Two Kinds of "Elements" and the Dialectic between Synthetic-deductive and Analytic-genetic Approaches in Mathematics*

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When talking about "Elements" in mathematics we usually think of *Euclid's* famous 13 books or, with respect to recent developments, we may refer to *Bourbaki's* influential "Eléments de mathématique", a series of books devoted to a comprehensive program of unifying modern mathematics. What links the two, is the *axiomatic-deductive approach*.

In these associations of ideas related to "Elements" as a mathematical key-word we neglect the fact that in the course of history other "Elements" have been written which do not follow Euclid's example. On the contrary, in these books strong criticisms against the Euclidean so-called *synthetic method* were put forward and their authors were mainly concerned with describing and elaborating a kind of opposite approach, called *analytic*, or later on *genetic*. Antoine *Arnauld's* "Nouveaux eleméns de géometrie" of 1667 and Alexis-Claude *Clairaut's* "Elémens de géometrie" of 1741 and "Elémens d'algèbre" of 1746 may especially be mentioned here. They belong to a new scientific movement heading towards discovery and development and looking for a method that would serve the foundation and justification of knowledge as well as its application and the finding of new truths. Other early representatives of this movement were Petrus *Ramus*, Francis *Bacon* and René *Descartes*.

Imre *Lakatos* has recently revived this counter-Euclidean position in "Proofs and refutations: the logic of mathematical discovery" [1976] and in "Renaissance of empiricism in the recent philosophy of mathematics" [1978]. Here he confronts *deductivist style* with *heuristic style*. In his paper "The method of analysis-synthesis" [1978] he particularly analysed the classical separation and combination of the two methods from his dynamic epistemological viewpoint. Lakatos comes to the conclusion that "school concepts of static rationality like *a priori-a posteriori, analytic-synthetic*" do not adequately apply in this epistemology: "These notions were devised by classical epistemology to classify Euclidean certain knowledge—for the problem shifts in the growth of quasi-empirical knowledge they offer no guidance" [Lakatos 1978(b), p. 41].

In this present paper we try, in addition to providing some interpretations of the two kinds of "Elements", to enrich the discussion especially by referring to a study by A.C. *Lewis* on the relations between Friedrich *Schleiermacher's* "Dialektik" and Herrmann *Grassmann's* "Calculus of extension" and the Hans-Niels *Jahnke's* elaboration of the role of *intended applications* as a part of any theory not only in physics—as was first explained, in contrast to the dominating "statement-view", by J.D. *Sneed* [1971]—but also in mathematics. The importance of a *complementarist philosophy* is indicated Schleiermacher's dialectic between what he called the *heuristic* and the *architectonic* method is depicted in Grassmann's *complementarity of rigor and overview*. Grassmann's understanding of his "calculus of extension" as "forming the abstract foundation of the theory of space (geometry)" can be interpreted as a Sneedian example of the relation between theory-kernel and intended applications, a relation which is actually a dynamic complementarity between theory development and application. Several implications for mathematics education are exhibited.

1. Euclid's "Elements" and the learner

One of the first sources in history for the identification of a domain of knowledge called "elementary mathematics" can be recognized in the 13 books forming *Euclid's "Elements"*. The greatest part of the geometry taught in the secondary schools of the 19th and 20th century stems from 9 of those "books". In Britain without much change they were even used as textbooks. Geometry as a school subject was simply called "Euclid", and it was only in the thirties of this century that the quarrel, which started around 1870, among individuals and groups like the "Anti-Euclid-Association" and the "Euclid Society", ended with the success of the former. Euclid has been called by van der Waerden the greatest school-master in the history of mathematics. In fact, by 1900, next to the Bible, Euclid's "Elements", of which more than a thousand edition have appeared since the invention of printing, was the book published and read most frequently in the Western World.

Euclid's is not the only work from the Plato period called "Elements". According to Proclus it terminates a series of similar texts. He mentions "Elements" by Hippocrates, by Leon, and by Theudius, all of which were most likely used as textbooks in the teaching of mathematics at the Platonic Academy [see van der Waerden, 1973]. However, Euclid was considered the master in this field and therefore in later times was often referred to by the title "the elementator" (ὁ στοιχειωτής)

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"Elements" (in Greek "στοιχεια") originally meant "letters" or "sounds" as members of a sequence or series. With Plato it denotes "*basic components*". Aristotle assigns four meanings, two of which are "*foundations of proof*" and "*supreme principles*" [see Lumpe, 1972]. In Euclid's *Elements* these foundations and principles are called "definitions", "postulates", and "axioms" [for details see Szabo 1960]. Mathematics can be built on them in a deductive way of which Euclid gave the first comprehensive example in history.

Proclus, seven hundred years after Euclid, in his commentary on the first book of the *Elements* [see Morrow, 1970] makes a distinction between "elements" and "elementary":

> We call "elements" those theorems whose understanding leads to the knowledge of the rest and by which the difficulties in them are resolved ... in geometry as a whole there are certain primary theorems that have the rank of starting-points for the theorems that follow, being implicated in them all and providing demonstrations for many conjunctions of qualities; and these we call "elements". "Elementary" propositions are those that are simple and elegant and have a variety of applications but do not rank as elements because the knowledge of them is not pertinent to the whole of the science: for example, the theorem that the perpendiculars from the vertices of a triangle to the sides meet in a common point (loc. cit., p. 59f.)

The role of books like the "Elements" in classical Greek *education* has been multifariously described for his time by Plato and commented upon later by Proclus. First, they can be used as a reference text for an introduction to hypothetic-deductive analysis as the specific method of *dialectic philosophy*. Second, according to their axiomatic organization of knowledge, they represent the best way of explaining that *mathematical truth* is eternal in nature and independent of empirical and sensual experiences, given that the basic concepts and axioms have been evidenced as possessing these properties. Third, insight into mathematical truth will best prepare the mind for *understanding the world of ideas* (comprising Truth, Beauty and Goodness), which is the utmost goal of higher education [see Reidemeister, 1949]. It should be added that Plato's philosophy with its theory of reminiscence of ideas (α'να'μνησιζ) and the central role of the parable of the cave also provides some kind of *theoretical learning basis* for the teaching of mathematics which is fundamentally connected with his ontological, epistemological and pedagogical theory [see Beth/Piaget, 1966].

In Proclus' commentary we read the following on *mathematics in general* and *geometry* in particular (see Morrow):

> The "Timaeus" calls mathematical knowledge the way of education, since it has the same relation to knowledge of all things, or first philosophy, as education has to virtue. Education prepares the soul for a complete (life through firmly grounded habits, and mathematics makes ready our understanding and our mental vision for turning towards that upper world. Thus Socrates in the "Republic" rightly says that, when the "eye of the soul" is blinded and corrupted by other concerns, mathematics alone can revive and awaken the soul again to a vision of being, can turn her from images to realities and from darkness to the light of intellect, can (in short) release her from the cave, where she is held prisoner by matter and by the concerns incident to generation, so that she may aspire to the bodiless and partless being (loc. cit. p. 17).
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> ... mathematics reaches some of its results by analysis, others by synthesis, expounds some matters by division, others by definition, and some of its discoveries bind fast by demonstration, adapting these methods to its subjects and employing each of them for gaining insight into mediating ideas. Thus its analyses are under the control of dialectic, and its definitions, divisions, and demonstrations are of the same family and unfold in conformity with the way of mathematical understanding. It is reasonable, then, to say that dialectic is the capstone of the mathematical sciences. It brings to perfection all the intellectual insight they contain, making what is exact in them more irrefutable, confirming the stability of what they have established and referring what is pure and incorporeal in them to the simplicity and immateriality of Nous, making precise their primary starting-points through definitions and explicating the distinctions of genera and species within their subject-matter, teaching the use of synthesis to bring out the consequences that follow from the principles and of analysis to lead up to the first principles and starting points. (loc. cit., p. 35 f)
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> Magnitude, figures and their boundaries, and the ratios that are found in them, as well as their properties, their various positions and motions—these are what geometry studies, proceeding from the partless point down to solid bodies, whose many species and differences it explores, then following the reverse path from the more complex objects to the simpler ones and their principles. It makes use of synthesis and analysis, always starting from hypotheses and first principles that it obtains from the science above it and employing all the procedures of dialectic (loc. cit., p. 46)

With respect to the *relation between the Elements and the learner*, Proclus emphasizes the strict building up from the primary elements which he calls an "*elementary treatise*":

> Of the purpose of the work with reference to the student we shall say that it is to lay before him an elementary exposition (στοιχειωσιζ) and a method of perfecting (τελειωσιζ) his understanding for the whole of geometry. If we start from the elements, we shall be able to understand the other parts of this science; without the elements we cannot grasp its complexity, and the learning of the rest will be beyond us. ... It is a difficult task in any science to select and arrange properly the elements out of which all other matters are produced and into which they can be resolved. ... in general many ways of constructing elementary expositions have been individually invented. Such a treatise ought to be free of everything superfluous, for that is a hindrance to learning; the selections chosen must all be coherent and conducive to the end proposed, in order to be of the greatest usefulness for knowledge; it must devote great attention both to clarity and to conciseness, for what lacks those qualities confuses our understanding; it ought to aim at the comprehension of its theorems in a general form, for dividing one's subject too minutely and teaching it by bits make knowledge of it difficult to attain.

With these remarks and quotations it is indicated that there are in particular *three aspects* related to the concept of "elements of mathematics" in the early historical phase:

1. The Elements are considered basic for a *comprehensive and systematic building up* of existing mathematical knowledge
2. They are rooted in the *logical and ontological foundations of mathematics*
3. In a fundamental way they are related to the *teaching of mathematics*

2. Bourbaki's "Elements", communication, and didactics

These aspects link history with recent developments. It was with respect to them the *Bourbaki* came to name his huge work on modern mathematics "Eléments de mathématique". As H. Cartan, one of the founding members, reports (in a talk given in 1958), the project grew out of discussions in 1934 among a group of about 10 young French mathematicians about their plan to write a new textbook on calculus for mathematics students at French universities. Classical standard textbooks like that by Goursat were no longer considered up-to-date. It was found that a truly modern approach would have to incorporate other fields of mathematics which were traditionally viewed as completely separated from analysis. Thus it was agreed to write a comprehensive presentation of important fields of contemporary mathematics which should start from scratch and make understandable the common foundations of all these fields. Cartan points out:

> From the very beginning Bourbaki was a convinced adherent of the axiomatic method for which he has occasionally been criticized. But with respect to his task it was a necessity . By strict utilization of the axiomatic method Bourbaki was consequently led to a completely new order in the various domains of mathematics. It was impossible to keep the traditional subdivision: analysis, differential calculus, geometry, algebra, number theory, etc. It was replaced by the concept of structure which permits the definition of the concept of isomorphism and in this way makes it possible to classify the fundamental disciplines of mathematics. . For his vast book Bourbaki has chosen the title "Elements of mathematics". At first sight this title looks rather unpretentious, but it is actually very ambitious because it reminds us of Euclid's "Elements" [Cartan, 1959: transl. by the author]

It should be clear that Bourbaki's axiomatics is somewhat different from Euclid's. His can be characterized as *abstract axiomatics* by means of which concepts like group, field, topological space, ordered group, etc., are defined as *types of structures* [see Steiner, 1964]. Structures are given by sets of (unspecified) elements together with one or several relations into which these elements enter; they belong to a special type if the characteristic type-properties listed as "axioms" are fulfilled. So a typical feature of *building up mathematics* in Bourbaki's "Elements" could be described by Bourbaki himself by saying:

> From this point of view, mathematical structures become, properly speaking, the only objects of mathematics. [Bourbaki 1950, p 226 f; see also Dieudonné, 1939; Cartan, 1943]

Concerning the rooting of his "Elements" in the *logical foundations* of mathematics, Bourbaki developed a differentiated attitude. On the one hand he took the *naive position* of a *working mathematician*, using the language of structures and morphisms as it grew out of previous developments in topology and modern algebra in a more unified and generalized way, especially by the consequent application of set-theoretical concepts. On the other hand, he was seriously concerned with the logical and foundational status of his theory of sets and structures and therefore carefully investigated at an early stage of his work the place of his position with respect to formal logic and metamathematics, the results of which appeared in the "Description of formal mathematics" in the first chapter of the first book of his "Elements" [see Dieudonné, 1939; Cartan, 1943; Bourbaki, 1949, 1950]. With respect to the logical completeness and the consistency of his foundations he made the famous pragmatic claim:

> On these foundations I state that I can build up the whole of mathematics of the present day; and if there is anything original in my procedure, it lies solely in the fact that, instead of being content with such a statement, I proceed in the same way as Diogenes proved the existence of motion; and my proof will become more and more complete as my treatise grows [Bourbaki, 1949]

If a science is not only understood as a body of knowledge represented in books, but rather as a social institution with human beings involved who in a cooperative way produce, organize, and change knowledge, it is quite natural to see the didactical problems related to a science as constitutive components rather than second order questions which only come up if one deals with institutionalized teaching. This *internal didactics* of a science is concerned with an economic and transparent *ordering of existing knowledge*, with the development, choice and standardization of concepts and language, with problems of *communication* and *distribution* of knowledge, solutions, tools, etc.

In this sense, E. Peschl in the discussion following Cartan's talk commented: "Thus Bourbaki's work as a whole is a necessary task from general didactical considerations as well and in this way represents a great achievement by itself". Cartan himself argued:

> While the members of Bourbaki considered it their duty to elaborate all mathematics according to a new approach, they did this with the hope and expectation of putting into the hands of future mathematicians an instrument which would ease their work and enable them to make further advancements. Concerning this point, I believe that their goal has already been reached as I have frequently observed that concepts elaborated by us with so much effort are now being practiced easily and artfully by the young who have learned them from Bourbaki's books [Cartan, 1959: transl. by the author]

In a rejoinder to strong criticisms of "modern mathematics", including Bourbaki's work, put forward by *R. Thom* [1971], Jean Dieudonné, another founding member of Bourbaki, especially emphasized the *role of communication* within the scientific community. Thom had questioned deeply the particular emphasis on rigor, axiomatization, and explication in "modern mathematics", to which *Dieudonné* responded:

> Communication between mathematicians by means of a common language *must* be maintained, as Thom himself acknowledges, and the transmission of knowledge cannot be left exclusively to the geniuses In most cases it will be entrusted to professors, who, in Thom's words, are "adequately educated and prepared to understand [the proofs]" As most of them will not be gifted with the exceptional "intuition" of the creators, the only way they can arrive at a reasonably good understanding of mathematics and pass it on to their students will be through a careful presentation of their material, in which definitions, hypotheses, and arguments are precise enough to avoid any misunderstanding, and possible fallacies and pitfalls are pointed out whenever the need arises ... It is this kind of expository writing that has been, I think, the goal of those mathematicians whom Thom calls "formalists" from Dedekind and Hilbert to Bourbaki and his successors. [Dieudonné 1973]

As is well known, Bourbaki had a strong influence on the *teaching of mathematics* at the university level and also on school programs, curricula, and the didactics of mathematics as a discipline. The role of Platonic philosophy which had stood behind the pedagogy of the classical "Elements" was taken over by a new *structuralistic philosophy* supported by *J. S. Bruner*'s emphasis on the *structure of the discipline* as the guiding paradigm for the teaching and learning of a science, as well as by *J. Piaget's genetic epistemology* which described the development of and individual's cognitive equipment as an elaboration of mental schemata whose constitutive components consist of elementary structures similar to Bourbaki's mother-structures [see Beth/Piaget 1966]. We may confine ourselves here to quoting Bruner:

> ... the curriculum of a subject should be determined by the most fundamental understanding that can be achieved of the underlying principles that give structure to that subject. Teaching specific topics or skills without making clear their context in the broader fundamental structure of a field of knowledge is uneconomical in several deep senses. In the first place, such teaching makes it exceedingly difficult for the student to generalize from what he has learned to what he will encounter later. In the second place, learning that has fallen short of a grasp of general principles has little reward in terms of intellectual excitement. The best way to create interest in a subject is to render it worth knowing, which means to make the knowledge gained usable in one's thinking beyond the situation in which the learning has occurred. Third, knowledge one has acquired without sufficient structure to tie it together is knowledge that is likely to be forgotten. An unconnected set of facts has a pitiably short half-life in memory Organizing facts in terms of principles and ideas from which they may be inferred is the only known way of reducing the quick rate of loss of human memory" [Bruner, 1963, 31-32]

3. Genetic-analytic approaches: Ramus, Arnauld, Clairaut

The interpretations of the concept of "Elements" as given by Euclid and Bourbaki, and the approaches to mathematics built on them, have received *strong criticisms* from a philosophical as well as methodological and didactical point of view; and *alternative interpretations* and *approaches*, frequently called *genetic*, have been asked for and developed. The first criticisms of Euclid's Elements go back to the 16th century when, in the period of transition from the Renaissance to the Baroque, the long-lasting influence of Scholasticism, which had integrated some parts of the "Elements" into a system of knowledge highly determined by a onesided interpretation of Aristotelian philosophy, was overcome by a new *spirit of discovery and development*

It was *Petrus Ramus* (Pierre de la Ramée) who in his "scholae mathematicae" [1569], in connection with his criticisms of the scholastical Aristotelian system, put forward weighty objections against the Euclidean method. He had an influence on Descartes and in France is considered a forerunner of Arnauld's [1667] and Clairaut's [1741] new type of "Elements" [Taton, 1952, 1958; Itard, 1939]

> Thus we read of Petrus Ramus (1515-1572) in France being forbidden on pain of corporal punishment to teach or write against Aristotle. This royal mandate induced Ramus to devote himself to mathematics; he brought out an edition of Euclid, and here again displayed his bold independence. He did not favour investigations on the foundations of geometry; he believed that it was not at all desirable to carry everything back to a few axioms; whatever is evident, needs no proof His opinion on mathematical questions carried great weight. His views respecting the basis of geometry controlled french textbooks down to the nineteenth century. [Cajori, 1914]

Actually Ramus, as with other Renaissance and humanist philosophers, mathematicians, and pedagogues, was looking for a *universal method* which at the same time would serve the *founding* and *justifying* of existing knowledge as well as the *finding of new truths*. Aristotelian syllogistics and the comprehension of Euclid in the related deductive spirit seemed useful only for systematizing propositions already known to be true. What was missing with respect to syllogisms was a heuristic tool for discovering the middle terms by means of which knowledge could be expanded into new frontiers On the other hand, concerning mathematics, the Greeks were understood to have possessed and implicitly used such an unknown method, as is especially suggested by the more advanced results to be found in treatises such as "Quadratics of the Parabola" or "On the sphere and cylinder" by Archimedes or in Apollonius' "Conics". According to *Descartes* [1629] the Greeks intentionally concealed it:

> Some traces of this true mathematics seem to me to be found in Pappus and Diophantus, who lived, if not in the earliest ages, at least many centuries before these present times Indeed I could readily believe that this mathematics was suppressed by these writers with a pernicious craftiness ... And I believe they preferred to show us in its place, as a product of their art, certain barren truths which they cleverly demonstrate deductively so that we should admire them rather than teach us the method itself. [quoted from Lenoir, 1979, p. 365]

As regards mathematics, Viète, Descartes, and Fermat were convinced they had found the missing "analytical" method (in completion of the "synthetic" Euclidean method, and even comprising it) in their "Analytic Art", consisting of symbolic algebra and the utilization of algebraic equation, by means of which they reformulated and transcended Greek geometric algebra. On the other hand, in his more general concern for "Rules for the direction of the mind", Descartes seemed to have been fundamentally influenced by Ramus and the school of Ramists:

> While the technical aspects of Descartes' analytical methods might have arisen from a consideration of classical problems, the ontological foundation of his mathematical and methodological thought traced its roots to the Ramist sources. The methodological works of Ramus grew out of a pedagogy movement in the Renaissance aimed at reforming Aristotelian logic within the arts curriculum. Like the Renaissance analysts ..., the pedagogues were interested in supplementing traditional syllogistics with a method of invention. To solve the problems of the logic curriculum, the pedagogues turned to the field of rhetoric. They attempted to fashion a dialectical method of discovery by fusing elements of the rhetorical works of Cicero and Aristotle, particularly the latter's "Topics". For their purposes the mosts important aspect of Cicero's rhetoric was the theory of memory. As mere devices for the storage and retrieval of arguments previously constructed there was no epistemological basis within Aristotelian theory for using the topics and loci as new sources of demonstrative knowledge. Therefore, in order to justify their method of invention, the pedagogues had to fashion their own epistemology. The fact that it came to be characterized as *dialectical* indicates that the new method was inspired by Platonic sources, but the epistemic justification for fashioning a general method of invention from the art of memory and the topics was provided by the Renaissance Neoplatonic movement. [Lenoir, 1979, p. 367].

Ramus' general influence was unusually strong. Lebesgue [1957] states that for about 50 years the majority of French mathematicians were his former disciples. It is known that Ramus was particularly busy with providing and distributing books. This was possible on the basis of the newly developed printing press. However, the invention of book printing also had another (indirect) effect on the broad spread of his ideas. Knowledge could now be thought of in terms of "input", "output", and "consumption", in some kind of analogy to the "commercial world"; it could be made universally available, thus serving the storage and retrieval functions according to Ramus' theory of memory [see Ong, 1958] and in this way also promoting the new *social-communicative aspect of knowledge*.

Both in the history of epistemology and the history of pedagogy, the place of the great promoter if not creator of the *genetic approach*, understood as a unity of a method for scientific exploration and a natural method of teaching, is assigned to *Francis Bacon* (1561-1626) (Cassirer, 1971; Gusdorf, 1977; Schubring, 1978). However, with respect to mathematics, and geometry in particular, Antoine Arnauld (1612-1694) and Alexis Claude Clairaut (1713-1765) are conceded special pioneering roles.

Arnauld, the author of "Logique, ou l'art de penser" [1662], the famous "Logique de Port-Royal", with an eye especially on Euclid, criticized the "mistakes which usually occur in the methods practiced by mathematicians". In his "Nouveaux éléments de géométrie" [1667], also known as the "Géométrie de Port-Royal", he tried to overcome the deficiences by rearranging the order of concepts and theorems in elementary geometry, by providing new approaches and proofs, especially by allowing reference to "exterior and visible things". His list of *mathematicians' mistakes* comprises the following:

- more care is devoted to certainty and persuasion of the mind than to evidence and clarification
- proving things that don't need a proof
  (exemplified by Euclid's proof that two sides of a triangle are longer than the third one)
- proof by deducing absurdity
  (instead of directly, and positively, explaining a matter from its constitutive principles)
- proofs that are too far-fetched
  (exemplified by the proof given by Euclid for theorem 5 of Book I, saying that the congruence of two sides of a triangle implies the congruence of the two adjacent angles: "Is it not ridiculous to imagine that this congruence depends on those artificial triangles (introduced in Euclid's proof)?"
- neglecting the true order of nature
  (with special reference to Euclid who "believed that there is no other order to be followed than: theorems derived first should be suitable for proving theorems to come next") [Arnauld 1972, p. 318ff: translated by the author]

Whereas Arnauld's approach might be classified as *logico-genetic*, *Clairaut*'s "Eléments de géométrie" (1741) is a first example of a *historico-genetic* method which uses the historical development of mathematics as an orientation for the organization of content and learning activities [see Schubring, 1978]. Clairaut, an outstanding mathematician, who also wrote "Eléments d'algèbre" [1746], criticized the "normal texts written for beginners" because of their "start from a huge number of definitions, postulates and tentative principles which promise the reader nothing but dry learning stuff. The theorems which follow do in no way direct the mind to the interesting subjects; and moreover, they are hard to comprehend so that beginners are usually bored and repelled before having grasped an idea of what they are supposed to learn". Also, Clairaut did not accept those texts which try to overcome the dryness of the normal presentation by having each theorem followed by some practical use: "In this way they demonstrate the applicability of geometry without improving the possibilities of learning it. Since every theorem always precedes its application, the mind gets hold of the sensibly perceptible concepts only after having endured the drudgeries of learning the abstract concepts." [Clairaut, 1920, preface]. His own method he describes as follows:

> According to my deliberations, this science (i.e. geometry), as all the other sciences, must have developed stepwise. Apparently there were particular needs that induced the first steps; and these first steps should not lie outside the scope of beginners because they were beginners who first took them. Engaged by this idea, I considered returning to what might have given birth to geometry; and I endeavoured to elaborate the principles

> by means of a truly natural method that can be assumed equal to that used by the first discoverers, only keeping in mind the need to avoid the incorrect trials they necessarily had to make Land-surveying appeared to me most suitable for generating the first principles of geometry ... In order to follow a path similar to that of the inventors, I intended to let the beginners discover those principles which make land-surveying possible, as well as measuring distances independently of whether they are accessible or not, etc ... If the first originators of mathematics presented their discoveries by using the "theorem-proof" pattern then they doubtlessly did this in order to give their work an excellent shape or to avoid the hardship of reproducing the train of thought they followed in their own investigations Be that as it may, to me it looked much more appropriate to keep my readers continuously occupied with solving problems, i.e with searching for means to apply some operation or discover some unknown truth by determining the relation between entities being given and those unknown and to be found In this way, with every step they take, beginners learn to know the motive of the inventor; and thereby they can more easily acquire the essence of discovery [Clairaut, 1920, preface: transl. by the author]

Arnauld's and Clairaut's understanding of "Elements" is shared by the authors of the famous French *Encyclopedia* [Diderot/d'Alembert, 1751-80]. In volume 5 of this work there is an article on "Eléments des Sciences" in which the treatment of the elements of a science is primarily left to the *analytic method* "which leads from composed to abstract ideas and ascends from familiar conclusions to the unknown principles" Another article is directly devoted to the analytic method:

> Analysis is what is called ... "the method to be followed in order to discover truth"; it is also called the "method of resolution". By means of this method one gets from the composite to the simplest; whereas synthesis leads from the simplest to the complex. ... Analysis consists in returning to the origin of our ideas, developing their order, decomposing and composing them in a variety of ways, comparing them from all points of view and making apparent their mutual interrelations. . In searching for truth it does not make use of general theorems, rather it operates like a kind of "calculus" by decomposing and composing knowledge and comparing this in the most profitable way with the intended discoveries. ... It is the only method which can give evidence to our inferences, and therefore the only one to be followed in searching for truth and even in instructing others, an honor which is normally given to synthesis. [Diderot/d'Alembert, 1751-80: author's trl.]

A very recent criticism of the Euclidean type of founding and presenting mathematics has been put forward by *I. Lakatos* in connexion with his quasi-empiricist philosophy of mathematics [Lakatos, 1978]. In his well known book "Proofs and refutations: the logic of mathematical discovery", the author is confronting the *deductivist* with what he calls the *heuristic approach* [see also Lakatos, 1978a, 1978b]:

> Euclidean methodology has developed a certain obligatory style of presentation I shall refer to this as "deductivist style" This style starts with a painstakingly stated list of axioms, lemmas and/or definitions The axioms and definitions frequently look artificial and mystifyingly complicated. One is never told how these complications arose The list of axioms and definitions is followed by the carefully worded theorems. These are loaded with heavy-going conditions; it seems impossible that anyone should ever have guessed them The theorem is followed by the proof ... An authoritarian air is secured for the subject by beginning with disguised monster-barring and proof-generated definition and with the fully-fledged theorem, and by suppressing the primitive conjecture, the refutations, and the criticism of the proof. Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof-generated definitions of their "proof-ancestors", presents them out of the blue, in an artificial and authoritarian way. It hides the global counterexamples which led to their discovery. Heuristic style on the contrary highlights these factors. It emphasises the problem-situation: it emphasises the "logic" which gave birth to the new concept [Lakatos, 1976, p. 142f.]

4. Dialectic between analysis and synthesis and complementarist epistemology

Above, we have identified *three aspects* related to the concept of *elements of mathematics* as they were suggested by the Euclidean type of "Elements". What would be the difference when considering the genetic or heuristic type? Apparently the basic role for a building up of mathematics (aspect 1) and the fundamental relation to the teaching and learning (aspect 3) also apply here However, there is a *significant difference* concerning what Arnauld called the *true or natural order* and what has been explicated in different ways by himself, Clairaut, Lakatos, and others.

As regards the learner, Arnauld employed arguments similar to those Bruner used with respect to structure: "It is without doubt, that what is being learned in the natural order is learned easier and is better remembered, because ideas arranged in the natural order make a fruitful texture in our memory. .. One can even say that what one has learned to know by penetrating its true foundation is not really kept by memory but through the power of judgement, and thus turns into one's property so intensely that one cannot forget it" [Arnauld 1972, p. 326]

The real difference between the two approaches concerns aspect 2 insofar as it is not a *global logico-deductive approach* and an *ontology of static concepts*, related to a *given world of absolutely existing mathematical entities*, that the heuristic or genetic method is based upon, but rather a *dynamic, operative and constructive attitude* in connection with a *"logic" of actions*, in which *concepts* are not fixed but *undergo changes* in the course of development.

In this somehow dualistic relation between the two types of "Elements", criticisms have also been put forward in the other direction, i.e. *against the heuristic-genetic type*. In fact, the already mentioned Thom-Dieudonné-controversy partly belongs here. With respect to Clairaut's "Elements" some critical voices of the 18th century point out several deficiencies, such as: prolixity, too many repetitions, too much emphasis on special cases and, connected to this, difficulties in reaching the intended generalizations, lack of precision, elegance and clarity. In his "Dictionnaire philosophique" [1746] *Voltaire* argues: "This method looked acceptable and useful. However, one did not follow it: On the teacher's side it demands great flexibility of his mind in being able to adapt to the various particular circumstances, and a rare elegance of those who normally are bound to the routines of their profession". And *Lacroix* in his "Essais sur l'enseignement en général et sur celui des mathématiques en particulier" [1805] emphasizes the difficulty of representing the analytic method in a text written for pupils: "Clairaut's elements of geometry organized according to the method of discovery are most suitable for guiding the teacher in this situation; there is no need for a book for pupils, and it appears to me almost impossible to write one for the first year of study in whatever science."

On the other hand it is important to observe that the difference between the two approaches was not generally understood as an unsurmountable dualism. Rather it was viewed by several authors as a kind of *complementarity*. This is already inherent in Plato's *dialectic* [see Szabo, 1960] and is articulated especially in Proclus' comments (see above) on what Plato meant in the "Republic" when he declared dialectic to be the "capstone of the mathematical sciences". It is true that, for the teaching of mathematics, Proclus advocated an "elementary treatise" written in the style of Euclid's "Elements". However, a consequent application of dialectic to the use of this text would imply not taking it for knowledge as such, but rather enriching it and relating it to external aspects: constructions, applications, proof analysis, analytic argumentations as exemplified in some Socratic dialogues, etc. Some of this can be found in what Freundenthal, in his pedagogy of *guided rediscovery*, has called *local ordering, Socratic method* and the *process of axiomatization* [see Freudenthal, 1973].

An important contribution to the analysis of the dualism between the two methods in mathematics and to constructively handling it was made by *H. Grassmann* in connection with the development and presentation of his *Ausdehnungslehre* (calculus of extension), especially the first version A_1 of 1844. A.C. Lewis [1977] has shown that Grassmann's related philosophy, characterized as the "*complementarity of rigor and overview*" [loc. cit., p. 123] has been influenced by the "Dialektik" [1839] of the German philosopher *F. Schleiermacher* who had described the dualism in terms of the *heuristic* and the *architectonic* method: "The method in the first direction—to find new knowledge from given knowledge—we call the heuristic; that in the other—to connect up the dispersed and isolated given material—the architectonic." For Schleiermacher "it will be in the oscillation between the two that we must progress" [quoted from Lewis, 1977, p. 114].

With respect to the two methods, Grassmann had observed "that in mathematics these series of development are most sharply distinguished from each other .... and the interpretation of the two appears more difficult than in any other science" [A_1, trans. acc. to Lewis, 1977, p. 132]. In spite of this difficulty, Grassmann believed in the possibilities of fundamentally linking the two methods:

> At each point of the development, the manner of further development is determined essentially through a *leading idea*, which is either nothing other than a conjectured *analogy* with related or already known branches of knowledge, or which—and this is the best case—is a direct *presentiment* of the truth to be sought. ... The presentiment appears to be foreign to the area of pure science and most of all to mathematical science. Alone, without it, it is impossible for someone to discover a new truth. ... This presentiment is, therefore, quite indispensable in the scientific field. In particular, if it is of the right kind it telescopes into one the whole sequence of development which leads to the new truth, but with the moments of development not yet separated from each other and thus in the beginning it is only an obscure presentiment. The separation of those moments contains at the same time the discovery of the truth and the critique of that presentiment. [A_1, transl. acc. to Lewis, 1977, p. 132; italics by the author]

In his A_1, at first glance, Grassman seems to have actually deepened rather than bridged the contrast, as he lays strong emphasis on a strict distinction between the abstract theoretical parts of the A_1 and their applications. Indeed, with his "Ausdehnungslehre" as "a new branch of mathematics" Grassmann had created the basic ideas of what is now called *multilinear algebra*. Anticipating Hilbert's "axiomatic method", he had interpreted it as an example of "*pure mathematics being the theory of form*" which ought to be strictly distinguished from the various *applications* ("to the other branches of mathematics, as well as statistics, mechanics, the theory of magnetism and crystallography" as stated in the subtitle of A_1), applications he had elaborated in the process of developing the theory: "My theory of extension is forming the abstract foundation of the theory of space (geometry), i.e. it is the purely mathematical science, detached from all spatial intentions, whose special application to space is geometry" (A_1)

However, as is already indicated in this quotation, the clear distinction between a theory and its applications makes possible and leads to the establishment of a *new, more generalized relation* between them. In the development of the theory of extension, as Grassmann has explained it, applications actually play the role of what he called the "leading idea" or "presentiment"; and—what is most important—such presentiments may well transcend normal applications in well known domains and anticipate *future new applications*: "The theorems of the theory of extension are not mere translations of geometrical theorems into abstract language, but have a much more general meaning; whereas spatial geometry remains bound to the three dimensions, abstract science is free of these limitations." [Grassmann 1894-1911, Vol. I, p. 297]

Lewis has commented on this in the following way: "What assurance is there that mathematics, developed along the lines of the A_1, would be anything other than mere phantasy? The answer which is suggested here is that the basis of the A_1 lies for Grassmann in the dialectical method of the work as a whole and not just in the purely abstract aspects. In this way Grassmann's concrete intuitional examples, operating through analogy and presentiment in the creative and learning aspects of the theory, enter into the foundations of the subject" [loc. cit., p. 136].

And, summarizing: "This analysis of the AII has shown that while Grassmann makes an explicit separation of pure mathematics, as pure theory of thought-forms, from its applications, he recognizes that such a separation itself implies a relationship of dependency between the two things being separated This general dialectical principle is reflected in every facet of the A_1" [loc. cit., p. 160]

In his profound study on the relation between the foundation and justification of knowledge on one hand and the growth and development of knowledge on the other hand, *H.N Jahnke* [1978] has shown that *J.D. Sneed's* [1971] fundamental investigation of the relations between theoretical concepts in mathematical physics and their role in concrete physical applications can be transferred to mathematics. In analysing the *role of theoretical terms* (non-observables such as "mass" and "force" in classical mechanics) in the applications or models of theories of mathematical physics, Sneed comes to the conclusion that the "*statement view*" of such theories is to be abandoned and replaced by a *dualistic view* comprising two components or a pair $< K,I >$: the *structural kernel* or core K of the theory and a (related) set I of so-called *intended applications*. K represents the logico-mathematical part. For each application K is extended by special laws or hypotheses which at the same time put certain *constraints* on the range of application; and only these are subject to empirical testing K determines the possible extensions and also the set I (in some general set-theoretical sense). However, only successful applications count and therefore the full meaning of a theory is left open with respect to as yet unexperienced applications; its factual meaning changes and grows with further development Thus justification and evidence of a theory is inseparable from its *development* and linked to *future knowledge*: "The more general, the more comprehensive, the further developed theory gives foundation to the less general, the less comprehensive, the less developed theory. With Sneed this means that a theory is justified and founded by the ensemble of all applications as a new quality." [Jahnke 1978, p 108f.]

On the basis of this interpretation Jahnke has pointed out the high degree of affinity which exists between Sneed's theory and Grassmann's epistemological viewpoint:

> In fact, Grassmann's "lineale Ausdehnungslehre" with its permanent subdivision according to "theory" and "application" and its use of applications as didactical means for exhibiting the leading idea that guides the formal development, i e its use as means of justification and foundation, represents an elaboration of a concept of theory which basically proceeds along the line of Sneed's conceptualisation of theories in mathematical physics. [Jahnke, 1978, p 110]

In his further investigations on the transferability of Sneed's reconstruction of the logical structure of mathematical physics to mathematics, Jahnke has identified a variety of similarities, especially concerning the role of *impredicative definitions* in the foundations of mathematics and the introduction of *ideal elements* in mathematics, which show a remarkable analogy to the problem of theoretical terms in physics, the *Hilbertian program* on the foundations of mathematics with its radical separation of the problem of justification from that of development, together with Hilbert's new understanding of the *axiomatic method*, the meaning of *Gödel's incompleteness theorem*, the dual understanding of the *concept of variable* in mathematics with its syntactic and semantic aspects and their fundamental interrelations. According to Jahnke this can be viewed as a confirmation of the assumption that in principle the *problem of foundation in mathematics* is a *subproblem* of the foundational problems in the *empirical sciences* and also that in mathematics one has to take seriously the *complementarity of theory and application* [loc cit., pp. 172, 280].

On this basis, the *development of a theory* means a double process: at the level of the formal structure it is the variation of the structural kernel K by sequences of stepwise extensions of K; at the level of intended applications it is the elaboration of new factual applications (though each extension decreases the related set of the possible intended applications). This also holds for the *development of scientific concepts* whose growth and change consists both in their *elaboration as theoretical terms* within developing theories and in their *gaining meaning through being involved in ever more applications*. The two processes are fundamentally interrelated and do mutually condition and further each other. The interrelation can be described in different ways; one way is by means of *Piaget's two kinds of abstraction*, simple and reflective, and their *mediation by actions*. [see Otte/Bromme, 1978; Jahnke/Otte, 1979]

There are many consequences at the level of didactical questions of which we only mention two. From an analysis of the *didactical problems of proof* and the role of a dynamic and varying reference to objects as well as the specific emphasis on intuition usually associated with finding and establishing proofs, Jahnke suggests interpreting *intuition as a set of intended applications*, which in turn "makes it possible to imagine the acquisition of intuition as a result of a learning process." [loc cit. p. 280; see also Jahnke/Otte, 1979] Furthermore he explains that *Freudenthal's practice of "local ordering"* is regulated by a process of *concept development* or *determination of theoretical terms* (which is in accordance with Thom's opinion that meaning and rigor in mathematics are local properties) and by bringing in specific references to objects and applications. [loc. cit., p. 213]

To *summarize*: In the manner described, the fundamental dualisms between synthesis and analysis, justification and development, theory and application, representation and operation, which specifically became apparent in the controversies about the "Elements of mathematics", have proved a vehicle for epistemological and didactical clarifications consisting in a *dialectic synthesis* of the original contrasts, based on the elaboration of *complementarist views*. For the *didactics of mathematics* this implies a general task of elaborating a concept of justification and of the foundation of knowledge which is compatible with a perspective of development and is regulated by it. It is necessary "to build up in a systematic way methods which allow one to emphasize the viewpoints of the more general, the more developed, future systems." [Jahnke, 1978, p. 252]

This reminds us of the goals for the *modernization of mathematical education* established in the late 50s and the 60s in connection with a new understanding of mathematics based on the axiomatic method, as the most elaborated conceptualization of the concept of theory and Bourbaki's mathematical structures. Obviously, grosso modo, some important aspects were neglected in this reform. Like Euclid, Bourbaki was often read as if his texts comprised the knowledge as such. Frequently structural concepts were only understood in a descriptive sense. Their operative and explorative meaning and their relatedness to intended applications both within and outside mathematics were neglected. Theories and concepts were not developed in a dynamic process with changing and growing meaning but rather were parachuted in. On the other hand, a variety of contributions in the right spirit have been made, and the didactics of mathematics has received many valuable impulses. Some of it is still waiting to be re-evaluated in a process that may overcome the still felt shock which was caused by the rather suddenly ascertained "failure of the new math", and which was actually a result of a variety of mutually intertwined factors involved in that huge and, in many countries, more than just curriculum-related reform.

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