Disruption and Disclosure: Learning to Model Spherical Geometry

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1 Modelling spherical geometry

Modelling the Earth's geometry has been an important practical issue since the sixteenth century, as explorers and traders plotted new discoveries of land in the process of exploration and exploitation (Holt and Majoram, 1973) Mercator's map is one of the earliest examples, using a combination of projection and co-ordinate transformation, of producing a chart that is accurate at the equator, but exaggerated at the poles

Stereographic projection is another common technique used to model spherical surfaces It is conformal (it preserves angles) but not distance measures, and illustrates how projective models preserve some, but not all, aspects of the geometry (Kreyszig, 1991).

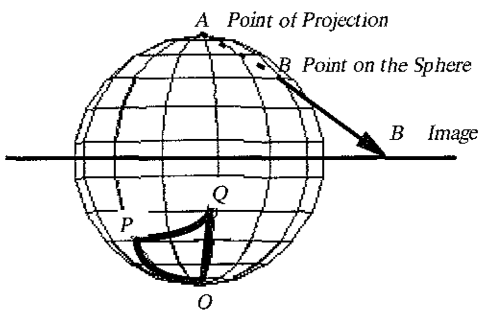

Figure 1(a) Stereographic projection

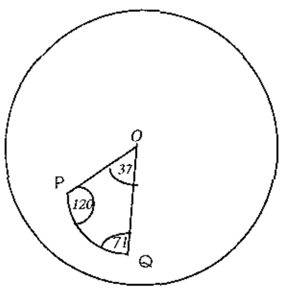

Figure 1(b) Equatorial image of the triangle

Figure 1(a) shows the process of projecting a sphere stereographically from its 'North pole' onto a plane through its 'equator', remembering that the 'North pole' is an arbitrarily chosen point on the sphere 'Great circles' (the 'straight' lines of spherical geometry) are mapped either to circles in the flat plane or to an infinite Euclidean straight line, if the great circle passes through the 'North pole'

Figure 1(b) illustrates what would happen if the triangle OPQ, created from the intersection of three great circles on the sphere's surface, were to be projected onto the 'equatorial' plane. In Figure 1(b), O is the image of the 'South pole', OP and OQ are part of the diameters of the equatorial circle and the arc PQ lies on a circle that cuts the unit circle at opposite ends of the same diameter. The angles shown are obtained by measuring the Euclidean angle made by the tangents to the circle at P and Q, and their sum is greater than 180 degrees

From a perceptual point of view, the resulting projection can be difficult to interpret because distance measures vary according to their position in the projected models, which runs contrary to our notions of Euclidean congruence. However, given the success of these projective models over the centuries, this is clearly not an insurmountable problem. The issue is how we learn to 'read' the variation in congruence and relate it to the original geometry of the sphere. In this article, I discuss some aspects of this learning-to-read process, since it touches on more general questions about how we appropriate and use models.

2 Learning and modelling

Trying to develop an understanding of terms such as 'straight line' in spherical geometry implies quite a complex set of transformations in which the linguistic token 'straight line' is retained but refers to an alternative set of images, actions and experiences. [1] In the case of stereographic projection, the situation is complex, since there are two sorts of visual referents for the term 'straight line', either the arc of a circle or a Euclidean 'straight line'

Probing the way(s) in which individuals make the connections between the flat model and the physical sphere raises deep questions about how learners make sense of models. Mellar and Bliss (1994), for example, distinguish between *learning about* a model - i.e. exploring someone else's - and the process of building a model oneself, through which learners can express their own understanding of a situation by trying to make a model of it.

Two distinct but inseparable kinds of learning are implied On the one hand, learning about a model implies that one has to appropriate the meanings embedded in the modelling context by the creator of the model. On the other hand, creating a model suggests that meanings are generated by learners actively engaged in externalising their understanding through the process of model building What is needed is a framework that will enable this expressive-exploratory

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dialectic to be articulated and analysed.

To explore this issue, I want to examine the idea of appropriating meaning, not seen as an individual constructing an object, as in 'making' or 'building' meaning, but rather as an induction into a structured social domain. Central to this idea is that meanings are immanent in the structure and function of specific social discourses, conditioning the particularities of that discourse through specific ways of speaking and acting (Wittgenstein, 1953).

Both exploratory and expressive modes of modelling are implied as learners are inducted dynamically into a social discourse through their internalisation of its structures, and externalise their understanding by increasingly confident and controlled participation. From an operational point of view, one needs a way of articulating the structure and function of a social discourse, so that one can locate the exploratory-expressive dialectic and chart its development.

In this context of spherical geometry, I am proposing to use the idea that meanings are generated by embedding linguistic signs in actions, on and with specific surfaces. Analysing the term 'straight line', for example, involves on the one hand a clear elaboration of how the term is used, both on the surface of a sphere and on its flat projection. On the other, charting how learners appropriate those meanings can be explored by paying careful attention to their linguistic and non-linguistic activities with both the sphere and its flat model.

3 "It must be circles"

To illustrate some of the issues associated with the appropriation of meaning in the context of learning about models, I am going to describe an episode from the iterative design and development of a computer-based microworld which implemented projective models for both spherical and hyperbolic geometry in Object Logo - an object-oriented version of Logo (Drescher, 1987). The aim of the study was to examine the interplay between learning non-Euclidean geometry using the flat models implemented in turtle geometry, and the design of a medium in which that learning could take place (Stevenson, 1999; Stevenson and Noss, 1999).

An iterative design methodology implies that tasks and resources are created for learners and their responses to carrying out the tasks are carefully monitored. Reflecting on the outcomes of the activities, both in terms of the individuals' understandings of spherical geometry and the contribution of the context to that understanding, provided the basis for creating new tasks (Stevenson, 2000; diSessa, 1989). The aim was to explore different aspects of the geometry in a computational context so that, from a design point of view, one could begin to understand what the viable opportunities for learning were.

During the development of the microworld for non-Euclidean geometry, there were three cycles of tasks spread over three years, with a total of six pairs of adult volunteers. The extract that follows comes from the second of the design cycles, involving Steve and Tim, two adults in their thirties who were training to be secondary mathematics teachers. Using the software, one of three turtles for spherical, Euclidean or hyperbolic geometry could be active at any given time. After a particular turtle was selected, all subsequent commands (e.g. FD 50) were executed according to the 'universe' that the particular turtle inhabited.

Inducting Tim and Steve into the model involved showing them how the stereographic projection was constructed, using diagrams similar to Figure 1 and a physical sphere. The sphere was overlaid with a grid, although it was possible to see the join between the two halves of the sphere through the grid - a fact that was to prove significant. They were also introduced to a metaphor from the book *Turtle Geometry*, in which Abelson and diSessa (1980) reformulate the idea of a straight line on a curved surface in terms of turtle steps:

> A [turtle] line is an equal-stride turtle walk (p. 204)

I take up the story at the start of the second session, with Tim and Steve discussing the screen they had produced. Figure 2 shows the path produced by positioning the turtle at 70 'steps' from the origin, turning 90° and letting it go forward for a large number of steps (e.g. FD 1000). The turtle path did not close and they interpreted this as the turtle "spiralling" inwards, although in fact the turtle's behaviour was produced by inaccuracies in the software - a 'happy fault' as we shall see.

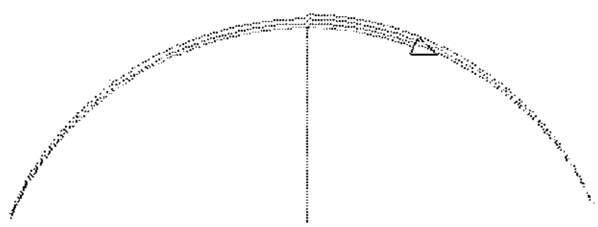


Figure 2 Tim and Steve's initial screen

As they were checking their understanding of the projection process, a discussion developed between Tim, Steve and the researcher about what straight lines were on the surface of the sphere. Tim and Steve were able to distinguish between lines of longitude and latitude using the 'turtle steps' metaphor. They recognised that lines of longitude were produced by equal steps, while the lines of latitude implied unequal turtle steps and so were not straight lines.

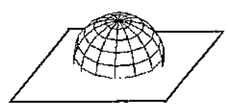
The transcript that follows has been split into two parts for the purposes of presentation and analysis and each consists of the spoken dialogue, accompanying actions and the surfaces used. It starts with Tim noticing the join between two halves of the sphere underneath the overlay of longitudinal and latitudinal lines. He expresses concern that he could not make sense of the screen images in light of his understanding of stereographic projection.

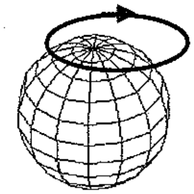
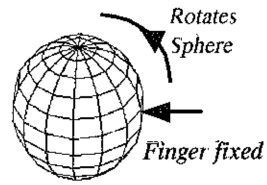
Dialogue	Action	Surface
T: I don't think there's anything different [from lines of longitude].	*Points to the line formed by the join of the two halves of the sphere.*	
Why does it [the screen image] ... why do they not come back and meet up?	*Points to the screen image*	
That line would be	*Points to the line on the sphere*	
I'm thinking that that line would be a circle..	*Points to the screen image.*	
R: Yes?		
T: But if you come to a point	*Points to the sphere.*	
forgetting where you've moved on, if you've just come to a point here	*Points to a place on the sphere but not on the join.*	
and go in a straight line in any direction .... why do you not ..		
just meet up?	*Moves finger across the sphere and round to indicate a complete circuit*	

Tim's rapid changes of attention give an insight into the connections that he has made between the sphere and its image. He seems to use the topological continuity of the sphere's surface as he traces across it and his sense of what a straight line is like in this situation to conclude that its projected image must be closed. Exploring the physical sphere through sight and touch seems to give Tim a sense of the properties that *must* be expressed by the screen model.

He seemed to know implicitly that the continuity and closure of the great circles on the sphere, which he experienced physically, had to be preserved by projection, and so concluded, correctly, that the computer model was wrong. In a context where the elements of the model did not cohere, Tim gave priority to his own perceptions and was prepared to disbelieve what he was seeing on the screen

In the next extract, which followed directly on from the first, the breakdown in the model provides the impetus for Tim and Steve to elaborate on their understanding. First, however, the researcher restates the turtle metaphor - *equal strides = straight line*.

Dialogue	Action	Surface
R: So thinking about that, that's what you think should happen Thinking of walking a straight line, that even if you were projecting onto this flat plane through the middle,	*Indicates flat surface parallel to the equator of the sphere.*	
which is what we are saying this is, that this should still meet up.		
T: Yeah, that's right. ... say you're anywhere, you're there	*Points to a spot on the top of the sphere*	

Dialogue	Action	Surface
and you walk in a straight line with a piece of string . . on the edge of that [sphere]	*Makes a circular motion in the air.*	
R: Yes?		
T: Surely you'd come back.	*Points the same point on the sphere.*	
S: Yeah .. it's like you'd said if you're starting there and kept going, you'd always come back to the same point, 'cos that's the starting point.	*Points to the sphere.* *Points to a point on the side of the sphere.*	
R: You're assuming its going in a straight line?		
S: Yeah		
T: You move to there but then ... it's like if you move it underneath there, it's still moving on a circle.	*Rotating the sphere so that a line of longitude is under the original point*	
So if you go in a straight line, why don't you come back to there? . [Pause] ... I think you should all be circles	*Points to the screen.*	

In a number of places, beginning with his statement of the problem and its restatement, Tim's circular motion with his finger on the sphere seems intended to match the motion of the turtle on the screen. Lying behind this identification appears to be two structures, one mathematical and the other metaphorical. There seems to be an implicit acceptance of the projective connection between the sphere and the screen, which both Tim and Steve had been told about just prior to this episode, and which formed the basis of a continuous map between the sphere and the screen.

However, Tim's objection to the screen image seemed to rest on his identification of the turtle's motion on the screen with the motion of his finger on the sphere and the fact that they do not coincide. It suggests that Tim has appropriated the transfer of meaning between the geometric domains, through a combination of physical manipulation of the sphere and a metaphor.

Tim provides two separate justifications of his own for the closure of the screen image. His first - "you walk in a straight line with a piece of string . . on the edge of [the sphere]" - captures the sense of a 'equal stride' turtle on the sphere leaving a trail corresponding to a great circle. Steve is keen to support this idea, and repeats it.

Tim's second justification - fixing a point and moving the sphere underneath the point through one revolution - is more dynamic and, taken together, they suggest an interesting equivalence for the turtle: moving across the sphere's surface is the same as moving the sphere under it. Being both local and intrinsic to the surface, the 'turtle' notion seems to support the equivalence quite naturally for Tim, in the sense that he produces the two possible situations, one after the other, and without prompting.

It is interesting to reflect on the equivalence between the static 'string trail' and the dynamic 'turning of the sphere while keeping the turtle fixed'. Clearly, it is a matter of speculation, but Tim's remarks seem to draw out the connections between turtle and differential geometry, indicated by Abelson and diSessa (1980). The static path and the dynamic sphere suggest that the turtle can be used to visualise tangent planes - a central concept in differential geometry - that 'roll without twisting' to produce geodesics in the surface.

4 Conclusion

To conclude, I would like to make two, necessarily tentative, general observations about the role of disruption in learning to model, one epistemological and the other methodological (Heidegger, 1962). This episode can be located at a crucial moment in the transition from an exploratory to a more expressive mode in modelling. It is characterised by learners marshalling resources - physical, computational and linguistic - to explain a lack of coherence in the context that they had been inducted into. From this point of view, the discrepancy that occasioned Tim's observations played a central role in disclosing the nature of the model.

The disruption simultaneously distanced him from the computational model that he had been introduced to and enabled him to articulate his difficulties, precisely as 'difficulties', against a background of practices that had coherence. It implies that in working with models, either in creating or learning with them, individuals engage in a dialectic of engagement and disengagement with the model, as their understanding develops (Ackermann, 1991)

Disruptions in the modelling context can provide a dynamism for learners to grasp the semantic and syntactic structure and function of a model while, at the same time, adopting a critical distance from it By trying to resolve the fractures in meaning, learners engage in a dynamic and iterative process as they attempt to draw together the fragmented discourse Metaphor, perception and action play crucial roles as resources for learners to draw on as they generate a more coherent account of the modelling context.

Methodologically, disruptions in learners' appropriations of models, intentional or otherwise, provide a site for probing the semantic connections that learners make. They help to disclose networks of meanings that learners have made in the process of induction and model exploration Examining the nature and extent of a learner's use of resources in a problematic situation - how they gesture, direct their attention and manipulate objects available to them - can enable researchers to chart learners' emergent understandings As the episode shows, tracking learners' responses relies, on the one hand, on provoking disruptions in that context and, on the other, on having a framework that can illuminate learners' interactions with the setting and the resources available to them

Note

[1] When discussing such phrases as 'spherical triangle' - such as OPQ in Fig 1(a) - Pimm (1987) points to the way in which metaphors are used to provide signification of terms, such as 'triangle', across domains of geometry.

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