Points, Lines, and their Representations

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Students come to mathematics lessons with some exisiting concepts about the topics being presented These concepts have been constructed by the students from information encountered in everyday situations, including lessons in school Each student's set of concepts is an individual construction and is likely to be different from that constructed by any other student The nature of these concepts significantly determines what is learnt and how it is learnt by the student. New information presented in school is likely to be interpreted differently by different students and so each student's concepts will be changed differently and perhaps unpredictably by the same instruction. Since existing knowledge is the student's own, often constructed over years of experience of a mathematical idea, it will probably be resistant to marked change, unless new information is seen by the student as intelligible, plausible, and more fruitful in explaining the relationships being investigated than the student's existing set of concepts [Hewson, 1981].

Part of the student's exisitng knowledge will be constructed on the basis of the ways in which words with specialized mathematical meaning are used in everyday situations For example, a student will have encountered the word "line" in common figures of speech such as "deadline," "a ticket line," "neckline," and "line of sight." The ways in which "line" is used in common expressions like these differ subtly from the ways in which mathematicians use this word. The concepts constructed by students are therefore likely to differ from the concepts used by mathematicians.

The set of concepts held by a student with respect to a science topic is called a "conceptual framework" [Driver, 1973] in the science education literature. This term has been used by science educators to emphasize that students' ideas are not misconceptions but rather preconceptions, not yet as complete and refined as the concepts held by scientists The existence of students' alternative frameworks with respect to numerous science topics is well documented [see Happs, 1983, for example]. the presence of such alternative frameworks means that the information about a topic presented by the teacher may interact with the student's existing knowledge in a number of different ways:

(1) The alternative framework held by the student persists unchanged by teaching
(2) Teaching results in a second view of the topic being acquired for use in school, but the alternative framework persists elsewhere.
(3) The alternative framework is strengthened by teaching which is now misapplied to support it
(4) Both views coexist in ways that are not integrated and may be self-contradictory.
(5) Both the alternative framework and the view presented by the teacher are extended, combined, and if necessary change to form a unified view

[Hewson, 1981]

If students' alternative conceptual frameworks exist for mathematical topics, then new information presented by teachers in mathematics lessons may interact with students' existing knowledge in any one of the five ways outlined above Clearly, the last of these five possibilities is the most desirable outcome of instruction.

Points and lines are abstract mathematical concepts which are not defined by mathematicians, but whose meanings are expressed in propositions that define relationships between them and other geometrical entities. From the everyday situations in which they hear "point" and "line" being used, students construct an intuitive understanding of these terms and their relationships. To find out some information about the conceptual frameworks that students have before they reach high school, I talked to ten able and articulate seventh graders about points and lines. In particular, I was interested to find out how these students represented points and lines in their thinking. I also discussed the same concepts with ten elementary education seniors. Their responses are compared briefly with those of the seventh graders

Points and their representations

The word "point" seemed to convey two different ideas to the seventh graders The first was suggested by common pointed objects such as the point of a pencil, a mountain peak, and the point of a diamond in a ring. The second and more common idea was to think of points as little round "dots." These are encountered in punctuation marks, as decimal points, and on line segments.

For most of the students, dots and points are much the same. Miki said that a point is just like a small dot. When a dot is placed on a line, the dot shows that there could be a point there. She thought that if a line is drawn on paper, the points are like little discs, but if the line is something like a piece of string, the points are like little beads. Nola claimed

For the Learning of Mathematics **5**, 3 (November 1985)
FLM Publishing Association, Montreal, Quebec Canada

that a point is the same size as whatever is used to draw it, if she thinks of a line in her imagination, Nola envisages the points as more like beads than discs Terry, Sandy, Paul, and Lynn also considered that points and dots are the same. Terry said that she thinks of them as being like beads; Sandy and Paul think of points as like discs. Sandy said that dots are a little larger than points Lynn said that she generally thinks of a "little circle" but thinks of points as like beads when a line is drawn on paper.

Owen also generally thinks of points and dots as the same: "I just kind of make a little dot with my pencil or something." Dots and points are the same when they are on a line segment When two lines meet, their intersection is a point but is not circular like a dot. Kim claimed that in everyday situations "point" is used more frequently than "dot" but in mathematics the reverse is true, although both words mean the same. Like Nola and Owen, she thought that the small mark made with a pencil is a point The point has the same size as the tip of the pencil.

Two students, Quin and Ricki, explored their ideas about points in some detail. Ricki said that she thinks of points as marking special places on a line so that the line can be used to solve equations. A point might also be the intersection of two lines. Generally she envisages points as shaped like small discs; they are two-dimensional markers on a one-dimensional line. She drew an analogy between the role of a point on a line and a saw horse across a road to mark construction or its end I asked Ricki whether she thought that a point and a dot are the same thing

> I guess. It's not exactly the same. I hate to say it [a dot] represents a point, but it's like marking where you want your point to be. .. It's just where the point is and the dot is the same place that you want your point to be. It's just that the dot is not the actual point It's just standing there telling you where it is

Quin remarked that she usually thinks of points in common phrases such as "at the point of going broke," as decimal points, and as angles If she is thinking of a point on a line, it is like a "teeny dot" and is marking a special place on the line. However, a point need not be marked If it is, the mark is symbolic of the point. Like Ricki, Quin also used an analogy to explain the role of a point on a line:

> Say that you were driving down Highway 75 and there was an old bent-up tree and that was your landmark to show that you were still on Highway 75 or something. This is like a point. .point A would show you that that was an important place on the line and that you'd probably need to remember it

Like the other students, Quin and Ricki seemed to think of points as added to lines to "point out" a particular location They made much clearer distinctions between points and their written representations than the other students.

I asked the students to draw a straight line, a wavy line, and a zigzag line and to say where there were points on these Owen and Paul thought that there were no points on the straight or curved lines except at the edge of the paper. Owen thought that there were points on the zigzag line where it changed direction He recognized that the curved line changed direction but described it as more "fluent." Five others thought that there were no points except at the vertices of the zigzag line. However, Miki said that a dot must be added to the line at those places to make them into points. I asked how it is possible that there are points on the zigzag line and not the others Terry said that it is "sharp," Lynn indicated the vertices and said "because they're pointy," and Sany said, "It's kind of like to a point. And these just curve, bend." Most of the students seemed therefore to associate points with pointed places on a line and perhaps with its ends.

Kim, Ricki, and Quin had different views. Kim thought that there would be points at the end of each line and that it is possible to add points anywhere along each line. Ricki and Quin thought that no line has endpoints or any other points until they are placed there Quin explained that on each line there are points "but you're just not acknowledging that they're there," just as in taking a trip it is possible to fly over cities without landing. It is also like a number line on which the integers have been labeled and the rationals are still there even though not labeled.

The association of a point with pointed places on a line was unexpected, but the genesis of this idea in common language is clear Points are often illustrated in mathematics textbooks by circular dots. These are sometimes wider than the illustrations of lines on which they are located I was therefore not surprised that the students thought of points as circular dots. These students might be helped to distinguish between the abstract idea of a point and its representation if textbooks were to refer to such illustrations as "dots." Since the students apparently thought of points as added to lines for specific purposes, they might not find lessons meaningful that refer to the location of points of line segments, such as midpoints and points of intersection

Lines, line segments, rays, and their symbols

Paul described a line segment as something that "just sort of stops and starts and stops." It is normally drawn without dots on the ends that signify that it stops. If a line is drawn without dots it is still a line segment that stops at the end of the drawing. Ricki and Quin both said the same thing as Paul Owen, Kim, and Lynn all emphasized that a line segment is a portion of a line and agreed that it is normally drawn with dots. I asked Owen about the difference between a line with dots and a line without dots or arrows. Owen: "Well a line segment is part of a line, it's a part of something, and a line...it's a whole, compared to just part of something"

Miki and Nola also thought that a line segment is part of a line and that the dots drawn on a line segment are not part of it but show that it ends. However, these students thought that if there are not dots, the line "could go on or it could stop You wouldn't know unless you had something else added to it." It would not be possible to tell if it were a line segment, a line, or a ray If the line had arrows added, it could go on forever Terry and Sandy similarly distinguished these three line forms Sandy was the only student who thought that the dots on a line segment are part of the segment. Neither Sandy nor Terry thought of a line segment as part of a line.

Almost all the students thought that if a line is drawn with arrows on its ends it is "endless" or "continues on forever." However, Lynn thought that the arrows indicate either that the line can go on further or that it stops where the arrows are. She described the number lines she had encountered at school as going only as far as the arrows. Terry claimed that a line with arrows would go around to the back of the page and then continue on as a line in the brickwork on the wall and travel around the room. I asked Terry if she could imagine a line going straight forever. Terry: "Yes, it would cut across the street, the trees, and on down through the campus. ... It would just go across country, across the world. It would probably meet the Great Wall of China." Quin's view of the relationship between line segments and lines was probably unique. I asked whether a line segment is part of a longer line.

Q: Yes, because you have one line, one big line going on forever and you just take little parts out and you put one little part in your math book and another little part in your social studies book for your graph.

I: So you can chop up a big line and use it for lots of purposes?

Q: Yes, but then you have to put them back.

I: Why do you have to do that?

Q: So you won't have a whole bunch of little line segments ... because lines are never ending... so you'd still have one long line.

Some students could recall what a ray is. Paul: "It has a dot on one end and an arrow on the other. One end can't move but the other end can." Owen: "It's like it stays here but it goes on for ever in another direction." However, Kim and Lynn thought that a ray has arrows on both ends. Both had previously stated that endless lines have arrows on both ends and evidently did not see any contradiction in their responses. Sandy also thought that a ray has arrows on each end but is drawn shorter than a line with two arrows. She thought that "ray" is used when someone gets tired of using "line."

The confusion of rays with other types of line forms by some students was not surprising. In school mathematics through the seventh grade rays are usually encountered only in units on angles. The term "ray" is not used as commonly as "line" in everyday situations. I was surprised to find that some students categorized line forms into three types: infinite lines, line segments, and lines for which it cannot be determined that they go on or stop. These students seemed to have constructed these categories from the normal written symbols that are used in school textbooks. Since line segments are illustrated with dots on the ends and infinite lines with arrows, then illustrations without dots or arrows must be neither infinite nor finite. They must therefore be indeterminate. Most students referred to such line forms as "lines." This prompted some unexpected responses. For example, I asked Owen how many straight lines could pass through two distinct points. He replied that there could be infinitely many since lines of different lengths could be superimposed on each other.

Lines and their representations

Sandy and Terry were discussing different ways to measure a line. Sandy: "You could measure the width, but that would not be big at all, maybe like a millimeter or something." Terry: "You could measure how dark the line is, like on a value scale in art." I asked whether a drawing of a line made with an extremely sharp pencil would have width. Terry said that it would, even if a microscope were needed to see it. A line folded in paper would also have a very tiny width. A line would look like a rectangle under a microscope. Any point on a line would have to be wider than the line so that the line could go through it. I asked Sandy and Terry to comment on the suggestion that pencil marks, a piece of string, a fold, and edges in brickwork are examples of things that are like lines and that lines as mathematicians think of them are abstract ideas. Both rejected the idea that lines are abstract. Sandy: "Well they're there. You know they've got to be something." I remarked that another student had said that if she is thinking about lines, the imagined lines must have width or it would not be possible to picture them. Sandy and Terry both agreed with this statement. Sandy said that she normally imagines a line as straight and thinks of its length not its width "even though I know it's got one." Terry thinks of curved lines as well as straight lines, like the lines on a basketball court.

This conversation with Sandy and Terry reflects views that were held by the majority of the students. Most were quite puzzled by the suggestion that a line is an abstract idea. To them, everyday objects to which the word "line" is applied, such as edges in brickwork, are real lines. So too are representations likely to be used in the classroom, such as folds in paper and pencil marks. A line therefore has some width which depends upon what is used to make it. To these students, the description of a line as one-dimensional was quite meaningless.

Although most students did not distinguish between lines and their representations, two students did. Nola thought that any line drawn on paper would have some width, even if a "really small, tiny little millimeter." However, a line in her imagination would not have width. Nola: "A line's just something you imagine, and if you draw it out, it's just a representative of the line you're imagining." She thought of pencil marks and folds as pictures of what she imagines.

Ricki and Quin discussed the distinction between a line and its representation at some length. To Quin, a line has "height" and cannot be seen if it does not. Ricki argued that a line has one dimension only and that it has no width. She added that a pencil line and a piece of string "form" lines but are not really lines themselves. Quin disagreed: "If you say it's forming a line, you're saying it *is* a line because any way you can do it, you can do it twisted up like that and it's still a line." Ricki drew an analogy with molding a chair from clay. The model would be in the form of a chair but would not be a real chair. Quin responded that the model would be a chair. She asked Ricki whether she thought that a rubber ball and the earth are the same thing because they are both spheres. Ricki replied that they both have the shape of a sphere but are not spheres. I proposed two statements, "the shape of the string is a line" and "the string is a line." To Quin, these statements meant the same and she preferred the second. Ricki remarked that there was a "hair's difference" between them and that the first summarized her view although she would normally say the second in ordinary conversation. I followed up this conversation with Ricki on a second occasion. She reiterated that a line has no width and that a pencil mark is not a line but merely indicates where the line is to be. She agreed with the suggestion that a line is an abstract idea and that a pencil mark is a representation of that idea.

The conversation between Ricki and Quin showed that these two students were grappling with quite profound ideas. Like Nola, Ricki had reached a very high level of abstraction. Quin did not accept Ricki's arguments and made no distinction between a line and its representation. In everyday situations, natural language usage would support Quin's view. Like most of the students, she was unable to transcend her everyday experiences.

The college students

Like the seventh graders, the college students often used "dot" to describe a point. Four students thought that dots and points are the same. For example, Ann thought that the only difference is that teachers prefer "point" because it sounds more important. Bette remarked that "to make a point you make a dot." Six thought that dots are used to represent points. Glenda explained that a point is really an abstract concept used to describe how a line is made up and cannot really be defined. I asked four students where there were points along a line segment, a wavy line, and a zigzag line. Glenda and Holly thought that there was an infinite number of points along all line forms. Holly remarked that the vertices on the zigzag line were no different from the other points "unless they tell it to turn around and they're the leaders." Ann and Bette did not think of lines as consisting of points and thought that the only points would be at the vertices of the zigzag line and at the ends.

Most of the college students had good recall of the distinctions between lines, line segments, rays, and their symbols. They did not distinguish between line forms with dots and line forms without dots or arrows, as the seventh graders had. Most thought that the dots drawn on the end of a line segment are not part of the segment. As Carol explained, "they're just like holding it so the line can't escape." However, Glenda thought that the dots are part of the segment because they are connected to it, and are represented as larger because they are more important.

I asked Joan to talk about the notion that a line in mathematics is an abstract idea. Joan: "When I imagine a line, it is not an abstract thing. I see it and so it is very concrete to me. .. It's not abstract." She discussed lines made of clay or drawn with a pencil and said that they can be felt, touched, drawn, and used. "Mathematicians wouldn't think of it if it wouldn't be of practical use." Half of the college students rejected as did Joan the notion that a line is an abstract idea. They gave examples of everyday objects to which "line" is commonly applied. Like the seventh graders, they also believed that lines have width. The other five students believed that a line is an abstract idea and that a drawing is a representation of the line. Carol explained it like this: "It already exists .. and I can't see it. But I can think about it. If I have to picture it in my mind, I have to picture it as something like this [a drawing] but that is not a line." Most thought that the mental image of a line would have to have width but Ella said that she could imagine a line without evoking a mental image.

The responses of the college students indicated that they had greater knowledge of standard line symbols than the seventh graders, as I expected because of their greater educational experience. They were also more likely to think of a line as a set of points. However, only half distinguished between points and lines and their representations.

The discussions reported here suggest that these seventh graders had constructed conceptual frameworks concerning points and lines that differed in significant ways from those that would be used by mathematicians. Chief of these differences were the identification of three different line forms, the idea that lines have width, the notion that points are entities added to lines, and the idea that points have a definite shape and size. All of these ideas seem to have originated in the common representations of points and lines that the students had encountered. The importance of everyday language usages of "point" and "line" in the construction of the students' conceptual frameworks can also be identified. If students do not distinguish between abstract geometrical concepts and their physical representations, then their concepts will include properties and relationships based on the features of those representations that are no part of the concepts as used by mathematicians.

The college students' frameworks differed in some ways from those of the seventh graders. No college student distinguished three different line forms, for example. Nevertheless, the college students' conceptual frameworks also embodied ideas that would not form part of a mathematician's concepts about points and lines, and this raises the question of how well these future teachers will be able to help their own pupils to construct abstract mathematical concepts.

The relationship between instruction and the students' concepts is unclear. The discussions with the college students suggest that school and college instruction had not entirely succeeded in helping them to replace their own constructions with abstract mathematical concepts. Future instruction may help the seventh graders to modify their conceptual frameworks and so arrive at a more sophisticated mathematical understanding of these ideas. On the other hand, the alternative frameworks constructed by these students may be adequate for school geometry instruction. Unless they encounter situations in which they identify a conflict between new information and their existing ideas, they may not recognise a need to modify their concepts. For example, Owen may at some time in the future encounter the mathematical statement that there is only one straight line that passes through two distinct points. Only if he recognises that this statement contradicts his belief that there are infinitely many is he likely to be able to modify his existing concept of a line. Alternatively, the new information may interact with his existing concepts in any of a number of other ways, as listed earlier [Hewson, 1981]

Few mathematics teachers are likely to dispute that the aim of their instruction is to enable their students to extend and modify their concepts to resemble more closely those of mathematicians. Whatever teaching strategies are developed and employed, they need to take account of the existing sets of concepts held by students Ignoring students' existing concepts matters for those students unable to modify their own views without assistance. Identification of the students' conceptual frameworks is only the first, although an important, step in designing instruction If it is desirable to change students' ideas about concepts such as points and lines, then a specific approach needs to be developed to teach about these ideas Since students rarely interpret instruction in the ways that their teachers intend, then it is difficult to predict what impact that instruction will have.

Another important step in instruction might be to make students aware that their ideas are different from the ideas of other students and the teacher This could be achieved through discussions designed to highlight and compare those differences Discussions like the ones reported in this study would not only provide valuable insight for the teacher into the concepts brought by the students to class, but would also enable the students to examine and perhaps refine their ideas in the light of the different ideas presented by others There is no doubt that in these discussions, most of the students were expressing ideas that they had never clearly articulated and inspected before Even when teachers have established the existing knowledge of their students and made the students aware of varying views, the students may not recognise a need to change their own concepts. New information may be learnt at a verbal level or may lead to conflict and confusion rather than the extension and refinement of ideas that are desired by the teacher The teacher needs also to provide opportunities for the students to test their newly-acquired knowledge in new situations and to confirm that the new knowledge is fruitful in explaining those situations

There is no doubt that students have some familiarity with the concepts of points and lines, built up over years of using "point" and "line" in everyday contexts and everyday figures of speech Because these everyday uses are rich in their variety, the concepts constructed by the students are also likely to be varied, as indicated by the conversations with the students in this study. Because points and lines are such familiar concepts to the students, teachers may not perhaps recognise that their students have such varied conceptual frameworks about these ideas and that these frameworks may ensure that students are interpreting lessons in geometry in different, unrecognised, and unforeseen ways. As teachers, we need to ask ourselves how well we are communicating with our students about fundamental ideas in geometry and what in fact we are teaching them.

References

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A *Poynt* or a *Prycke* is named of Geometricians that small and unsensible shape, whiche hath in it no partes, that is to say: nother length, breadth nor depth But as this exactness of a definition is more meeter for onle Theorike speculacion, then for practise and outwarde worke (consideringe that myne intente is to applye all these whole principles to woorke) I thynke meeter for this purpose, to call a poynt or prycke, that small printe of penne, pencyle, or other instrumente, which is not moved, nor drawen from his fyrste touche, and therefore hath no notable length nor bredthe.

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