# WHICH PROBLEMS DO TEACHERS CONSIDER BEAUTIFUL? A COMPARATIVE STUDY

ALEXANDER KARP

What kinds of problems seem beautiful to teachers? Are there differences between the mathematical tastes of teachers in the context of different educational systems and, if so, in what do these differences consist? The study of teachers' tastes and aesthetic values seems important if only because they influence the development of the students' attitudes toward mathematics. Ward [1] says that "there is no established theory [on aesthetics of mathematics] and the existing related literature is modest" (p. viii). At the same time, research mathematicians have named the beauty and elegance of mathematics as one of the main sources of their creativity and as one of the most important motivations for their studies (Hardy, 1956; Krull, 1987; Poincaré, 1956). Meanwhile, as Sinclair [2] points out,

> not only have educators paid little attention to the aesthetics, but many have questioned the extent to which it can and should play a role in a majority of students' acquisition of mathematical knowledge and competency. (p. 6)

The actual concept of beauty in mathematics is usually not defined; or rather, it is defined only in passing (for example, Krull, 1987, noted that the word "beautiful" can be used to describe a clear and compelling presentation of results) and through virtually synonymous expressions (thus, Poincaré, 1956, writes about "the harmony of numbers and forms"; p. 2047). The beauty of mathematics is said to consist in the fact that it allows mathematicians to see a "hidden order" in a "seemingly confusing picture" (Krull, 1987, p. 52), and it is particularly apparent when "underlying mathematical relationships [are expressed] in an extremely simple and transparent way" (*ibid.*, p. 50). Mathematical literature contains various attempts (Birkhoff, 1956; Davis, Hersh, Reuben and Marchisotto, 1995; King, 1992) to describe the elements of mathematical aesthetics, to formulate the principles of mathematical aesthetics, and even to derive formulas for aesthetics (including the aesthetics of mathematics).

Dreyfus and Eisenberg (1986) studied the aesthetic reactions of university-level students – rather than professional mathematicians – to solving problems. Their research showed that students do not usually find more beautiful solutions and that they see no difference between these solutions and others. Dreyfus and Eisenberg (1986), therefore, believe that:

- taking pleasure in the beauty of mathematics is not a feeling that arises spontaneously;
- there is a failure on the part of mathematics educators to cultivate such a feeling in their students.

Even so, they write that they "believe that an aesthetic appreciation can be nurtured" (p. 9) and, with this aim in mind, call for a substantial change in the pedagogical approach to problem solving – including devoting more attention to problem solving.

Silver and Metzger (1989) discovered aesthetics playing a substantially greater role in the process of problem solving. However, they studied the problem-solving behavior of a group of people with a substantially higher level of expertise. Their subjects were university professors and graduate students. The research was able to identify an "aesthetic emotion" (p. 60) that served "as a guide in decision making" (p. 66) – similar to what was described by Poincaré (1956) – and a role for aesthetics in the "post hoc evaluation of solutions or problems" (p. 69).

The work of Sinclair ([2], 2004) has enabled a deeper examination of the approaches to defining the beautiful in mathematics. In particular, she draws attention to the relations between objective, subjective, and social-historical elements in the definition of the beautiful. Thus, for example, Sinclair has pointed out that Dreyfus and Eisenberg (1986) based their analysis on an objectivist conception of the beautiful, treating mathematical beauty as an objective attribute of the problem and, in effect, viewing the opinions that develop within the mathematics community as universal truths. In response to the question "why do mathematicians so often agree on aesthetics judgments in mathematics?", Sinclair (2004) notes that "aesthetic choices might be at least partially learned from their communities" (p. 276). Accordingly, in different communities different approaches may well take shape. In particular, as Sinclair argues (2004), students can develop their own aesthetic preferences. (It should be noted that the specific processes by which aesthetic preferences are communicated within different communities have not yet been sufficiently studied.)

Consequently, it would be interesting to investigate aesthetic preferences in mathematics within different communities – and first and foremost within the community of mathematics teachers. Such an investigation would make it possible better to understand teachers' views and conceptions of mathematics. It would also set the stage for a more detailed analysis of the ways in which such preferences are formed and the concrete influences to which they are subject.



*For the Learning of Mathematics* **28**, 1 (March, 2008)
FLM Publishing Association, Edmonton, Alberta, Canada

## Some theoretical considerations

Views about the beautiful in mathematics are a part of teachers' beliefs. In recent times, the beliefs of mathematics teachers, and the influence of their beliefs on their teaching practice, have been the object of extensive research (see, for example, Leder, Pehkonen and Törner, 2002). Teachers' beliefs are seen as one of the most important indicators (along with their goals and knowledge) that make it possible to characterize and even to predict their behavior in class (Schoenfeld, 1998). The methods used to study the beliefs of teachers are themselves an object of discussion.

Teachers' beliefs are divided into two categories: *attributed* beliefs and *professed* beliefs, that is, beliefs that are attributed to teachers by the researchers who study them, and beliefs expressed by teachers themselves, respectively (a large survey of the relevant literature can be found in Speer, 2005). Such a distinction cannot be considered to be entirely accurate, however, since researchers invariably introduce their own intuitions into their findings (Speer, 2005), whether by deriving theoretically formulated beliefs when analyzing lessons or by posing direct questions in some particular form and interpreting the answers given. Teachers who are interviewed may understand both the researcher's questions and their own answers in a different way from the researcher. Consequently, although it seems enticing simply to ask the teachers and to find out what they think, this approach is by no means always fruitful.

Nor is this the end of the matter. Tolstoy's (1984) ironic words are well known:

> If the critics already understand what I want to say and can formulate it in a newspaper editorial, then I congratulate them and can confidently assure them *qu'ils en savant plus long que moi* [that they know more about it than I do]. (p. 785) [3]

In a similar way, what can be achieved by studying those teachers' views that can be expressed in the form of a "newspaper editorial" – that is, as answers to direct questions in a research interview – must of necessity be rather limited. (Naturally, this does not negate the fact that the use of interviews and questionnaires can frequently be quite effective.) Texts must be analyzed "by guiding readers through that infinite maze of connections which is the essence of art" (Tolstoy, 1984, p. 785). The analysis of videotaped lessons – widely in use today – represents just such an attempt at textual analysis (in the broad, semiotic sense of the term).

Also understandable is the tendency to enlarge the corpus of observed cases, which usually leads to the use of increasingly varied questionnaires. However, it is also possible to employ a different methodology. Historians offer examples of studying the views – or rather, the mentality – of people who can no longer be interviewed (for example, Hutton, 1981). The notion of a 'mentality' encompasses that which is not verbally articulated and even that which is not always consciously recognized, but which is for all that no less important. What must be analyzed in such cases is the totality of various indirect indicators and products of people's activities: these are the facts that make it possible to study the behavior of relatively large groups of people. Analyzing the kinds of furniture or clothes that people made, say, in the seventeenth century tells us not a little about the mentality of the people living at that time. Bearing this in mind, it is worth noting that the products of the labor of teachers – tests or problems – are quite rarely subjected to analysis; and even when they are analyzed, this is typically done from a mathematical point of view and not as a way to achieve a deeper understanding of the teacher.

The most important reason for this, it may be argued, consists in the fact that researchers frequently doubt – without always articulating this doubt – that these materials really are the products of teachers' labor. In his study of a teacher named Christopher, Skott (2001) writes that Christopher "did not make curricular decisions with respect to the mathematical content and tasks" (p. 10). The problems were taken from the textbook and from supplementary materials that were recommended in the same textbook. Christopher simply did not have the energy to select assignments on his own.

Unfortunately, this is true of many teachers, so that their approach to writing and selecting assignments can in and of itself be considered an important characteristic of their mentality. Still, in most cases, the teacher does make choices, to one degree or another, even if this involves nothing more than picking and arranging problems from the textbook that is used in class. It is also possible to make specific research interventions by deliberately asking teachers to write or select assignments in the context of a study. Clearly, such techniques cannot replace the continuous observation of teachers in practice; nevertheless, it may be argued that they are a natural step in gathering information about the products of teachers' labor, and a means and a method for studying them. The present study was conducted within the framework of such an approach.

A study that draws conclusions about the mentality of teachers by analyzing their mathematical activity is undoubtedly far from traditional quantitative research. Lawrence-Lightfoot and Davis (1997), describing the methodology of a researcher-portraitist, frequently compare the researcher-educator with a painter. A comparison with a literary or art critic, who helps readers to grasp the meaning of a writer's or painter's activity and its sources, appears equally justified. The use of such a framework seems particularly justified in those cases that involve the analysis of the perception of the beautiful, reminding us that mathematics and the teaching of mathematics are not for nothing sometimes regarded as forms of art.

## Design of my research

The point of my study was to analyze examples of beautiful problems submitted by middle and high school teachers, that is, teachers of 11-15 year-old students and teachers of students older than 15-16, respectively (or future middle and high school teachers). The examples were collected in three stages. During the spring and autumn of 2004, two groups of American graduate students in mathematics education were asked to submit examples of problems "which they particularly enjoyed and which they considered especially beautiful." (The notion of "problem" was conceived broadly – problems could involve, for example, proofs, constructions and computations.) The study was conducted as part of the course *Problem Solving* offered to in-service and pre-service mathematics teachers at a large university on the east coast of the US. Each of the participants in the study had a bachelor's degree in mathematics or its equivalent. The task was given as part of a homework assignment; consequently, students were free to use any sources that were available to them. The first group contained 23 people, the second group 34. (Note that there were 9 pre-service teachers in the first group, 20 in the second.) The respondents were also asked to provide a solution for each problem that they submitted and to indicate why the problem seemed particularly attractive to them. At the time of this assignment, the topic of *mathematical beauty* had not been in any way touched upon or discussed in class.

It was assumed that the system of education through which the teachers themselves had passed, and in which they continued to work, exerted a considerable influence on their perception of mathematics. Therefore, it seemed interesting to compare the results obtained in this study with results obtained in a parallel study involving teachers who had been educated and continued to work in a different system of education. For this reason, in the summer of 2004, we asked mathematics teachers, pursuing postgraduate studies at the St. Petersburg Academy for Postgraduate Education (Russia), to submit an example of a problem that they viewed as particularly enjoyable and beautiful and to explain their choice (they, too, submitted problems with solutions). This study was conducted as part of the course *Problem Solving*, which was offered at this academic institution. The assignment was given to 72 participants in total, all of whom had university degrees in mathematics or mathematics education. This assignment was likewise given to them as a homework assignment, and the problem of *mathematical beauty* had not been touched upon in class discussions. On the whole, the conditions under which the study was carried out in Russia and the US were quite similar (which, naturally, does not negate the substantial differences in the participants' educational backgrounds). In both cases, the participants had completed their higher education and were working as mathematics teachers in schools (or, in the case of some of the American participants, were still preparing for such work). The courses in the context of which the studies were carried out were sufficiently similar as well: in both cases, they included the analysis of different problem-solving strategies.

The differences between American and Russian curricula are largely expressed in the fact that different amounts of time and attention are devoted to different topics. Therefore, I found it expedient to analyze the collected materials from this point of view – by determining the topics from which 'beautiful' problems were drawn. The collected materials were also analyzed from a different perspective, in effect using the approach described by Glaser and Strauss (1967) as the "constant comparative method of analysis" (p. 101). In accordance with this methodology, the collected materials were simultaneously analyzed and compared, which made it possible to identify similarities, differences, and patterns in the tastes of the respondents, and thus facilitated a better understanding of which particular aspects of their problems appealed to them.

The American participants in the study professed certain beliefs about the beauty of their problems and provided explicit explanations for their choices. However, practically all of the Russian participants confined themselves to offering mathematical assignments without any explanations whatsoever as to why the problems that they submitted were attractive and beautiful. I could only attribute such beliefs on the basis of their materials. The Russian participants clearly only took the strictly mathematical component of the given assignment seriously. This contrast between the two groups of teachers is quite interesting in and of itself (the explanation for it must be sought, in my view, in the fact that the Russian audience is much less accustomed than the American audience to questions about personal opinions, particularly in classes on mathematics). Without going too deeply into an examination of the root causes of this contrast, however, there is methodological importance in the extreme difficulty or impossibility of collecting materials in certain cases when studying professed beliefs.

## What is a beautiful problem? The problems submitted by the respondents

I want to focus on two problems. The first, was submitted by an American teacher.

> Perform the following operations on your calculator: (1) enter the first three digits of your phone number, (2) multiply the number you obtain by 80, (3) add 1, (4) multiply by 250, (5) add the number formed by the last four digits of you phone number, (6) add this number again, (7) subtract 250, (8) divide by 2. You will obtain your phone number. How does this work?

The solution to this problem consists of a simple manipulation of variables. Letting $x$ denote the number formed by the first three digits of the phone number, and letting $y$ denote the number formed by the last four digits of the phone number, we obtain the numbers $80x$; $80x + 1$; $20{,}000x + 250$; $20{,}000x + 250 + y$; $20{,}000x + 250 + 2y$; $20{,}000x + 2y$, $10{,}000x + y$, and evidently this will be precisely the number formed by the digits of the phone number.

The second problem was submitted by a teacher in Russia: Solve the equation

$$\sqrt{16 - 3x} - \sqrt{9 - x} = 2x - 13 + \sqrt{(16 - 3x)(9 - x)}.$$

To solve this problem, it is useful to note that

$$\begin{aligned}&(\sqrt{16 - 3x} - \sqrt{9 - x})^2 = \\ &= 16 - 3x + 9 - x - 2\sqrt{(16 - 3x)(9 - x)} = \\ &= 25 - 4x - 2\sqrt{(16 - 3x)(9 - x)} = \\ &= -2(2x - 13 + \sqrt{(16 - 3x)(9 - x)}) - 1.\end{aligned}$$

Therefore, using chunking, and letting $t = \sqrt{16 - 3x} - \sqrt{9 - x}$, the given equation can be rewritten as $t = -(t^2 + 1)/2$. This new equation is easier to solve, obtaining the answer $t = -1$, and the final equation $\sqrt{16 - 3x} - \sqrt{9 - x} = -1$ is now also not difficult to solve, by squaring both sides. Thus, $\sqrt{16 - 3x} = \sqrt{9 - x} - 1$, $16 - 3x = 9 - x - 2\sqrt{9 - x} + 1$, $2\sqrt{9 - x} = 2x - 6$, $\sqrt{9 - x} = x - 3$, $9 - x = x^2 - 6x + 9$, $x^2 - 5x = 0$. Therefore, $x = 0$ and $x = 5$; but the number 0 is an extraneous solution and the only answer is $x = 5$.

The two problems seem extremely different, although in essence both involve algebraic manipulations. The manipulations involved in the second problem, however, are undoubtedly more difficult than those in the first. In addition, although the first problem could also not be described as in any way connected to the real world, it nonetheless contains some reference to real-world objects – something is done to a phone number. The second problem, by contrast, is 'purely' mathematical. The situation with these two problems is typical. Differences between 'American' and 'Russian' problems are quite conspicuous (while differences between the two American groups, or between American pre-service and in-service teachers, are practically nonexistent). Russian teachers usually submitted problems with longer solutions, demanding greater algebraic skills.

Figure 1 shows how the 'American' problems broke down by topic. Such a division into topics is to some extent merely nominal, reflecting what a problem 'looks like' rather than what is involved in its solution. Thus, the category *Real-life situations and applications* (see Fig. 1) includes problems devoted to determining the area of a figure, a problem that involves solving a linear equation, problems with content from physics, as well as simple calculations – but the formulations of all of these problems make reference to real-world objects of some kind or other. Furthermore, because each problem has been assigned to only one category, the situation has been oversimplified, at times rather substantially, since in some cases the respondents deliberately pointed out that they found the problems attractive because their solutions involved the use of different sections of the course. For example, certain problems, which were classified as problems in probability, also involved graphing and determining the areas of curvilinear figures. In comparing these problems with the 'Russian' ones, it is also important to note that only one of the problems listed here in the category *Algebra* was devoted to equations; most of the rest were devoted to various facts from elementary number theory (proved in some cases using variables); there was also one (quite elementary) problem devoted to matrix algebra. The category labelled *Analysis* – a label that is perhaps too strong – contains a problem on the transformations of graphs, as well as problems devoted to derivatives, limits, and integrals. *Puzzles* included mainly classic problems about, for instance, pouring liquids into and out of vessels of given sizes; getting the wolf, the goat, and the cabbage across the river; cryptarithms (wherein each letter represents a digit, and different letters represent different digits). In general, this kind of playful, humorous mathematics is generously represented in other categories as well. Here is an example of one such problem that was classified above under *Analysis*:

> Find the mistake in the following proof of the equality $1 = 2$. Let us write down the equation $x = 1$. From this it follows that $x^2 = 1$. Going back to the first equality, we obtain $x = x^2$. Now, by finding derivatives, we obtain $1 = 2x$. Substituting $x = 1$, we obtain $1 = 2$.

Figure 2 shows how the 'Russian' problems broke down by topic. The fundamental change in topics is striking. Problems in *Trigonometry* appear – a category entirely lacking in the American sample. On the other hand, many fewer problems are devoted to analysis and combinatorics. The overwhelmingly dominant category is *Algebra* – and, moreover, the content of the problems belonging to this category is entirely different from the content of the 'American' problems on the same topic. Overwhelmingly, the 'Russian' problems in algebra feature equations and inequalities (irrational, logarithmic, exponential …) that must be solved using various intricate techniques (as in the problem analyzed, for example).

Furthermore, a considerable number of problems involve transformations. For example, one problem asks students to derive the value of

$$A = \sqrt[3]{5\sqrt{2} + 7} - \sqrt[3]{5\sqrt{2} - 7}.$$

Cubing the number $A$, $A^3 = 14 - 3A$. Since $A^3 + 3A - 14 = (A - 2)(A^2 + 2A + 7)$ and the equation $A^2 + 2A + 7 = 0$ has no real solutions, there exists only one real number that satisfies the relation – this number is 2. Therefore,

$$\sqrt[3]{5\sqrt{2} + 7} - \sqrt[3]{5\sqrt{2} - 7} = 2.$$

In general, the role of transformations in virtually all of the problems is substantial. Thus, the one problem that has been listed under the category *Analysis*, in essence comes down to executing a transformation. Its formulation is analytic, "Determine the intervals where the given function is increasing and decreasing." However, the given function is expressed by the formula

$$y = ((x^3 - x^2 - 14x + 24)(x + 1))/(x^2 + x - 12),$$

which, after the numerator and the denominator have been factored and the appropriate factors have been cancelled, can be rewritten as $y = (x - 2)(x + 1)$, on the condition that $x \neq -4$, $x \neq 3$. Therefore, the strictly analytic component of this

| Topic | Algebra and number theory | Logic, probability, combinatorics | Analysis | Geometry | Real-life situations and applications | Puzzles |
|---|---|---|---|---|---|---|
| Number of problems | 11 | 9 | 8 | 9 | 12 | 8 |

*Figure 1: Distribution, by topic, of problems submitted by American teachers.*

| Topic | Algebra and number theory | Combinatorics | Analysis | Geometry | Puzzles | Real-life situations and applications | Trigonometry |
|---|---|---|---|---|---|---|---|
| Number of problems | 40 | 1 | 1 | 11 | 1 | 9 | 9 |

*Figure 2: Distribution of problems, by topic, submitted by Russian teachers.*

problem consists in its reference to the properties of a quadratic function.

The category labelled *Trigonometry* is likewise filled with problems involving the transformations of trigonometric expressions and solving trigonometric equations. The calculations in the 'Russian' problems in geometry are also more complicated (and require greater knowledge). Even the 'Russian' problems about real-life situations usually present greater technical difficulties than their 'American' counterparts.

## What makes a problem beautiful?

The explanations that were offered by the American respondents to justify their selection of one or another problem as particularly beautiful can be divided into several groups, as presented in Figure 3. The first column lists the principles on the basis of which responses were grouped together, and the second column offers examples of responses from the corresponding groups.

Also, several respondents explained why they found a problem enjoyable by noting that its solution made it possible to establish a "clear pattern." Finally, a number of respondents in effect gave no answer to explain why they were attracted to a problem, but simply paraphrased the fact that the problem in question was beautiful. For example, one respondent made the following comment: "This problem is simply very elegant."

Analysis of the problems offered by the American teachers reveals a greater quantity of problems with real-world content and 'non-standard' problems, in accordance with these teachers' professed beliefs. Reality, however, turns out to be more complicated than their explicitly articulated opinions. For many teachers, non-standard ways of thinking inevitably turn out to be connected to non-standard topics. Among the problems offered by the American teachers, many were puzzles. Naturally, this is only one tendency among many and cannot be described as universal and absolutely dominant. For example, one of the respondents named an algebraic problem on a 'standard' topic as an enjoyable one; the respondent was clearly attracted by the fact that the long verbal formulation of the problem could be briefly written down using equations. Still, it hardly seems accidental that many of the submitted problems, even those devoted to 'ordinary' topics, involve something like a trick (as in the problem about the phone number) or a patently false assertion. In such cases, non-standard form seems to act as a synonym for non-standard content.

A number of cases contain a patent contradiction between the reasons given for considering a problem beautiful and the

| Unifying principle | Examples |
|---|---|
| Usefulness in the teaching process | "It gets students to use algebraic representations"; "it reinforces the students' knowledge of all the properties of parallelograms"; "clearly exercises solving simultaneous equations"; "a great example of experimental mathematics"; "it tests both the students' knowledge and their conceptual thinking"; "it invokes storytelling and visual imagery"; "it demonstrates the value of technology"; "a common misconception is cleared up." |
| Useful in practical life or comes from the real world | "Matrices are used in business, government, and technology. It is simple and can be applied to the real world"; "I like it because this problem arose from pop culture, from a game show." |
| Non-standard and cannot be solved using ordinary methods that are regularly discussed in school | "The problem can not be solved by learning"; "students do not need to memorize a formula to solve it but they can use reasoning to find the answer"; "it doesn't require rote memorization of formulas and can be calculated using many ways"; "there is no simple formula that one can use to 'plug in'"; "it doesn't require equations"; "it is a logical thinking problem that doesn't involve numbers"; "this problem can be solved by looking at a picture"; "encourages a new way of looking at it"; "seems difficult, but works out so neatly when you stop and think about the problem." |
| The unexpectedness of the solution | "It is somewhat surprising"; "it might seem that a lot of information is given, but it is actually a fairly straightforward algebra problem"; "many may be tempted to say that no information is given to answer the question." |
| Openness of the problem | "A variety of cases need to be considered. There are different ways to approach the problem"; "the mathematical skills are limited but there are several solutions and there is no pressure to find a particular one"; "there are multiple ways in which students can solve it." |
| There is a combination of methods and knowledge from different fields of mathematics | "It combines so many areas of mathematics"; "it involves many different types of mathematics"; "incorporates Algebra, Calculus and Numerical methods"; "ties together many concepts"; "it relates mathematics to physics." |

*Figure 3: American participants' explanations.*

actual character of the problem. An example of such a problem is the one that involved determining the sum of two matrices, submitted because "it is simple and can be applied to the real world." Nowhere in this problem is the connection with the real world mentioned, so the connection remains a connection in name only. The respondent would have liked to submit an example of such a problem but in reality did not know of any examples.

For the Russian educators, since the teachers' own explanations for their choices are not available, I can only guess at what it was about one or another problem that appealed to the teachers. Such guesses, however, are grounded in the analysis of the problems presented by the teachers, as well as in comparisons between these problems, and between these problems and the 'ordinary' problems found in standard textbooks. It would seem that, for the Russian teacher, the beauty of a problem is connected with a kind of "overcoming of chaos" that is involved in its solution. A problem that initially appears to be unbelievably difficult and entangled is solved through a successfully applied technique that makes it possible to see order and regularity in any muddle. In one of his novels, Nabokov (1964) writes that

> there was mysterious sweetness in the fact that a long number, arrived at with difficulty, would at the decisive moment, after numerous adventures, be divided by nineteen without any remainder. (p. 17)

It would seem that a similar feeling also motivates many of the Russian respondents. In the examples above, a complex fraction whose numerator and denominator contain polynomials of comparatively high powers turns out to be a square trinomial. An expression containing several radicals turns out to be equal to 2. Expressions with roots that are chaotically scattered in an equation turn out to be exactly what they need to be in order for the equation to be reducible to a quadratic, and so on.

Another characteristic, which for many Russian respondents clearly constitutes a basis for considering a problem to be beautiful, is its non-standard nature. A source of pleasure is found in the contradiction between what one is usually supposed to do in problems of a given type and what turns out to be most effective in the present case. To test functions for properties of increasing and decreasing, it is usual to take the derivative, but, in the example analysed, it turns out that there is no need to do this, and that it is only necessary to know how to transform expressions. In order to simplify expressions containing radicals, it is usual to raise them to the appropriate power or to guess that the number or expression under the radical is a perfect square or a perfect cube, but in one of the problems given above it turns out to be possible to do this by constructing and analyzing some equation, and so on.

It seems highly indicative that the beautiful problems chosen by Russian teachers come from traditional fields, which they themselves have covered rather intensively in their own studies – and which continue to be the focus of considerable attention in their courses. It is little wonder that among the submitted problems there are practically no problems in combinatorics or the theory of probability, since these fields typically are not studied in Russian schools. What is much more interesting is the fact that, in effect, there are no problems in *Calculus* either, which in Russia – by contrast with the United States – forms part of the required curriculum for all high school students. Most likely, this is a result of the fact that *Calculus* is not part of the program for the so-called entrance examinations – the examinations that must be passed by any student wishing to go to college in Russia (Goldberg and Swetz, 1977). The problems given on such examinations are usually relatively challenging (and sometimes also beautiful). Although, formally, the school curriculum is developed independently from the entrance examinations it is obvious that school teachers become more or less familiar with the materials of the entrance examinations, and that, to a greater or lesser extent, they orient their lessons around them. Consequently, they have a source of comparatively difficult problems in many areas of high school mathematics. But they have no source of such problems for *Calculus*. Meanwhile, in school, *Calculus* is usually studied in a rather routine and superficial fashion. As a result, teachers are not exposed to beautiful problems in this area.

A similar effect, in different topics, may be observed among American teachers: they lack experience with examples of vivid problems from ordinary mathematics. This phenomenon, which leads to a distorted conception of problem solving, has been noted in the literature (Cooney, 1985). 'Ordinary' mathematics often appears to be a field in which there can be nothing except for rote memorization of commands and routine drills, so that genuine problem solving is something that is inevitably confined to special sections, and first and foremost to puzzles.

The answers of Russian and American teachers were contrasted by pointing out that most of the answers in the two groups fit two distinctive patterns. At the same time, it is important to emphasize, once again, the diversity of answers within each group. In particular, note that many of the collected responses express opinions that are close to what has been written and said by research mathematicians. For example, the feeling of being surprised by the result is singled out as being particularly enjoyable, as is the possibility of seeing a regular pattern. Some of the respondents in both groups submitted problems and arguments that constitute classic examples of mathematical beauty (such as the proof of the fact that there is an infinite number of prime numbers or that $\sqrt{2}$ is irrational).

## Reflections

On the whole, the differences between the "beautiful problems" submitted by American and Russian teachers correspond quite well to the differences between the curriculum these teachers teach their students (Burmistrova, 2004a, 2004b; NCTM, 2000). The American course is broader and attempts to cover a greater number of concepts. It devotes much more attention than the Russian course to sections on such 'new' topics as discrete mathematics, while devoting substantially less attention to traditional topics such as trigonometry. The Russian course is deeper in certain areas, and in particular, assumes a higher level of skill and greater knowledge of mathematical techniques. In each case, the educator's attitude toward mathematical beauty has been nurtured by the curriculum offered in school.

Obviously, it is futile to hope that the mathematical beauty of a problem could ever be measured objectively in any way. Not everyone's views about the beauty of a problem are identical: people exhibit conspicuous personal preferences. Thus, to an outside observer, 'Russian' problems often look artificial – they do not come up naturally in the course of some more general theoretical or applied investigation, but are constructed specifically for the purpose of demonstrating the force of the applied technique. Nevertheless, it cannot be denied that the teachers who submitted these problems possessed a consistent and well-defined conception of mathematical beauty (regardless of whether or not this conception agrees with, say, mine).

Many of the American teachers also expressed characteristic preferences in their views about beauty. Their heightened attention to the usefulness of one or another problem contrasts an extremist understanding of beauty as something directly opposed to usefulness (an understanding stemming back to classical antiquity and illustrated by the famous story of Euclid chasing away a student who wanted to "make a gain from what he [learned]"; Eves, 1990). However, it appears more accurate to interpret the American respondents' predilection for problems that are useful in the teaching process as indeed a striving for beauty – only not a mathematical beauty, but a pedagogical one (implying a teaching process organized in the most natural and simultaneously the most effective way – in a "clear and compelling way," to use Krull's words). These educators would like to introduce various rules or concepts through problems rather than explaining them directly; consequently, they are particularly drawn to those problems that facilitate such an approach. Attention to problems as a means of instruction can be viewed as a highly promising phenomenon. On the other hand, it is true that in and of itself such attention tells us little about the teachers' attention (or lack of attention) to beauty in mathematics specifically – in place of this concept, some of the respondents substitute the more general concept of the value of problems. The teachers' conception of the connection between these two concepts merits further investigation.

I do not possess the data that would allow me to describe the manner in which the aesthetic preferences of the respondents took shape. It would indeed be interesting to undertake a more detailed study of the formation of such preferences among teachers. For example, we do not know which preferences develop while teachers are being educated, and which preferences developed in the process of teaching. On the whole, however, it may be concluded that most teachers do develop distinctive aesthetic preferences in mathematics.

Finally, the analysis conducted in the course of this study also has a methodological aspect. We have aimed to assess the teachers' views not so much on the basis of their professed opinions as on the basis of the mathematical texts that they had written – on the basis of their problems. It may be argued that such an approach facilitates a deeper analysis, and contributes to a more vivid and multi-faceted picture of the facts – it is not even always possible to elicit explicit responses. Mathematical texts are far more frequently used to assess subjects' knowledge of mathematics than to illuminate their attitude toward it. In fact, they can be telling and informative in this respect as well. Devoting greater attention in research on mathematics education to the products of teachers' mathematical activity can facilitate a better understanding of teachers themselves.

These findings indicate the importance of the aesthetic in mathematics for many teachers and confirm the similarity between the aesthetic perceptions of many teachers and research mathematicians. At the same time, it is possible to make clear differentiations among various types of perceptions of mathematics that take shape as a result of the influence of the educational systems established in different societies. The beauty of mathematics turns out to be neither as universal nor as contingent on time and social conditions as is commonly believed. It is unlikely that teachers anywhere in Russia or the United States are explicitly taught what they should consider beautiful in mathematics, but the indirect influence of each system is evident. It would be worthwhile to develop a better understanding of the mechanisms of this influence; in particular, it is noteworthy that teachers sometimes assimilate the declarative part of an ideology – for example, a faith in the importance of practical applications – without any corresponding substantive examples.

In conclusion, the analysis of what teachers consider beautiful in mathematics is important not only for a better understanding of teachers' mentality: it also directs our attention to very practical issues. The formation of the aesthetic perception of mathematics proves impossible when one or another section must be taught and studied too quickly, superficially, and by relying on mindless, rote memorization of rules. The fact that for many teachers the beautiful lies outside the bounds of the ordinary program is, surely, an alarming signal. Students' attitudes toward mathematics are formed under the influence of their teachers; it is the teachers who can and must convey to their students a conception of the beauty of mathematics and the beauty of solving problems in particular. It is an important challenge for the mathematical community to reorganize the ordinary course in mathematics so as to make the teachers see the beauty in it. Then the students have the chance to see it there as well.

## Notes

[1] Ward, J. (2002) *The aesthetics of mathematics in teaching and learning: a case study*, unpublished doctoral dissertation, The Florida State University, FL, contact e-address, jssward@hotmail.com.

[2] Sinclair, N. (2002) *Mindful of beauty: the roles of the aesthetic in the doing and learning of mathematics*, unpublished doctoral dissertation, Queen's University, Kingston, Ontario, Canada, contact e-address, nathsinc@sfu.ca.

[3] All translations from Russian are by the author.

## References


Birkhoff, G. (1956) 'Mathematics of aesthetics', in Newman, J (ed.), *The world of mathematics* **4**, New York, NY, Simon and Schuster, 2185-2195.

Burmistrova, T. (2004a) *Theme planning in mathematics. Grades 5-9* [*Tematicheskoe planirovanie po matematike. 5-9 klassy*] Moscow, Russia, Prosveschenie.

Burmistrova, T. (2004b) *Theme planning in mathematics. Grades 10-11* [*Tematicheskoe planirovanie po matematike. 10-11 klassy*] Moscow, Russia, Prosveschenie.

Cooney, T. (1985) 'A beginning teacher's view of problem solving', *Journal for Research in Mathematics Education* **16**, 324-336.

Davis, P., Hersh, R. and Marchisotto, E. (1995) *The mathematical experience*, Boston, MA, Birkhäuser.

Dreyfus, T. and Eisenberg T. (1986) 'On the aesthetics of mathematical thought', *For the Learning of Mathematics* **6**(1), 2-10.
Eves, H. (1990) *An introduction to the history of mathematics: with cultural connections*, Philadelphia, PA, Saunders College Publishers.
Glaser, B. and Strauss, A. (1967) *The discovery of grounded theory. Strategies for qualitative research*, Chicago, IL, Aldine Publishing Company.
Goldberg, J. and Swetz, F. (1977) 'Mathematical examinations in the Soviet Union', *The Mathematics Teacher* **70**, 210-218.
Hardy, G. (1956) 'A mathematician's apology', in Newman, J. (ed.), *The world of mathematics*, **4**, New York, NY, Simon and Schuster, 2027-2040.
Hutton, P. (1981) 'The history of mentalities: the new map of cultural history', *History and Theory* **2**(3), 237-259.
King, J. (1992) *The art of mathematics*, New York, NY, Plenum Press.
Krull, W. (1987) 'The aesthetic viewpoint in mathematics', *The Mathematical Intelligencer* **9**(1), 48-52.
Lawrence-Lightfoot, S. and Davis, J. (1997) *The art and science of portraiture*, San Francisco, CA, Jossey-Bass.
Leder, G., Pehkonen, E. and Törner, G. (eds) (2002) *Beliefs: a hidden variable in mathematics education?*, Dordrecht, The Netherlands, Kluwer Academic Publishers.
Nabokov, V. (1964) *The defense*, New York, NY, Putnam.
National Council of Teachers of Mathematics (2000) *Principles and Standards for School Mathematics*, Reston, VA, NCTM.
Poincaré, H. (1956) 'Mathematical creation', in Newman, J. (ed.), *The world of mathematics* **4**, New York, NY, Simon and Schuster, 2041-2050.
Schoenfeld, A. (1998) 'Toward a theory of teaching-in-context', *Issues in Education* **4**(1), 1-94.
Silver, E. and Metzger, W. (1989) 'Aesthetic influences on expert mathematical problem solving', in McLeod, D. and Adams, V. (eds), *Affect and mathematical problem solving*, New York, NY, Springer-Verlag, 59-74.
Sinclair, N. (2004) 'The roles of the aesthetic in mathematical inquiry', *Mathematical Thinking and Learning* **6**(3), 261-284.
Skott, J. (2001) 'The emerging practices of a novice teacher: the role of his school mathematics images', *Journal of Mathematics Teacher Education* **4**, 2-28.
Speer, N. (2005) 'Issues of methods and theory in the study of mathematics teachers' professed and attributed beliefs', *Educational Studies in Mathematics* **58**, 361-391.
Tolstoy, L. (1984) 'Letter to N. N. Strakhov', in *Collected works* [Sobranie sochinenii], **17-18**, Moscow, Russia, Khudozhestvennaya literatura, 784-785.

---

There is one major difference which has not been discussed. Problems are usually serious and demanding on a cognitive level. The satisfactory resolution of a problem frequently provides new learning or a new rearrangement of old learning in the problem solver. Dealing with problems involves creating a learning environment and the energy generated and consumed ensures that the learning is retained. This would suggest that becoming more efficient at coping with problems would give positive pay-off in terms of life style. Further, the link between problem solving and learning suggests that this experience can valuably be gained in the classroom. Puzzles, on the other hand, are diversionary and, as long as they do not create too much tension and frustration, they are "fun". There is not necessarily any new learning required in their solution – a shift in perception is frequently all that is necessary – and they do not have application or relevance to the world of the puzzler.

(Leone Burton (1980) 'Problems and puzzles',
*For the Learning of Mathematics*, **1**(2), 20-23)

---