THE IMPACT OF INFORMATION AND COMMUNICATION TECHNOLOGY ON OUR UNDERSTANDING OF THE NATURE OF MATHEMATICS

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The incorporation of information and communication technology (ICT) into mathematics education constitutes one of the most important themes in contemporary mathematics education. For example, one of the six principles of school mathematics as espoused by the National Council of Teachers of Mathematics (NCTM, 2000) is that

> Technology is *essential* in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning. (p. 24, emphasis added).

The purpose of this article is to review research findings on the effectiveness of the use of some selected ICT tools in mathematics teaching and learning. [1] This review is then set against a summary of the actual use of ICT tools in real classrooms. Various issues related to the incorporation of ICT into mathematics education arise, including:

- the reasons for the discrepancy between the potential and actual use of ICT
- the impact of ICT on our understanding of the nature of mathematics
- the implications of this understanding for mathematics teaching and learning.

Potential effectiveness of ICT tools

It is beyond the scope of this article to present a comprehensive review of ICT tools. In this section, research findings on the use of a common hardware (graphing calculators), a general purpose software (spreadsheets) and two mathematics-specific software (one for algebra: Computer Algebra System (CAS) and one for geometry: Dynamic Geometry Software (DGS)) will be reviewed to illustrate the potential effectiveness of ICT tools for mathematics teaching and learning.

Graphing calculators

The graphing calculator can play different roles in the teaching and learning process. It can be used merely as a computational tool (when used this way, the graphing feature is not capitalized on – it is used just as an ordinary scientific calculator) and it can also be used as a transformational tool, a data collection and analysis tool, a visualizing tool and/or a checking tool. Doerr and Zangor (2000) summarized the different roles the graphing calculator plays and the corresponding student actions as follows:

Role of the Graphing Calculator	Description of Student Actions
Computational Tool	Evaluating numerical expressions, estimating and rounding
Transformational Tool	Changing the nature of the task
Data Collection and Analysis Tool	Gathering data, controlling phenomena, fiding patterns
Visualizing Tool	Finding symbolic functions, displaying data, interpreting data, solving equations
Checking Tool	Confirming conjectures, understanding multiple symbolic forms

Figure 1: Table showing the patterns and modes of use of a graphing calculator (p. 151).

In the research literature, it has been reported that the graphing calculator contributes positively to mathematics teaching and learning. For example, Embse (1992) found that

> the graphing capability of the graphing calculator helps students make connections among the numerical, symbolic, and graphical representations of a mathematical relationship. (p. 78)

and Ruthven (1990) reported that students using graphing calculators made a significantly stronger linkage between the algebraic form and the graphic form of functions than students not using it. Shoaf-Grubbs (1995) also reported that in using the graphic calculator to teach algebra, students' performance has been improved, especially on spatial visualization and mathematical understanding. Penglase and Arnold (1996) found the graphing calculator to be a powerful tool for teaching and learning the concept of function.

Spreadsheets

Spreadsheets were originally developed as (and are still commonly used as) accountancy tools. They are in essence computer programs for making multiple calculations that do not require the use of a programming language. Spreadsheets first found their use in mathematics teaching and learning in the area of algebra. Pioneer work was done in the 1980s by researchers such as Healy and Sutherland (1990), who found that spreadsheets helped students develop powerful mathematical ideas such as generalization, symbolization and functional relationships.

Spreadsheets explicitly demonstrate in numerical form values and relationships in any problem or content domain. Identifying values and developing formulas to interrelate them enhance learners' understanding of the algorithms used to compare them and of the mathematical models used to describe content domains (Jonassen, 2000). Students' willingness to monitor solution methods and their results were increased considerably by being released from computations and algebraic manipulations and by being able to relate to the meanings attached to the problem situations.

Studies in the 1990s (*e.g.*, Filloy, Rojano and Rubio, 2000; Kieran, 1992; Rojano, 1996; Sutherland and Balacheff, 1999) analyzed spreadsheets as an intermediate expression between natural or numeric language and algebra. It was claimed that spreadsheets would enable students to cope with the transition from a numeric or verbal representation to a symbolic representation; from the specific to the general, from the known to the unknown, and from intuition to abstraction. Studies of students working with spreadsheets on arithmetic or beginning algebra problems found students' communicative power enhanced, and interesting and powerful thinking strategies evolved from students' use of spreadsheets as a problem-solving tool (Ainley, 1996; Sutherland and Rojano, 1993). Spreadsheets were also found to contribute positively to students' mathematization in the domain of beginning algebra, and they were successfully used as a technological tool for other areas in mathematics as well as in science (Hershkowitz *et al.*, 2002).

Computer Algebra System

A computer algebra system (CAS) is software that works with strings of symbols. Artigue (2002) argued that CAS provides a means of facilitating and extending experimentation with mathematical systems, including generalization. Graphic and symbolic reasoning through using CAS influences the range and form of the tasks and techniques experienced by students, and so the resources available for more explicit codification and theorization of such reasoning.

Keller and Russell (1997) found that students who used CAS technology were more successful at having correct solutions and at producing correct answers given that they had a correct solution method, and they were more able to concentrate on developing their conceptual understanding of mathematics and their problem-solving skills. Shaw *et al.* (1997) also argued that students using CAS technology were better able to develop "their mathematical skills by freeing themselves to focus on understanding the problems and doing the mathematics" (p. 179).

Dynamic Geometry Software

Dynamic geometry software (DGS), as the name suggests, is software that deals with geometry in a dynamic manner. A typical example of DGS is *Geometer's Sketchpad*. Healy and Hoyles (2001) pointed out that *Geometer's Sketchpad* allows students to use a mouse to interact directly with the tools provided by the system so that they can build, manipulate and explore figures. Through using *Geometer's Sketchpad*, students are able to make conjectures that can be tested. It offers fast and non-judgmental feedback to students, and opens up their minds to accept lots of possibilities. Many of the problem solving strategies students use and the experimental stance to challenging problems adopted were unimaginable before the advent of DGS.

Osta (1998) argued that DGS can provide a valuable means for visualizing geometrical situations. The animation capabilities of DGS provides ways for constructing, moving and rotating configurations, for observing them under various angles, and for modifying some of their features. Laborde (1998) maintained that DGS provide visual evidence in solving geometry problems. She argued that visual evidence plays an important role in the problem solving process. Visual evidence is interpreted in geometrical terms and generates questions that are solved by means of geometry. Geometrical analysis triggers new questions, which may be empirically explored, giving rise to experiments with the software.

Another popular example of DGS is *Cabri-Geometry*. *Cabri-Geometry* provides a 'real' model of the theoretical field of Euclidean geometry in which it is possible to handle, in a physical sense, the theoretical objects which appear as diagrams on the screen. The behavior of *Cabri-Geometry* is based on geometrical knowledge in two ways (Laborde, 1998):

1. diagrams can be drawn, based on geometrical primitives which take into account relevant geometrical objects and relationships, and

2. it offers feedback which can distinguish diagrams drawn in an empirical way from diagrams resulting from the use of geometrical primitives.

DGS and geometric proof

Hanna (1998) differentiated two kinds of proofs:

- *informal*: proofs that show only that the theorem is true, by providing evidential reasons

- *deductive*: proofs that explain why the theorem is true, by providing a set of reasons that derive from the phenomenon itself.

It can be argued that DGS only enhance the first kind of proof, and do not provide the second kind of proof. However, DGS, in affording students greater access to exploration, heuristics and visualization, actually increases their understanding of the limitations of informal approaches and thus of the need for deductive proof.

Mariotti (2000) conducted a long-term teaching experiment for students in the 9th and 10th grades (about 15 to 16 years old) aimed at analyzing the process of semiotic mediation related to the emergence of the meaning of proof, and also the specific role played by DGS. She found that when a construction problem is presented in a dynamic geometry environment, the justification of the correctness of a solution figure requires description of the procedures used. The intrinsic logic of a dynamic geometry figure, expressed by its reaction to the dragging test (to see whether the geometric properties are changed when an object in the figure is moved) induces students to shift their focus onto the procedure, and, in doing so, opening up a theoretical perspective.

So, experience of geometrical constructions in the dynamic geometry environment could effectively facilitate 'semiotic mediation', which was mainly achieved by using the dragging function as a tool to check the correctness of the constructions (Mariotti, 2000). This 'semiotic mediation' helps students make sense of the process of exploring, conjecturing and arguing as a way of arriving at a valid proof, and contributes significantly to their understanding of 'theoretical' geometry. Thus, it seems that DGS has the potential of breaking down the traditional separation between action (as manipulation associated to observation and description) and deduction (as an intellectual activity detached from specific objects).

Strässer (2001) summarized the roles of DGS in the following terms: teaching and learning geometry through DGS is a human activity integrating the use of modern instrument; DGS strengthens students' problem-solving capacity; and it deeply changes geometry and its teaching and learning.

Summary: essential features of ICT

From the literature summarized above, it can be seen that there are three essential features of ICT tools:

1. Efficiency in mathematics manipulation and communication

2. Multiple representation of mathematics, especially the efficient coupling of visual representation with other forms of representation

3. Interactivity between the learner and mathematics.

These features have important implications for our understanding of the nature of mathematics, and for the teaching and learning of the subject. We will elaborate on this point later in the paper.

Actual use of ICT tools in the classroom

Given the great potential ICT offers to mathematics teaching and learning as reported in the research literature, to what extent is ICT incorporated into the real classroom? The actual use of ICT tools in mathematics teaching and learning will be summarized from two perspectives: the official policies of ICT use in different countries, and surveys on the actual use of ICT tools in the classroom, based on the TIMSS 1999 results. Firstly I will recount the official positions in various countries as reported in the TIMSS 1999 study, and will then report the results of the teacher and student questionnaires of the Study for the actual use of ICT tools in the classroom (Mullis *et al.*, 2000).

National policies on calculator use

Official documents from 23 of the TIMSS 1999 countries included an explicit policy on the use of calculators. Wide variation was reported across countries in their official policies on calculator use, ranging from encouraging unrestricted use, use with restrictions, to banning calculator use entirely. Out of the 23 countries, seven allowed unrestricted use of calculators, 14 permitted some restricted use, and two countries had policies that varied across different regions in the country. In several countries, calculators were not permitted in lower grades of primary school. In others, use of calculators in these grades was limited so that students could master basic computational skills, both mentally and using pencil and paper (see Figure 2 for a summary of these policies).

Exhibit 6.15 Calculator Use in Mathematics Class*

	Percentage of Students Having Access to Calculators in Class	Policy on Use of Calculators During Mathematics Lessons for Students Having Access					
		Unrestricted Use		Restricted Use		Calculators Not Permitted	
		Percent of Students	Average Achievement	Percent of Students	Average Achievement	Percent of Students	Average Achievement
International Avg.	73 (0.5)	21 (0.5)	490 (2.2)	67 (0.7)	488 (1.2)	12 (0.6)	464 (3.5)

Figure 2: Policy on calculator use (Mullis et al., 2000, p. 213)

Actual use in the classroom

In TIMSS 1999, teachers and students were surveyed on their use in the classroom of handheld calculators, computers and other ICT tools such as the internet. Some of the results are presented below.

How are calculators used in the classroom? Results of TIMSS 1999 show that in 14 countries teachers reported that nearly all students (more than 90 percent) had access to calculators in class. For students in classes with access to calculators, most teachers reported some type of restricted use (for about two-thirds of the students on average internationally; see Figures 3 (reported by teachers) and 4 (reported by students) for the different ways calculators were used in mathematics classrooms of TIMSS countries and their reported frequencies of use).

It can be seen from Figures 3 and 4 that the calculator is not heavily used in the mathematics classroom internationally. Teachers of 28% of students worldwide reported never or hardly ever using the calculator in the mathematics classroom (Figure 3). This is consistent with the figure as reported by the students (32%, Figure 4). Also, only less than 20% of the students reported that the calculator was "almost always" used in their mathematics lessons, and another 20% reported using the calculator "pretty often".

TIMSS 1999 also developed an index of emphasis on calculators in mathematics class (ECMC), "based on students' and teachers' reports on the frequency of calculator use", and found that across countries there was enormous variation in the emphasis on calculator use as measured by ECMC. For example, the Netherlands, Singapore, and Australia had more than 4/5 of their students (from 84 percent to 95 percent) in the high ECMC category, but Chinese

Exhibit R3.14 Ways in Which Calculators Are Used*

	Percentage of Students					
	Never or Hardly Ever Use Calculators	Ways in Which Students Use Calculators At Least Once or Twice a Week				
		Checking Answers	Tests and Exams	Routine Computations	Solving Complex Problems	Exploring Number Concepts
International Avg.	28 (0.5)	44 (0.6)	20 (0.4)	43 (0.6)	43 (0.6)	26 (0.5)

Figure 3: Teachers' report on ways calculators are used. (Mullis et al., 2000, p. 300)

Exhibit R3.12 Students' Reports on Frequency of Calculator Use in Mathematics Class*

	Almost Always		Pretty Often		Once in a While		Never	
	Percent of Students	Average Achievement	Percent of Students	Average Achievement	Percent of Students	Average Achievement	Percent of Students	Average Achievement
International Avg.	19 (0.2)	469 (2.0)	20 (0.2)	482 (1.1)	29 (0.2)	488 (1.2)	32 (0.3)	476 (2.2)

Figure 4: Students' reports on frequency of calculator use. (Mullis et al., 2000, p. 298)

Taipei, Iran, Korea, Japan, Malaysia, Romania, Thailand, and Turkey had half or more of their students in the low ECMC category.

For the 24 countries that participated in both the 1995 and the 1999 TIMSS studies, the 'trends' for the frequency of calculator use were also computed (see Figure 5). There is a slight but statistically significant decrease in the percentages of students who "almost always" use calculators in the mathematics classroom (and there is also a slight but significant increase of the percentages of students in the "once in while" category). This shows that, at least for these 24 countries, the use, between 1995 and 1999, of calculators in the mathematics classroom is actually on the decrease.

Exhibit R3.13 Trends in Students' Reports on Frequency of Calculator Use in Mathematics Class*

	Almost Always		Pretty Often		Once in a While		Never	
	Percent of Students 1999	1995-1999 Difference	Percent of Students 1999	1995-1999 Difference	Percent of Students 1999	1995-1999 Difference	Percent of Students 1999	1995-1999 Difference
International Avg. ³	23 (0.3)	-3 (0.5) ▼	25 (0.3)	-1 (0.4)	27 (0.3)	4 (0.5) ▲	25 (0.4)	0 (0.5)

▲ 1999 significantly higher than 1995
No significant difference between 1995 and 1999
▼ 1999 significantly lower than 1995
Significance tests adjusted for multiple comparisons

Figure 5: Trends in calculator use. (Mullis et al., 2000, p. 299)

How are computers and internet used in the classroom? Results of TIMSS 1999 show that across the TIMSS countries, the vast majority of students (80% on average internationally) reported never using computers in mathematics class. When compared with the data in 1995, it is found that there is a small but statistically significant shift from the "never" to the "once in a while" category, showing that although the use of computers in the mathematics classroom is very scarce in countries all over the world, there is sign of a slight increase in the usage between 1995 and 1999.

On access to the internet, TIMSS 1999 found great variation across countries. About a quarter of the students across all the countries had access to the internet at school. But the international average for using the internet to access information for mathematics teaching and learning in the classroom on even a monthly basis was only 10 percent (less than half of those reporting having access to the internet).

In summary, we can see that although there are ample research findings testifying to the great potential of the effectiveness of using ICT tools in teaching and learning mathematics, the actual use of ICT tools in mathematics classrooms across the world is still very limited. What are the reasons for this discrepancy between the potential and the realization of application of ICT tools in mathematics teaching and learning?

Discrepancy between the potential and actual use of ICT

From the results reported in the previous section, it can be seen that for many countries, the problem is not in the access to ICT tools, but in their actual use in mathematics teaching and learning. Commenting on the situation in Sweden, Lingefjärd *et al.* (2004) found that the Swedish classroom falls short of what is expected in the official documents in terms of incorporation of ICT into mathematics teaching and learning. They stated that access to technology is not the problem. It is not the access to technology that limits the use of technology in the teaching and learning of mathematics, instead it is the views and opinions of the user. Samuelsson concluded that

> the computer-supported teaching of mathematics (in Sweden) is still infantile. The possibilities and conditions for teaching that the technology bring does not always meet appreciation from teachers, who probably are far too busy with all issues involved in their everyday teaching. (Samuelsson, 2003, p. 223 [2]).

Cuban (2001) described ICT in the Californian classroom as "oversold and underused", and reported that

> despite abundant access to information technologies [...], and contrary to the expectations of promoters, teachers made infrequent and limited teacher use of computers in classrooms. (Cuban, 2001, p. 97)

As Hoyles and Noss (2003) observed,

> changes in the computational domain open up only the potential for change, not actual change in the didactical field. (p. 324)

On the particular tool of the microcomputer, Bottino and Chiappini (2002) lamented that the high expectations regarding its potential to drive change and innovation in schools appear to have remained largely unfulfilled. And in reviewing the practice across countries, Pelgrum and Plomp (1993) concluded that "computer use has had a limited impact on schooling throughout the world" (p. 100).

Why is there such a great discrepancy between the potential and the actual use of ICT tools in mathematics teaching and learning? Can this be explained simply by conservatism of teachers? What in essence are these ICT tools? Are they merely computational and presentational devices?

A new paradigm

As pointed out in the summary of research findings on the potential effectiveness of ICT tools earlier in this article, not only does ICT increase the efficiency in manipulation and communication in the teaching and learning of mathematics but, more importantly, it allows multiple representation of mathematics and enhances the interaction between learners and the mathematics that they learn.

The advent of ICT does not merely increase the repertoire of tools available for mathematics teaching and learning. It represents the emergence of a new paradigm, which has significant impact on mathematics education (Khait, 2005). As Kaput (1999) put it,

> the computer heralded a new kind of culture – a virtual culture – which differs crucially from preceding cultural forms. Not only is there a new representational infrastructure but also the externalisation (from the human mind) of general algorithmic processing.

An ICT tool is not just an artifact, it is an 'instrument' in the sense of an artifact plus a cognitive scheme (Artigue, 2002).

Recognizing these essential features of ICT tools, the implications for the teaching and learning of mathematics is momentous, since they

> entail fundamental shifts in the teacher's and students' roles, the social organization of the classroom, and power relationships between teacher and students. (Cuban, 2001, p. 134).

In the introduction of ICT tools to the mathematics classroom, what is involved is not simply the addition of one more computational or presentational tool, but the actualization of a paradigm shift in teaching and learning mathematics.

What is mathematics?

Sherin (2001) suggested that programming (in *Boxer*, a text editor for editing, for example, HTML files, program source code and database files) could shift the ontological foundations of school physics and mathematics, and that

> the nature of the understanding associated with programming-physics might be fundamentally different than the understanding associated with algebra-physics. (p. 1)

The argument should hold true for mathematics as well. Guin and Trouche (1999)) pointed out, "there is an unavoidable gap between 'real' mathematics and the image reflected by calculators (p. 198)". The question is: are there two 'mathematics', one as understood through 'traditional' methods (traditional-mathematics, say) and one arrived at through the use of ICT (ICT-mathematics)? If there are two mathematics, which is the 'real' one (Guin and Trouche seem to suggest that traditional-mathematics is the 'real' mathematics)?

For example, Weigand and Weller (2001) reported that their

> investigations did not show a better understanding (of the function concept) for students working with a computer, but they got different understanding compared to students working with pencil and paper. (p. 109).

Yerushalmy (2005) provided further evidence that students' conception of function in following the *VisualMath* curriculum, a curriculum that made use of new technological tools, was essentially different from the traditional conception. The new conception aided some "critical transitions" in learning, but generated new cognitive obstacles:

> In typical algebra instruction, solving an equation in two variables means isolating a variable: shifting from a non-explicit form ($x + y = 2x - y$) to an explicit form: $y = x/2$. For our function-approach students, the equal sign of the equation represents a symmetric-comparison sign and the function-equal sign represents an asymmetric-assigning sign [...] They would rather keep viewing equations as comparisons of two functions. (Yerushalmy, 2005, p. 40)

So we can see that students' conceptions of functions and of solving equations were essentially different from the traditional conceptions.

What is spreadsheet algebra?

Take the learning of algebra through spreadsheets as an example. Is 'spreadsheet algebra' (*i.e.*, the algebra as learned through spreadsheets) the same as the algebra we learned in school before the advent of ICT tools, or is it something else?

Heid (1995) argued that spreadsheets demand new visions of school algebra that shift the emphasis away from symbolic manipulation towards conceptual understanding, symbol sense, and mathematical modelling. No longer can the main purpose of algebra be the fine-tuning of techniques for by-hand symbolic manipulation or the acquisition of a predefined set of procedures for solving a fixed set of problems. Students will spend far less time on many of these techniques, will execute a majority of them with computing technology, and will completely forgo the study of others.

The concepts of variable and function in a technological world are much richer than those found in some current school text-books or in the minds of many of today's students. For these students, their understanding of the very nature of school algebra may be changed because of the impact of ICT.

Dynamic geometry and the nature of proof

Similarly, is dynamic geometry the same kind of geometry as Euclidean geometry? Goldenberg (1995, 2001) has stated that dynamic geometry is not merely a new interface to Euclidean construction. It is a new style of reasoning and it generates new heuristics. Dynamic geometry mediates the nature of explanation, verification and proof.

DGS raises questions about the place of proof (Hershkowitz *et al.*, 2002). Since conviction can be obtained quickly and relatively easily with DGS, and the dragging operation on a geometrical object enables students to apprehend a whole class of objects in which the conjectured attribute is invariant, and hence convince themselves of its truth, does DGS then conceal the need for proof?

The nature of mathematics

These questions touch on the philosophical issue of the nature of mathematics. If a Platonic view of mathematics is taken where mathematics is considered as having real existence, and mathematical knowledge is regarded as absolute, infallible truth (Ernest, 1991), then there is *ipso facto* only one mathematics. Different ways of approaching the same mathematics (*e.g.*, through using different learning tools) may give rise to different representations of mathematics. But the tools won't change the mathematics under study, and the different approaches will only add to the richness of our understanding of the mathematical truth. Taking this approach, ICT provides an alternative way for us to understand mathematics and enriches our understanding of the traditional, absolute mathematics.

But there is an alternative to this understanding of the nature of mathematics based on a fallibilist view of mathematics:

> the only access that human beings have to any mathematics at all […] is through concepts in our minds that are shaped by our bodies and brains and realized physically in our neural systems […] the only mathematics we can know is […] embodied mathematics. (Lakoff and Núñez, 2000, p. 346)

The mathematics we know is thus conditioned by the way our bodies and minds acquire it, and so it is not surprising that mathematics learned through the medium of ICT as embodied in our minds is different from that learned through the traditional medium of paper and pencil.

Understood this way, mathematics is defined by the language and the tools used to study it. As Artigue (2002) remarked, "the development of mathematics has always been dependent upon the material and symbolic tools available for mathematics computation" (p. 245). With a change of tools as drastic as the introduction ICT tools, the very nature of the mathematics as understood by us will inevitably be changed. So the mathematics studied through ICT is no longer the mathematics in the pre-ICT era, just as the mathematics after the invention of language is fundamentally different from pre-written-language mathematics. The very use of ICT in teaching and learning mathematics changes the nature of the subject matter that we teach and learn, and challenges us as to what mathematics is. This has fundamental implications for what mathematics students should learn, how mathematics should be learned, and how mathematics should be taught. What it amounts to is a paradigm shift in the teaching and learning of mathematics.

Concluding remarks

ICT represents a fundamental paradigm shift in mathematics education, and such a shift demands fundamental justification. The justification must be based on the ultimate question of what education in general is for, and what in particular the aims of mathematics education should be. Goals of education and aims of mathematics education of course change with time. They need to be set in accordance with the ever-changing environment in which students are situated. It is only when we have a clear idea what we want to achieve in education in this information era that we can evaluate whether the use of ICT will contribute to those goals of education. Doing it the other way round, that is, just looking at the potential of what ICT can offer and exploiting its use in different areas of mathematics as understood in the traditional sense, may lead us astray from what we want to achieve in education.

Notes

[1] This article is developed from a lecture given at ICME10 in Copenhagen, Denmark on 9 July 2004. The lecture was on behalf of the ICME10 Survey Team 5, with team members comprising: Colette Laborde, Frederick Leung (Chair), Thomas Lingefjärd, Teresa Rojano and Jeremy Roschelle.

[2] The English version of Swedish quotations and papers in this article are all taken from an English translation of a newsletter from the Swedish association of researchers in mathematics education, supplied to the author by Thomas Lingefjärd, one of the authors of the newsletter (Lingefjärd, Holmquist, and Bergqvist, 2004).

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