ALGEBRAIC FORMULAS, GEOMETRIC AWARENESS AND CAVALIERI'S PRINCIPLE

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> It seems appropriate in a new journal of mathematics education [*For the Learning of Mathematics*] that there should be an attempt to tackle a perennial problem [the loss of geometry] with new terms. (Tahta, 1980, p. 9)

One of the fundamental tensions at work in both learning and teaching mathematics is that between fluency and understanding. [1] Formulas are frequently offered as an aid to fluency, as a concise and manipulable summary of an invariance, an indifference to change and to the contingent. Formulas contain a tacit 'always' (with 'under suitable conditions', sometimes said *sotto voce*). One unfortunate result can be students at various levels seeing mathematics as being 'all about formulas'. More than twenty-five years ago, Hart (1981) observed:

> mathematics teaching is often seen as an initiation into the rules and procedures which, though very powerful (and therefore attractive to teachers), are often seen by children as meaningless. (p. 118)

The presence of formulas is particularly prominent in the teaching of geometric topics such as area and volume. This, Janvier (1994) explains, has the consequence of making people think of area and volume as easy topics to teach, where it "simply consists of leading students to memorise some formulas" (p. *ix*, our translation). In these instances, learning about area and volume is reduced to knowing the formulas, recognizing which one to use and applying it in a problem to obtain an answer.

The starting point for this article is an exploration of the complex set of connections between algebraic formulas and geometric awareness, in particular the limited extent to which the latter is either embedded in or necessary for the former. We do so within the core school mathematical topic of area (and, to some extent, of volume), but we wish to go beyond what is now explicit in curriculum documents of various countries, namely that such formulas should be motivated or even proven using geometric arguments (*e.g.*, NCTM, 2000). One complaint in relation to this minor move (as we see it) is that it leaves unchallenged the prominent place taken by formulas themselves as the primary or even sole goal of teaching and learning about area and volume. We wish to examine a possibly unfamiliar way (namely Cavalieri's principle, our mediating third of the title) of staying with the geometry on its own terms. In doing so, we take seriously Tahta's (1980, p. 7) formulation that "the geometry that can be told is not geometry", noting that area and volume formulas are unreservedly about such tellings. The moment the word 'measure' is uttered, the instant that variables are deployed, geometry has vanished (despite the etymological origins of the word itself).

We argue that offering and focusing upon formulas in this context transports the mathematical work from the geometric realm of area and volume to that of substitution of numerical values into formulas – a quite different mathematical arena of potential technique, fluency and skill. (One of us, for instance, remembers classroom work involving 'changing the subject of the formula' – algebraic tasks that were quite independent of the meaning and referents of the particular formula in question and, indeed, whose unavowed purpose was precisely such semantic suppression.) Significantly, however, this arena has nothing to do with the *geometric* notions of area and volume themselves: when students work with area or volume formulas, they are *not* working on understanding concepts of area or volume, they are working within a particular sub-domain of algebra, namely substitution.

Underlying understanding is awareness, a term much drawn upon by Caleb Gattegno who offered the challenging proposition 'Only awareness is educable'. In this journal, Hewitt (1999, 2001a, 2001b) has published a series of articles concerned with teaching and the educating of mathematical awareness, in relation to a distinction between the arbitrary and the necessary, as well as examining the challenge of memory for mathematics. Hence, lying behind this article's more surface concerns is a desire to find ways to work more geometrically (which often means more directly) with much of school mathematics. We would argue pedagogically for approaches that attempt to educate geometrical awareness of the structure of some area and volume concepts. To do this, we bring forth a powerful historical mathematical resource, an early seventeenth-century mathematical theorem, the curiously named Cavalieri's principle. [2]

Cavalieri's principle

Bonaventura Cavalieri (1598–1647) was an Italian mathematician whose work is often mentioned in accounts of the emergence in Europe of infinitesimal calculus and infinitesimal calculations in the early part of the seventeenth century. One of his theorems reads:

> If between the same parallels any two plane figures are constructed, and if in them, any straight lines being drawn equidistant from the parallels, the included portions of any one of these lines are equal, the plane figures are also equal to one another; and if between the same parallel planes any solid figures are constructed, and if in them, any planes being drawn equidistant from

the parallel planes, the included plane figures out of any one of the planes so drawn are equal, the solid figures are likewise equal to one another. […] The figures so compared let us call *analogues*, the solid as well as the plane [...] (Cavalieri, 1635)

Gray (1987) glosses the two-dimensional version of this claim as follows: “The principle asserts that two plane figures have the same area if they are between the same parallels, and any line drawn parallel to the two given lines cuts off equal chords in each figure” (p. 13). Figure 1 presents two plane figures illustrating Cavalieri’s principle.

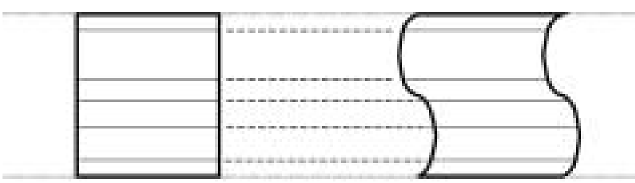

Figure 1: An illustration of Cavalieri's principle in two dimensions

The principle seems to allow qualitative assertions of same or different only (although other circumstances might allow greater or lesser to be argued). Questions of *how much* bigger or smaller lead to quantification *of the relation*, though not necessarily quantification of the area itself. [3]

Plane geometry through Cavalierian eyes

As a very straightforward application of Cavalieri’s principle, here is a parallelogram and a corresponding rectangle: as both figures lie between the same parallels (resulting in the fact that each figure has the same height, a consequence of Euclid’s Fifth Postulate [4]) and for every cross-section the segments have equal length, then by Cavalieri’s principle, they have the same area (see Fig. 2). Notice in particular that for all Cavalieri figures this condition applies at the two extremes, the top and the bottom of the shape.

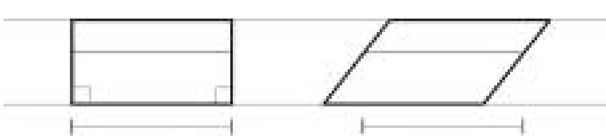

Figure 2: Cavalieri's principle at work with a rectangle and a parallelogram

Consequently, particular pairs of parallelograms can be associated: any parallelogram has a single associated rectangle (seen as a rectangular parallelogram) *relative to a given base* (choose the ‘other’ side as the base and there is a different family of analogues) or *relative to a given pair of parallels*. Any parallelogram so identified is a member of an infinite family of parallelograms with the same height and equal cross-sections all the way up. These parallelograms are more or less ‘oblique’ or skew, more or less sheared to left or right. The degree of obliqueness in this case does not change a thing in terms of the area of the parallelogram. In addition, only in the particular case where a parallelogram has identical height and base will there be a square in this family.

Figure 3 shows members of the family of equivalent parallelograms obtainable from a given one (there is a unique rectangle in this family). Working geometrically with Cavalieri allows us to *know* directly that these figures have the same area, without any computation.

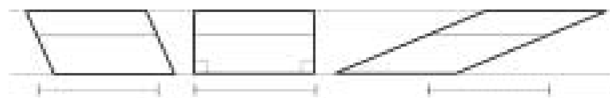

Figure 3: An infinite family of Cavalieri-equivalent parallelograms

A rhombus is a parallelogram with all four sides of the same length. In fact, in the particular infinite family of associated parallelograms indicated in Figure 3 (that all lie within the two given parallel lines, with the same base), there are two equivalent rhombuses (one left-facing and one right-facing) that could be drawn. So, any rhombus belongs to a unique family of parallelograms and so can be associated with a single rectangle of the same area, that is, the rectangle that has the same base and the same height as the rhombus. But not all families of parallelograms contain a rhombus. For example, if the shorter side of a parallelogram is selected as the base of the parallelogram and the height is larger than this length, then no rhombus can be found in this family. Thus, all rhombuses are associated with a family of parallelograms, but not all families of parallelograms contain a rhombus. In addition, no rhombus family (other than the one associated with a square rhombus) contains a square.

One thing Cavalieri provides, then, is a reason to attend to related figures drawn between the same pair of given parallels. It also invites attention to the notion of an infinite *family* of figures that resemble one another in terms of some attribute (in Cavalieri’s terms, the family of *analogues*). In addition, the principle points to the freedom this equivalence allows us to be able to select any member of this family to be used as a referent to which to relate other figures. For purposes of area (and of volume as we will shortly see), when dealing with categories of shapes for which there are general formulas, these reference figures are often chosen to be rectangular. These features illustrate examples of awarenesses that Cavalieri’s principle stresses and brings to the fore.

The realm of quadrilaterals is complex, with various figures named (and some not) as a consequence of particular schemes (*e.g.,* classification by number of pairs of parallel sides, exclusive or inclusive; classification by means of number of equal side lengths; etc. – see de Villiers, 1994). The result is a set of distinguished, named shapes, but one which does not necessarily reflect a single coherent scheme with respect to area. In consequence, there is a tacit implication and disseminated understanding that every different named quadrilateral deserves its own area formula (in order to be able to calculate its value). Hence, among many other things, Cavalieri’s principle appears to offer ways to structure and inter-relate different quadrilaterals. For example, rectangles, squares, parallelograms and rhombuses become inter-related (in fact, parallelograms and rhombuses could be seen as oblique rectangles!), being part of the same group with regard to their area. [5]

Another commonly named quadrilateral is a trapezoid (or trapezium). Cavalieri provides us with a means to identify families of trapezoids, which also always contains a 'rectangular trapezoid' (see Fig. 4). Again, there are two such 'basic' trapezoids, one left-facing and one right-facing.

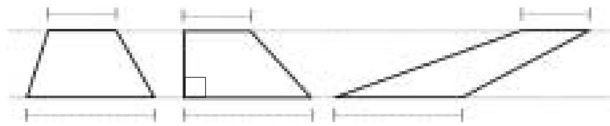

Figure 4: A family of Cavalieri-equivalent trapezoids

Any family of equivalent trapezoids will not contain any rectangles (unless 'inclusive' definitions are used, namely '*at least* one pair of parallels' rather than '*exactly* one pair of parallels'). However, there is a fixed relationship with an associable rectangle. By rotating the rectangular trapezoid about the mid-point of the remaining slant side, we can produce a rectangle that is the double of the trapezoid. (It is for this reason we made the earlier note about Fowler's work and ratio-numbers, of which the 'double' or 'twice' offers one instance.) As it stands, this invokes the duplication of a figure rather than a dissection of one, although a mental shift of figure and ground allows us to see two identical copies of a trapezoid sitting inside any rectangle (see Fig. 5).

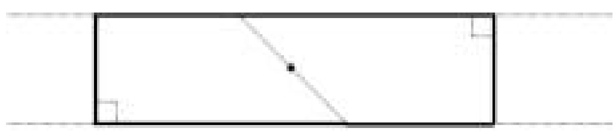

Figure 5: Two copies of a right trapezoid creating a rectangle

While we used the right-angled trapezoid to do the rotation here toward a rectangle, we could use any trapezoid of the family and rotate it to obtain a parallelogram, one which would still be twice the area of the trapezoid as well as equivalent in area to the rectangle just obtained above from the rotation/doubling of the rectangular trapezoid.

Triangles in relation to rectangles

Cavalieri's principle can, of course, also be applied to triangles and connects any given triangle to an infinite family of area-equivalent triangles. As Figure 6 indicates, all the triangles in this family are also equivalent to a right-angled triangle – again, there are two such triangles, one left-facing and one right-facing. Any triangle can be placed on any of its sides determining a 'base', thereby leading to three different and independent families of triangles from the same initial figure.

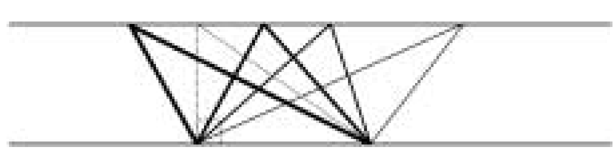

Figure 6: An infinite family of Cavalieri-equivalent triangles

There are no rectangles in this family either, but once again it is possible to establish a relationship between the right-angled triangle and an associated rectangle, which is once more its double (obtained by rotation about the mid-point of the hypotenuse, see Fig. 7). [6]

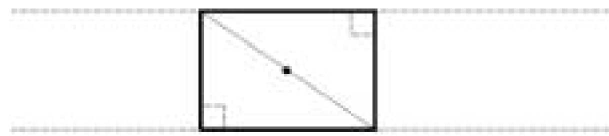

Figure 7: Two copies of a right-angled triangle forming a rectangle

The last two examples have indicated how Cavalieri can be used with repetition of a particular figure, in order to focus on its fixed relationship with a more familiar figure – in this case, one half of a specific rectangle.

Cavalieri's principle highlights families and relationships. Earlier on, the term 'associated' meant a figure that belonged to the Cavalieri-equivalent family, and hence the area relationship was 1:1 (rectangle, parallelogram, rhombus, square). However, in the latter two examples, we extended the notion of 'associated figure' to one in which in each case the areas were in the ratio of 1:2. Hence, Cavalieri enables the establishment of 1:1 relations among rectangles, parallelograms and rhombuses, and 1:2 relations between trapezoids/triangles and rectangles or parallelograms. These ratios and families relate and connect different figures together through geometrical links, not by formulas or calculations; nothing was calculated in these examples. However seemingly transparent though, these latter examples show by their very obviousness one possible strength of understanding areas in terms of families. There are fascinating geometrical links between planar figures that no set of isolated algebraic formulas is able to do justice to in its full breadth. Such fascination is fundamentally geometric, not algebraic.

Cavalieri and associatedness in the third dimension

The previous discussion about Cavalieri's principle and area is equally 'applicable' to volumes. [7] For volumes, the principle can be cited as follows: two solids have the same volume if they lie between two parallel planes (hence have the same height) and any section made by a plane parallel to the plane of the common base of both solids creates cross-sections of equal area. This is illustrated in Figure 8.

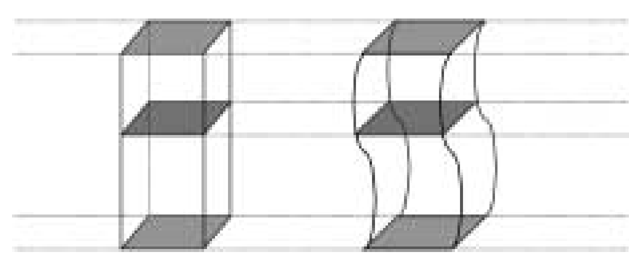

Figure 8: An illustration of Cavalieri's principle for three-dimensional solids

Janvier (1992a, 1992b, 1994, 1997) did important work along these lines in his teaching experiments on volume of solids with students, work that importantly influenced this article. We summarize some of his ideas, but we also extend them to align with our previous discussion of area and families.

Janvier defined the volume of prisms in terms of a piling up of layers of area (which can be visualised via sheets of paper or cards to create a pile). Drawing on Cavalieri's principle for three-dimensional objects, he argued that oblique prisms have the same volume as their associated straight prisms that have the same height and the same base (Fig. 9 provides three illustrations of this).

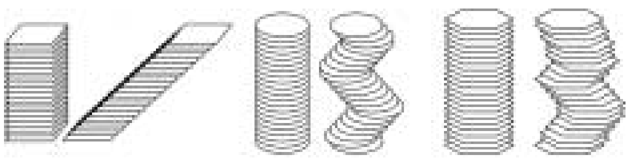

Figure 9: Illustrations of pairs of equivalent volumes of solids from piles of layers, both straight and oblique

To extend from these instances using the previous language of families, there are families of prisms (with the same height and same base), where slanted prisms are of equivalent volume to vertical ones. For example, Figure 10 illustrates a family of rectangular prisms with the same base and same height, where the rectangular straight prism can be taken as the reference form.

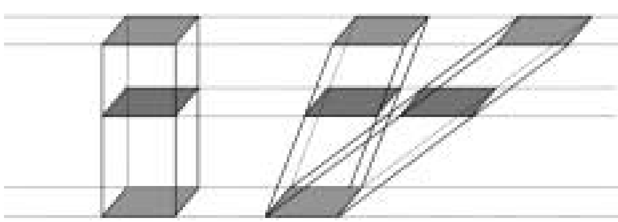

Figure 10: A Cavalieri-equivalent family of rectangular prisms

The same could be said for pyramids, where each plane, parallel to the common plane given by the bases of the two pyramids, right up to their their apices, results in a cross-section of equal area. Therefore, the considerations leading to families of pyramids with the same base and same height (Figure 11 presents an example of a family of squared-based pyramids).

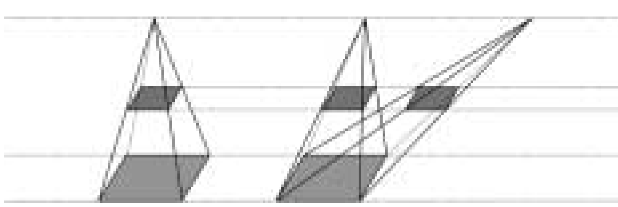

Figure 11: A Cavalieri-equivalent family of square-based pyramids

In his work, Janvier also illustrated for students an association between pyramids and prisms of the same height and the same base in order to draw out the relationship/ratio existing between them (in a similar way to the previous work we described with triangles and trapezoids in relation to their associated rectangles). He did this by using the familiar decomposition of the cube into three identical pyramids (Fig. 12).

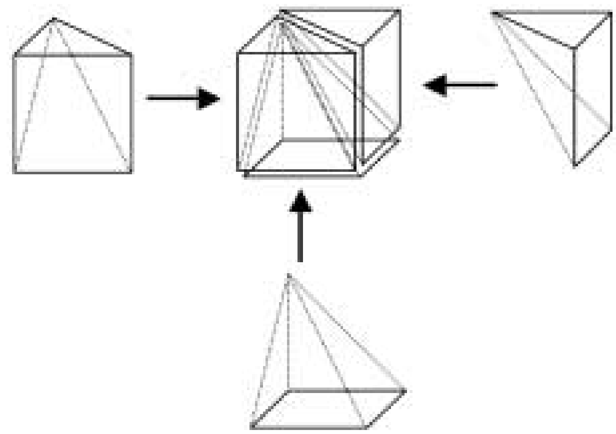

Figure 12: A cube decomposed into three right-angled pyramids

Janvier also extended this approach to other prisms, like the triangular prism and its associated triangular-base pyramids (although he never referred explicitly to the notion of associatedness). In some cases involving prisms and pyramids, the decomposition leads to pyramids of identical volume, but not identical shape, requiring use of Cavalieri's principle to establish equivalences. All this work on pyramids and prisms enabled him to establish a link or a relationship of associatedness between pyramids and prisms that share the same height and the same base, namely 1:3 or 'triple': the prism is the triple of its associated pyramid. [8]

It is possible to offer alternative ways of looking at cylinders and cones, ways that align with this previous work on prisms. For example, by thinking of cylinders as 'prisms with an infinite number of sides', cylinders can also be seen as created by means of piling up area layers, which are circles. In addition, one can establish a link between straight and oblique cylinders of same base and same height. Hence, it leads to seeing the possibility of families of cylinders (straight and oblique) with the same base and same height (see Fig. 13).

Moreover, thinking of cones as 'pyramids with an infinite number of sides' also enables the creation of links

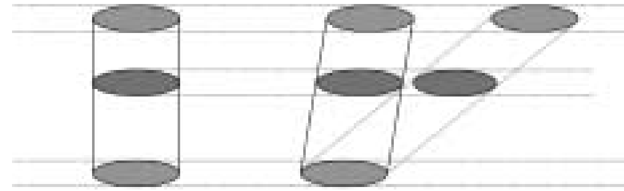

Figure 13: A Cavalieri-equivalent family of cylinders

between oblique and slanted cones of the same height and same base, as was done with pyramids. But most importantly, it allows one to establish the relationship between cylinders and cones in the same way as the one between pyramids and prisms, where every cone has its associated 'prism' (a cylinder) of the same height and same base and the latter is related to the former by the ratio of 1:3. [9]

As is possible to see from what Janvier offered with Cavalieri, and the possible extensions to his work that we have briefly offered, similar conclusions to what was done for area can be drawn concerning the focus on geometry and the geometrical work on volumes of solids. This work is not directly linked to calculations and formulas, illustrating how it is possible to work with concepts of volume within geometry itself at a qualitative geometric level of comparison and not solely at a quantitative one. Once again, Cavalieri's principle enables the establishment of geometric links between solids, drawing them closer together rather than isolating them.

Direct comparison of figures: Cavalieri and relations of associatedness

Up until now, we have stayed away from a discussion of the decomposition or dissection of figures as a second independent means of working directly with area and volume as geometric entities. One historical lesson comes from ancient Greek geometry. Those mathematicians handled geometric questions quite differently from the way we do today, as they worked directly on area (developing what might be called a finite theory of content) as a fundamentally geometric object (an area *was* a geometric square, a square root *was* a line segment). If there were any comparing of figures to be done, it could be achieved by direct comparison of the squares that were the two respective figures' areas. Nowhere did any measuring or calculating come into play. An ancient Greek student faced with the request to "Find the area of …" would construct a square (by given permissible means). This too is what Cavalieri's principle provides to some extent: that is, a means to relate and compare figures.

Obviously, as has been discussed, relations between two plane figures or solids belonging to the same family can be established, as can ratios between different pairs of plane figures or solids. But this geometrical awareness of relations can also lead to powerful and significant understanding of comparisons between similar figures with the same side-lengths, without requesting explicit calculations or measurements.

Is Cavalieri's principle a geometric principle with mathematical power to assist in solving a range of problems? While we do not go into detail here, we sketch three examples. The first relates to a geometric question made famous by its inclusion in one of the Japanese lessons published as part of the TIMMS video study (Stigler *et al.*, 1999; DoE, 1997) and also as further discussed by Lopez-Real and Leung (2001) in this journal. The problem first concerns two people the border between whose land is a bent line. The challenge is to redraw the boundary as a single line so land is neither gained nor lost by either. Later, a related problem becomes to change a quadrilateral into a triangle with the same area.

We simply note in passing that there is a Cavalierian point of view involved in the solutions offered in the Japanese class video (by finding a triangle area-equivalent to another between the same parallels and then showing how to change a quadrilateral into a triangle while preserving areas, again dependent on use of parallels). It provides a good example of how Cavalieri's principle could function as a way of thinking and reasoning about geometric problems in the classroom, rather than simply being an illustrative principle for exploring quadrilateral figure areas. Additionally, although there is no space here to discuss significant questions raised about this approach in relation to dynamic software environments, shearing of a figure along a parallel is straightforwardly performable within them.

We may want to compare the areas of two figures, without specifying what the area is. Might we be able to use more than one Cavalieri transformation, thereby cutting across families? In such circumstances, rectangular figures are no longer privileged. Figure 14 shows a series of transformations which

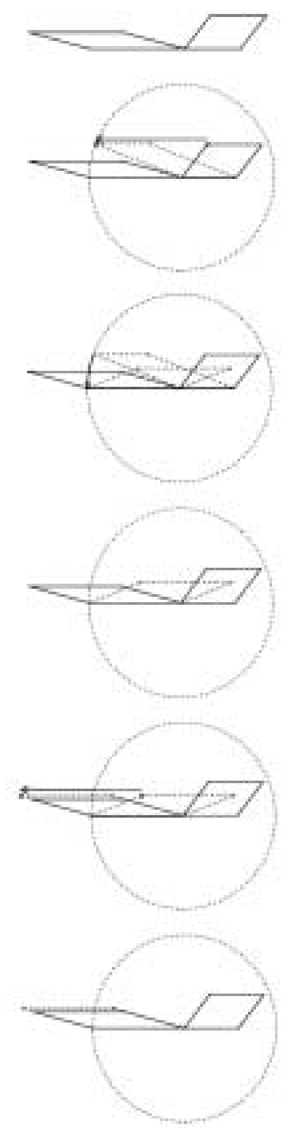

Figure 14: A series of transformations for comparing two parallelograms (the right-hand one has a larger area than the one on the left)

allows any two parallelograms to be directly compared. Apart from a translation and rotation, there are two transformations drawing on Cavalieri's principle. In order to support explicitly our assertion of the importance of 'staying with the geometry,' we have refrained from providing a verbal description and only offer the image sequence for the reader to make sense of. [10]

The third situation concerns a question that is seldom raised as questionable. 'The' formula for the area of a triangle is usually given as 'half the base times the height'. Leaving aside questions of different ways of seeing that connect to different 'algebraically equivalent' versions of this result (see Appendix 1), our actual question here is on the one hand more fundamental and on the other more mathematical in some respects, namely why is this area formula well-defined? The definite article 'the' hides the fact that any triangle actually has *three* potential 'bases' and three corresponding 'heights' (as we referred to earlier in this paper in discussing Cavalieri families in relation to triangles). Why should $\frac{1}{2}b_1h_1 = \frac{1}{2}b_2h_2 = \frac{1}{2}b_3h_3$? (Why is it always so – a fact, once shown, which allows us to be indifferent with regard to any given triangle as to which corresponding base-height pair we choose, and can therefore pick the most convenient with respect to some goal?) Such a practice is commonplace and usually passes unremarked. [11]

In other words, why is this formula well defined mathematically? What do we need to know about area itself – and triangular area in particular – to become aware of this *fact* about triangles (see Mason, 1991)? 'Unpacking' formulas in this way offers one means of reclaiming the geometric nature of area, by offering "ways of thinking and acting geometrically" (Tahta, 1980, p. 9).

A sequence of transformations (see Fig. 15 opposite) allows one to move from one rectangle to a second and thereby pass to the third and argue they must all have the same area. [12] This question raises again the issue of what is achievable not just by means of one Cavalieri transformation, but by a whole sequence of such transformations. Is there a sequence of transformations that will generate the rectangle area-equivalent to any polygon or one which turns any rectangle into an equivalent square? (See the discussion following Problem 35 in Henderson, 1996, pp. 113–121.)

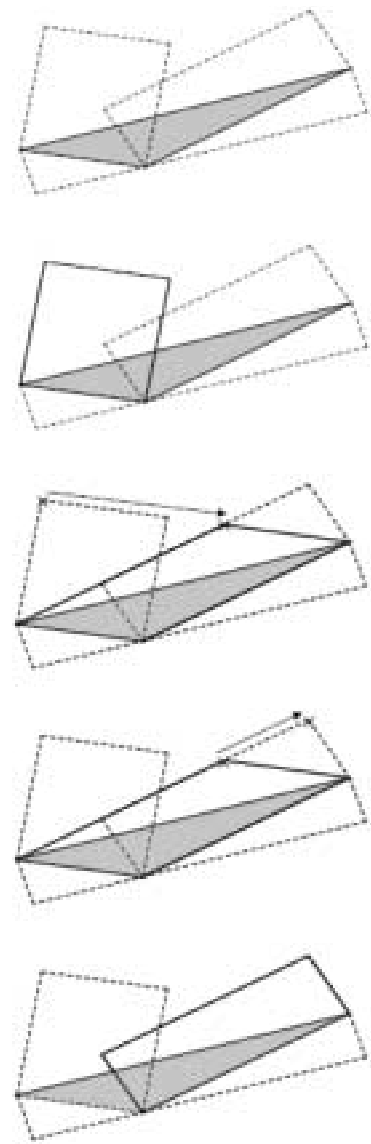

Figure 15: A sequence of transformations to transform one base × height rectangle into another one related to the same triangle

An alternative to the rapid arithmetisation of geometry

One loss that results from inattention to history is in mathematical perspective. Being attuned to and informed by the history of mathematics can inform us and provide resources for resisting either mathematical or pedagogical hegemonies. It can also keep alive earlier rich intuitions about – as well as profound awarenesses of and ways of working on – school mathematical topics. However, the goal is not to reproduce history, but rather to see how history can provide a rich ground of (mathematical) inspiration concerning the concepts themselves (Charbonneau, 2002; Proulx, 2003). [13]

The arithmetisation of area and volume, which results in a topic widely taught in schools, is not the only viable approach to area and volume – and could be argued to be conceptually weak. As a result of familiarity with this part of mathematical history, we retain the opportunity to stress aspects of such a geometric approach in classroom settings. We do this in order to provide a greater geometric sensibility to questions of area and volume and, among many other things, to develop a way to avoid the proliferation of seemingly independent formulas for each possible and different planar and three-dimensional shape taught as discrete items to be remembered. The Cavalieri principle draws together (seemingly different) figures and solids that are often seen as separated through their different isolated formulas.

We entitled this section with a probably unfamiliar hybrid term: 'arithmetisation of geometry'. Because the arithmetisation of the whole of mathematics is so familiar to us, it is often not singled out and named (except in historical works, *e.g.*, the arithmetisation of calculus or of analysis, as well as evoking the powerful and significant disagreement over the notion of Greek 'geometric algebra' – see Unguru, 1975). But by doing so, we mean to suggest that this is not inevitable. Our primary concern about the current school embodiment of area is that it is almost all arithmetic geometry, where the presence of and focus on formulas push students into substituting numbers into them (thus embedding the mathematical work in the non-geometric realm of algebraic formulas and numbers), leaving the geometry far behind. But such an approach is *not* pedagogic with respect to geometry: it does not make anything obvious about area as a geometric notion, it just sets it to one side and masks its meaning to some considerable extent. What we have tried to do here is to illustrate how it is possible to work on the 'same' material (areas and volumes of figures), but with a different orientation and intent.

In conclusion

> The spirit of geometry circulates almost everywhere in the immense body of mathematics, and it is a major pedagogical error to seek to eliminate it. (Thom, 1973, p. 208)

> In any case it would seem that geometry might be more successfully pursued if it were to be explored in its own terms (Tahta, 1980, p. 7)

It was mathematician William Thurston (1994) who claimed the job of the mathematician is "finding ways for *people* to understand and think about mathematics" (p. 162; our italics). It is our contention that this job ought not be carried out exclusively by professional mathematicians, but also by others equally deeply invested in providing insight into school mathematical topics. It may well be the case, however, that when individuals do undertake such work as this, they are acting *as* mathematicians. We see this type of conceptual work as part of theoretical mathematics education, an activity which takes serious note of the presence of the word 'mathematics' within 'mathematics education'. As efforts are made to render the study of mathematics both more comprehensible and tolerable for students, as well as to enable them to develop their significant mathematical powers, it seems simply mistaken to restrict the study of rich geometric domains to realms of arithmetic techniques and algebraic formulas.

Appendix 1: The area formula for a triangle

What are some possible geometric awarenesses behind the over-familiar formula for the area of a triangle? Algebraic fluency leads to presenting 'the formula' as half base times height, even though there are three formally distinct (if presumably equivalent) versions of 'this' formula, namely:

- one half of (the base times the height);
- (one half of the base) times the height;
- the base times (one half of the height).

We have chosen to 'revert' to primarily verbal (enhanced by brackets rather than using words like 'then' or 'first' to indicate scope) rather than syncopated or symbolic means of expression (which led to the embedding of formulas in algebraic activity), in order to draw attention to how quickly algebraic equivalence disregards potentially significant differences, to use Thom's suggestive term, in *vision*. [14]

To understand an area or volume formula requires staying with the geometry. A meta-precept we hold to is that any such 'saying' can be reconnected to an underlying 'seeing'. How might each of these 'verbal formulas' be related to different 'seeings'? What is sought is an algebraic equivalent, not of the *doing* of arithmetic but writing down what arithmetic you *would* do, in order to bring out the structure of the underlying algebra which comes initially from the way of seeing (before transformed into a 'simplest' form, an act which destroys the link with the originating seeing). But this too is potentially absorbable into the quasi-conservative view that such seeings should simply be deployed in order to generate the correct algebraic formulas.

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Notes

[1] For more on this tension, see Pimm (1996).

[2] For Cavalieri (1635), it was a theorem, for which he offered a sophisticated proof. Exploring the confused meta-mathematical term 'principle' (Bernoulli's principle, the principle of mathematical induction, the principle of least action, the pigeon-hole principle, …), while a potentially interesting project, is beyond the scope of this article.

[3] There are comparable issues with the ancient Greek quantification of ratios (see Fowler (1987/1999)) involving what he terms 'ratio' numbers: once, twice, three-times, etc. Fowler makes a very strong case for a 'lost'-because-superseded ancient Greek theory of ratio and a geometry that was not arithmetised. This idea is discussed in more detail later, as well as in Gouvêa's (1999) review of Fowler's book.

[4] In fact, the assumption of parallels being everywhere equidistant from one another is equivalent to Euclid's fifth postulate. Al-Haytham (also known as al-Hazen) in the tenth century proved that the parallel postulate is equivalent to the assumption that the equidistant curve to a straight line is itself straight.

[5] In fact, rectangles, parallelograms, rhombuses and squares could even be linked in relation to their potential area formulas. Instead of having $L \times W$ for rectangles, $B+h$ for parallelograms, s^2 for squares and $(D+d)h/2$ for rhombuses, as we frequently see in textbooks, they could all have the formula: $B \times h$.

[6] We need not insist on moving to the 'rectangular' object first: for instance, any triangle can be 'doubled' to produce a parallelogram which then has an associated area-equivalent rectangle. It can be an interesting geometric exercise to show that this is the same rectangle as the one generated by going the other route. This suggests a sort of commutativity principle between 'family' and 'relationship'. The right trapezoid or right-angled triangle route is family first and then relationship (*i.e.*, stay within the family of equivalents and choose a rectangular-referent one, then relationship), whereas the above example illustrates 'relationship' first (*i.e.,* here, doubling) and then finding a rectangular-referent element within the new family (*i.e.,* here, of parallelograms).

[7] It is in handling volume that Cavalieri's principle is better known, discussions of which can be found in some older books on solid geometry – for instance, *Solid mensuration with proofs* by Kern and Bland (1934/1947). These authors, both associate professors of mathematics at the US Naval Academy, organised much of their book around Cavalieri's principle. Interestingly, from our perspective, one main organising principle for the various solids treated is that of their corresponding algebraic formula for volume: Chapter 3 is entitled 'Solids for which $V = Bh$', Chapter 4 'Solids for which $V = {}^1/_3Bh$' and Chapter 5 'Solids for which $V = (\text{mean } B)h$'.

[8] It is important to appreciate here the distinction between this decomposition and an argument based on pouring water from a pyramid into its associated cube. Geometrically, the physical decomposition not only demonstrates that the volume of the pyramid goes into the cube exactly three times, it also illustrates that three pyramids of identical volume can be physically combined to make a cube, hence creating a strong relationship between them *geometrically*. The pyramid and the cube not only have a ratio of 1:3 with respect to one another; they become geometrically linked

in regard to their forms – just as triangles and rectangles were. (See, also, Pallascio, Allaire and Mongeau, 1993.)

[9] This could even lead to a different classification of three-dimensional solids, where cylinders and cones would be in the same category as prisms and pyramids, moving aside from the usual binary split into 'round objects' and 'not-round objects'. This is reminiscent of de Villiers (1994) alternate classifications of quadrilaterals, which differ from the ones usually worked with in schools.

[10] We are grateful to Dave Hewitt for his keen interest in these ideas and the production of various drawings to represent them. Figures 14 and 15 reflect his own constructions, which he generously agreed we could incorporate into the article.

[11] Theorems (almost always tacit and often unrecognised) about an invariance can be seen as an indifference allowing an optimal choice for a given purpose frequently lie behind a number of school practices (*e.g.*, work with algorithms involving equivalent fractions). The upshot, however, is that users seldom notice they have made a choice (which potentially may have made a difference).

[12] Another example of Cavalieri at work can be found in the work of his contemporary seventeenth-century mathematician Gilles de Roberval (1602–1675), in particular Roberval's ingenious determination of the area under a cycloid in relation to the area of the generating circle – the former is exactly three times the latter. Roberval talked of areas 'made of equal lines' and apparently claimed he had invented Cavalieri's technique independently. (See Fauvel and Gray, 1987, pp. 376-7, for a translation of Roberval's original text, as well as the discussion in Gray, 1987, pp. 12-14. See also: www.maths.uwa.edu.au/~schultz/3M3/L17Roberval.html.)

[13] Note that this is different from what has been termed 'the genetic approach to teaching mathematics' (see, *e.g.*, Toeplitz, 1949/1963; Fauvel, 1991), namely teaching mathematics in the developmental order that it emerged historically. In this context, and simply put, the ways ancient Greek mathematicians handled geometric questions (in particular, their notions of area and volume) differed significantly from the ways we do now. For instance, much of Euclid Book I can be seen as an attempt to create a *geometric* theory of the area of polygons, with Pythagoras's theorem a culminating key result in such a theory necessary to combine any two squares into a third.

[14] René Thom (1971, p. 697) has noted: "And according to a long-forgotten etymology a theorem is above all the object of a vision".

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What is the lowest multiple of 360 that can be formed by arranging all ten digits (1, 2, 3, 4, 5, 6, 7, 8, 9, 0) in any order?

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