italics added)

I thanked Hadley, and asked the class something like "Is that OK?". I then asked whether Hadley's examples enabled them to 'see' why the argument would 'work' with any prime-square number. What I actually asked them was whether, for them, Hadley's example was indeed 'generic', having introduced them to the term in a previous course on mathematical processes, and referred to it often in discussing proofs.

It then occurred to me, not to tell them what we would do next, but to ask what they thought we might do. Emma replied, "Look at *p*-cubed". It has to be said that I was pleased, more than pleased, because this was precisely what I had in mind. I had a sense that their agenda for enquiry—well, Emma's, at least—was the same as mine.

I asked them how we might go about 'looking' at cubes of primes. What approaches came to mind? Hadley suggested, presumably in keeping with the earlier approach to squares of primes, that we might first evaluate $\varphi(27)$, then perhaps $\varphi(125)$, and try to make a conjecture. What else might we do, I asked. Abby suggested that we could look at Hadley's proof for p^2, and see whether it could be adapted.

Inside, I was elated, for these were the two approaches—the only two—that had occurred to me seconds before I had asked them for suggestions. The sense of unity of purpose was amazing. I invited them to go ahead with Hadley's suggestion, or Abby's, whichever appealed to them, working in pairs or threes if possible. The subsequent class discussion centred on Abby's approach. The numbers in this list 1, 2, 3, …, 27 that are not coprime with 27 are the multiples of 3, and there are 9 of those because 27 divided by 3 is 9. That leaves 27 – 9 numbers that are coprime, so $\varphi(3^3) = 3^3 - 3^2$. And *it would be the same* for 5^3. So the argument about 3 and 3 cubed really was seen as a 'characteristic representative' and $\varphi(p^3) = p^3 - p^2$ is proved by generic example. Whereas it has sometimes felt so difficult, this time it seemed so easy, so natural. To complete this account of the mathematics, suffice to say that they conjectured that $\varphi(p^k) = p^k - p^k - 1$ for all positive integers *k*, and they could see how the proof would be accomplished for any particular value of *k*.

I took a moment to let them know how all this felt from my perspective. I didn't make too much of it, because that would have embarrassed them and, probably, me too. I just said that it felt very good to realise that they had been able to see the session as a process of enquiry to which they could make a significant contribution as opposed to positioning themselves as passive receivers of 'my' mathematical knowledge. That there seemed to be some spin-off ('transfer'?) from the processes course we had engaged in the previous term, in that they could see in a mainstream 'textbook mathematics' course the possibility of asking "What next?", and consider different options for *how* to proceed.

There is, of course, a dilemma here. The teacher wants the students to engage in a process of enquiry, but—at the same time—the teacher knows what they should come to know as a consequence! Some time ago, in the context of science teaching, Rosalind Driver (1983) detected a kind of 'intellectual dishonesty' in this dilemma. She wrote, "On the one hand pupils are expected to explore a phenomenon for themselves, collect data and make inferences based on it; on the other hand this process is expected to lead to the currently accepted law or principle" (p. 3).

For the most part, what Driver writes is true. Whereas—as I have tried to show—the genetic sequence underpinned my pedagogy (and that of some other mathematics colleagues) in the lecture room, the underlying intention was that the students would come to know and to understand the topics and results to be found in the standard undergraduate textbooks on Real Analysis, Group Theory, Vector Mechanics, Number Theory, and so on. In the 'lesson' described here, my aim was for the students to know and to understand results about the Euler function that they would find in textbooks.

Induction, time, and the syllabus

In the session described above, it took about an hour to conjecture that $\varphi(p^k) = p^k - p^{k-1}$ and to prove it. By conventional norms of undergraduate teaching, that was an extravagant use of time. The previous term, I had 'taught' Wilson's Theorem to the same class of first-year undergraduate students. That also took the best part of an hour. This brief account of that session should suffice to indicate why it took so long.

I began by asking the students to evaluate 4! mod 5, 6! mod 7, 10! mod 11, and to write down a conjecture. The most common version of the conjecture was $n! \equiv n \bmod (n + 1)$. (The 'for all *n*' seemed to be implicit.) I asked them to evaluate 5! mod 6. They did, and they were visibly surprised by the refutation. I asked whether they could modify the conjecture. At first they homed in on the even/odd distinction between moduli, but trying $n = 8$ (*i.e.,* modulo 9) led to further refutation and eventual restriction to prime values of $n + 1$ and $n = 12$ provided a further confirming instance. This time, I proceeded to an interactive presentation of a generic proof, inviting Sonia to pick a prime between 11 and 19. She chose 19. I got them to list 1 to 18 and work on inverse pairs in table-groups, during which Simon spontaneously explained to his colleagues why 18! *had* to be 18 mod 19. I asked him to repeat his reasoning to the class, and to write his explanation on the whiteboard. They dutifully copied it. Later, I enquired what would have happened if we had looked at 28! mod 29. Abby talked about inverse pairs mod 29, and argued that 28! mod 29 would have to be 28. "Does everyone agree?", I asked. They agreed. One shouldn't read too much into such consent, however pleasing; sometimes, they just want to get a good place in the queue for lunch. Nevertheless, Abby, at least, had convinced me that she had appropriated the proof-scheme (in the sense of Harel & Sowder, 1998).

The next day, at a tutorial meeting, I asked five members of the class to write out the proof, that $p! \equiv p - 1 \bmod p$ (for primes p), in conventional generality. Their responses were unaided and individual. Hannah's response, which was typical, was as follows.

> $(p - 1)(p - 2)(p - 3)(p - 4) \ldots 2 \times 1$
>
> Every element of M_p has an inverse, because M_p is a group. [2]
>
> We know (from work on primitive roots) that only $p - 1$ has order 2. Therefore $p - 1$ is self-inverse. All other members of M_p apart from 1 must have a distinct inverse.
>
> Each inverse pair when multiplied gives 1 mod p.
>
> This gives $(p - 1)(1^{½(p-3)})1 \equiv (p - 1)! \bmod p$
>
> Therefore $(p - 1)! \equiv (p - 1) \bmod p$

But what about the time? Of course, for Wilson's Theorem and for the Euler function, I could have stated the result and proved it formally in five minutes. Time is precious, increasingly so as lecturers' time is spread thinly across courses and programmes. Moreover, their research will determine their prospects of promotion more than their teaching. Elsewhere (Rowland & Hatch, 2007) I have described an experience of 'mathematical investigation', in the preparation of a paper submitted to a mathematics teachers' journal. This was three years after gaining my first degree in mathematics, by which time I was employed at a tertiary college, teaching mathematics to young prospective elementary school teachers. I wrote:

> In retrospect, this paper shows that I had gained an awareness of mathematical enquiry at the college that I am sure I had not acquired by study of advanced mathematics at university. It was, in effect, the seed of a fascination with inductive reasoning that has stayed with me ever since. (pp. 79–80)

Somehow I had come to experience the genetic sequence, and acquired a determination that, if it were possible, my students would experience it in 'my' classrooms.

What kind(s) of mathematical knowledge?

So what do we want our mathematics students to learn? Answers to this question might be different for students at different stages in their learning—elementary, secondary, tertiary. One approach to an answer might address what they will do with their mathematical knowledge and expertise. If workplace application is an ambition, then knowledge of some of the ways that mathematics is applied—in science and engineering, in finance, in technology—will be prioritised. In any case, it will be good for students to come to know the distinctions between knowing *that*, knowing *why* and knowing *how*. For the mathematician, inductive reasoning is a form of discovery and a cause of delight, like the metal detectorist's discovery of ancient coins buried in a field. To take an elementary example, consider sums of consecutive odd numbers: we find 1, 1 + 3 = 4, 1 + 3 + 5 = 9, 1 + 3 + 5 + 7 = 16 … Yes, we have to recognise that the sums are square numbers, but what a surprise! Of course, we have yet to find whether every such sum is a perfect square, and if so, why.

Many years ago, Joseph Schwab (1978) made the distinction between what he called substantive knowledge and syntactic knowledge. For the most part, the detail of these kinds of knowledge will look different in different fields of knowledge, The first of these includes the key facts, concepts, principles and explanatory frameworks of a discipline—so they look different in history, say, as opposed to mathematics. One would expect to see all of them included in undergraduate mathematics education: explanatory frameworks, for example, could include the purpose of proof, and different approaches to achieving it. The second, Schwab's 'syntactic knowledge', is knowledge about the nature of enquiry in the field, and the mechanisms through which new knowledge is introduced and accepted in the relevant disciplinary community. This second kind of knowledge would be expected to be acquired in graduate study and research, and yet syntactic knowledge of how learners encounter and acquire new knowledge is vital to all *teachers*. For teachers of mathematics, it includes knowledge about inductive and deductive reasoning, the affordances and limitations of exemplification, and problem-solving heuristics and proof. I suggest that at all grades, and in tertiary education in particular, the acquisition of substantive *and* syntactic knowledge should be on the agenda. Both were evident in that lesson on the Euler function and as I saw both of them flourish, 'joy' was the word that captured my response to that shared experience.

Closing thoughts

In 1661, Isaac Newton came as a student to Trinity College, Cambridge. He paid his way by working part-time as a college servant until he was awarded a scholarship in 1664. In the following year, Newton gained his Bachelor's degree, and the Great Plague broke out in England, with 100,000 deaths. Cambridge University closed as a precaution, and Newton was reduced to studies-in-isolation at his home in Woolsthorpe, where his twice-widowed mother had raised him. In the next two years, seemingly without the benefit of online resources from his university, Newton began to develop his theories of calculus, optics, and of gravitation [3]. He returned to Cambridge in 1667 as a senior member of his College.

Well, it would be optimistic to generalise from the particular case of Isaac Newton. But somewhere in his pre-pandemic years, Newton had evidently learned how to learn—to be curious, to know what might be a good question, a good problem, and how to set about solving problems and answering questions, to develop 'answers', or theories, to explain why the world was the way he observed it to be. Which is where I came in—in the same city at least—mathematics teaching, including university mathematics teaching, is at its best when it is more than telling, more than 'delivering' content for the student to rehearse and reproduce. Research by Burn and Wood (1995) might justify some optimism as we try to teach and to learn in 'lockdown'; in their study of undergraduate mathematics student experiences at two universities, just one course was found to have earned unqualified student admiration. It was given by a lecturer who acknowledged that he could not provide a worthwhile student experience within the given time constraints. He had developed interactive lecture notes and had toyed with the idea of abolishing lectures altogether. The Digital Education Research Group at the University of Edinburgh seems to have anticipated—and welcomed—the current prevalence of online teaching and learning by a decade. Their 'Manifesto for teaching online' (Bayne *et al.,* 2020) expounds 21 statements, beginning with "Distance is a positive principle, not a deficit. Online can be the privileged mode". They argue that, in several respects, online teaching and learning is more equitable than face-to-face instruction, and that "freedom from the requirement for physical and temporal co-presence can work to the benefit of many, much of the time" (p. 12).

Person-to-person instruction, whether online or face-to-face, is both a cognitive and an affective experience for all participants—teacher(s) and student(s). In her 2011 *FLM* article, Julie Long considered what it might mean to 'care' for a student by attending to their learning of mathematics. I have tried to describe 'care' of that kind in my management of a class encounter with the Euler function. In her recent book, Anne Watson writes, "If I can communicate my enthusiasm for mathematics within a positive and caring relationship, those for whom I am caring can understand that such enthusiasm is possible and may even join in" (2021, p. 79). At its most joyous and enjoyable, mathematics teaching—whether in one room or at a distance—can enable the student to explore, to investigate and engage with examples, to find inductive reasoning irresistible, to know the epiphanic experience of constructing a generic example. Drawing on Nel Noddings (2003), Long reminds us that "reciprocity is one feature of a caring relationship. Reciprocity traces the movement of care—literally the give and take of the relationship. Care is not unidirectional; when the person receiving the care—for example, a student— acknowledges and recognizes the care, he or she completes the relationship" (2011, p. 2). In retrospect, I believe that reciprocity of this kind was present in the engagement of Abby, Emma, Hadley, and others perhaps, thereby enabling their 'teacher' to experience joy—in that moment, and even now, in retrospect.

Notes

[1] Attributed to Charles A. Beard in 'Condensed history lesson', *Readers' Digest,* February 1941, p. 20

[2] Recall the notation M_p for the group {1, 2, 3, …, p – 1} under multiplication mod p.

[3] Some scholars have questioned the veracity of Newton's own account of his *annus mirabilis,* which is held in the archives of the Cambridge University Library (Westfall, 1980).

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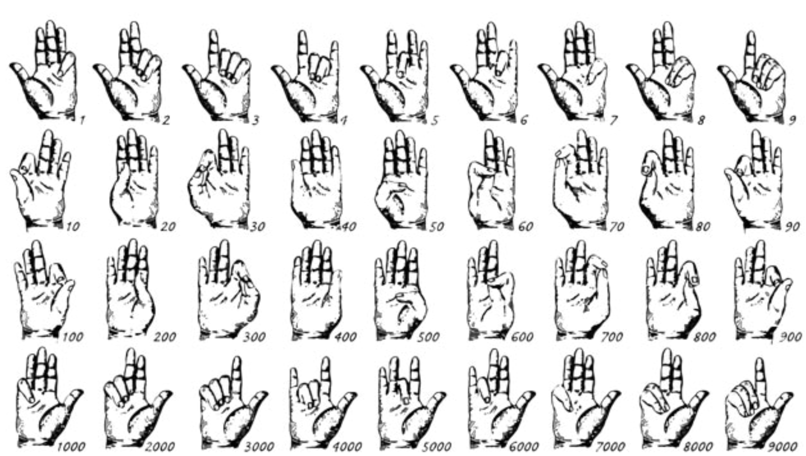

Finger numerals from a woodcut printed in 1520, based on those described by the Venerable Bede in 'De temporum ratione' (725). Such numerals were important to trade in the medieval world.