ENACTIVISM, FIGURAL APPREHENSION AND KNOWLEDGE OBJECTIFICATION: AN EXPLORATION OF FIGURAL PATTERN GENERALISATION

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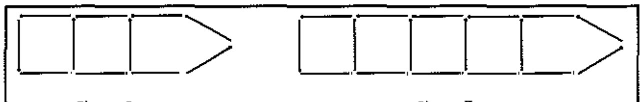
To what extent are pupils able to generalise pictorial patterning tasks in multiple ways? What is the relationship between the embodied processes of pattern generalisation and the visualisation of pictorial cues? This paper seeks to explore these questions through a novel combination of theoretical ideas: *enactivism, figural apprehension,* and the theoretical construct of *knowledge objectification*

Our initial interest in the way pupils visualise pictorial patterns was sparked by an earlier investigation into the extent to which question design influenced pupils' approaches to pattern generalisation tasks (Samson, 2007). We were fascinated by the remarkable diversity of expressions which pupils were able to generate for the general term of a pictorial pattern. While some of these expressions were arrived at through numerical considerations, we were particularly interested in those that were based on visually mediated strategies. In this earlier study, pupils were only required to come up with a single expression for the general term. We thus began to wonder to what extent individual pupils would be able to articulate multiple expressions for the general term using different modes of visualisation. A literature review revealed that very little empirical research has been reported in this area.

Multiple representations have long been recognised as being of fundamental importance in developing an appreciation for the interconnections between different aspects of mathematics. Within the specific context of figural pattern generalisation, a multiple representational view has the potential not only to lead to an exploration of the notion of *equivalence*, but also to encouraging pupils to critically engage with the underlying physical structure of the figural cue as seen from alternative points of view. By way of example, a typical pedagogical strategy is to suggest various equivalent algebraic expressions for the general term of a pictorial pattern. Pupils are then encouraged to arrive at plausible explanations for each expression by referring to the physical structure of the provided context. Figure 1 illustrates a typical example.



Shape 3 Shape 5

Look at the diagrams shown above. Shape 3 requires 11 matchsticks while shape 5 requires 17 matchsticks

(a) Write down an algebraic formula that will determine how many matchsticks are needed to build the n^{th} shape.

(b) Terry says the formula is $2(n+1)+n$ while Jeff says the formula is $1+3(n-1)+4$. By referring to the diagram, explain how Terry and Jeff arrived at their respective formulae.

Figure 1. Typical question encouraging a multiple representational view of pattern generalisation

While such a pedagogical approach may be useful for some pupils, for others it may well create additional complications. Not only is there an inherent ambiguity in the structure of the algebraic expressions - in Figure 1 for example, $2(n + 1)$ could represent either 2 multiples of $(n + 1)$ matchsticks or $(n + 1)$ multiples of 2 matchsticks - but an additional cognitive demand is placed on pupils who write algebraic expressions in non-standard format (e.g., n x 3 as opposed to $3n$).

Our interest in a multiple representational view of pattern generalisation lies at a more fundamental level and seeks to explore the inter-relationship between the embodied processes of pattern generalisation and the visualisation of pictorial cues. While our present interest lies in establishing a framework of analysis through a novel combination of theoretical ideas - *enactivism, figural apprehension,* and *knowledge objectification* - our ultimate objective is to provide practical strategies which support and encourage a multiple representational approach to pattern generalisation in the pedagogical context of the classroom

Enactivism

From an enactivist perspective, the act of perceiving something is not a process of recovering properties of an external object. Rather, "perception consists of perceptually guided action" (Varela, as quoted in Lozano, 2005, p. 26). Thus, we perceive things in a certain way because of the manner in which we relate to them through our actions (Lozano, 2005). This idea is succinctly stated in Maturana and Varela's (1998, p. 26) aphorism: "All doing is knowing, and all knowing is doing". Thus, knowledge depends on "being in a world that is inseparable from our bodies, our language, and our social history - in short, from our *embodiment*" (Varela, Thompson & Rosch, 1991, p. 149). For Varela *et al* (1991), who build on Merleau-Ponty's phenomenology, it is critical that we see our bodies as both physical (biological) structures as well as lived, phenomenological structures Thus, *embodiment* has an important double sense: "it encompasses both the body as a lived, experiential structure and the body as the context or milieu of cognitive mechanisms" (Varela *et al.*, 1991, p. xvi). While the phenomenological level is accessible to consciousness, there is a large *cognitive unconscious* below the surface of consciousness (Johnson, 1999, p. 82; Lakoff & Johnson, 1999, p 103)

Enactivist theory brings together action, knowledge and identity so that there is a conflation of *doing, knowing,* and *being* (Davis, Sumara & Kieren, 1996) Within an enactivist framework there is a purposeful blurring of the line between thought and behaviour (Davis, 1997, p. 370), and cognition is thus viewed as an embodied and co-emergent interactive process where the emphasis is on *knowing* as opposed to *knowledge*. Rather than the representation of a pre-given world by a pre-given mind, enactivism sees cognition as "the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs" (Varela *et al*, 1991, p. 9), thus foregrounding the mind-body unity that lies at the heart of enactivism.

Discussing mathematics cognition and education, Simmt (2000) draws attention to the distinction between behaviour as *caused* by features or constraints in the environment and the notion of understanding as being *occasioned* by one's interactions with the environment. Enactivist theory rejects the notion of optimal knowledge or ultimate Truth in favour of *effective* or *adequate* action - a "survival of the *fit*" logic (Maturana & Varela, 1998) In terms of individual cognition this position suggests that learning "[ ] is not a process of selecting 'correct actions' but of discarding those actions that do not work" (Davis *et al.*, 1996, p. 166). This notion of viable action as opposed to optimal action is thus one of proscription rather than prescription (Varela *et al.*, 1991) In the absence of ultimate criteria, the enactive view of cognition can be seen as one of transformational potential (Campbell & Dawson, 1995).

However, in order for this transformational potential to be realised, these potentialities for change must first be "recognised" in the environment through interaction (Proulx, 2004, p 116). If one is unable to "see" the triggers in the environment then one cannot be "affected" by them. In other words, "we can only experience what our embodiment allows us to experience" (Johnson, 1999, p. 81) Or as Proulx (2004) succinctly puts it, "You get triggered by what you CAN get triggered by" (p. 119)

As Davis (1995) comments, language and action are not merely outward manifestations of internal workings, but rather they are "visible aspects of [.] embodied (enacted) understandings" (p. 4). For Davis *et al.* (1996), enactivism prompts us not only to consider the formal mathematical ideas that emerge from action, but to give close scrutiny to those preceding actions - "the unformulated exploration, the undirected movement, the unstructured interaction, wherein the body is wholly engaged in mathematical play" (p. 156). As Núñez, Edwards & Matos (1999) argue, the nature of situated cognition cannot be fully understood by attending only to social and contextual factors. Learning and thinking are also situated "within biological and experiential contexts, contexts which have shaped, in a non-arbitrary way, our characteristic ways of making sense of the world" (Núñez *et al.*, 1999, p 46) Indeed, Núñez *et al.* go further to suggest that while our understanding of mathematics may well be socially and culturally situated, it is "the commonalities in our physical embodiment and experience that provide the bedrock for this situatedness" (p. 63).

Perception and knowledge objectification

From an enactivist stance perception needs to be considered as a fully *embodied* process - a complex activity related to the *manner* of our acquaintance with the objects of perception, in other words the *activity* that mediates our experience with objects (Radford, Bardini & Sabena, 2007). An interrogation of the *embodied* processes of perception thus needs to focus on the phenomenological realm of students' experience in order to emphasize the subjective dimension of knowing (Radford, 2006). Radford (2008, p 87) refers to the process of making the objects of knowledge apparent as *objectification*, a multi-systemic, semiotically mediated activity during which the perceptual act of noticing progressively unfolds. The objects, tools, linguistic devices and signs used by individuals in social meaning-making processes to achieve a stable form of awareness, he refers to as the *semiotic means of objectification* (Radford, 2003, p 41) Such semiotic means of objectification could include: words and linguistic devices; metaphor and metonymy; gestures; rhythm in speech and gesture; graphics and the use of artefacts.

Such a multi-semiotic view of knowledge objectification takes cognizance of the *principle of non-redundancy* (Benveniste, 1985, p. 235), the notion that different semiotic systems allow for different forms of expressivity (Radford, 2006, p. 7). Each semiotic system opens up a different space of possible action and thus has the potential to shape enactive processes of construction (Lozano, Sandoval & Trigueros, 2006, p 90; Radford, 2003, p 41). In addition, such a multi-semiotic view displays an enactivist sensitivity for the process of objectification in which the interplay of a variety of semiotic means/systems is seen to have a fundamental role in knowledge formation and in which cognitive activity is seen as being "embodied in the corporality of actions" (Radford *et al*, 2007, p. 508)

Visually mediated approaches to pattern generalisation tasks set within a pictorial context provide for an interesting interplay between two different modes of visual perception: sensory perception and cognitive perception (Rivera & Becker, 2008) These different modes resonate with Fischbein's (1993) theory of figural concepts, and the notion that all geometrical figures (or *figural objects*) possess, simultaneously, both conceptual and figural properties. Mariotti (as cited in Jones, 1998, pp 30-31) stresses the dialectic relationship between figure and concept as an important interaction in the field of geometry, a relationship that can create tension from a student's perspective We suggest that a similar tension is likely to underlie visual strategies applied to pattern generalisation tasks set within a pictorial context. In order to explore this visual tension, the notion of figural *apprehension* as espoused by Duval (1995, 1998, 1999) was identified as providing a useful framework.

Figural apprehension

Duval (1998, p. 41) makes the pertinent point that most diagrams contain a great variety of constituent gestalts and sub-configurations – far more than those initially identified at first glance or those made explicit through construction or accompanying discursive statements. Critically, it is this surplus that constitutes the *heuristic power* of a geometrical figure, since specific sub-configurations may well trigger alternative solution paths.

Although Duval's notion of figural *apprehension* was developed within a more classical geometry context, with only slight modifications it can readily be adapted to other contexts involving geometrical figures. The four different modes of apprehension are *perceptual, sequential, discursive* and *operative*. These are explained in reference to Figure 2 which shows an initial visual stimulus (the 2nd term of an arithmetic sequence) along with possible outcomes (*i.e.*, algebraic expressions for the nth term) for each of the four modes of figural apprehension:

- *Perceptual apprehension*: the initial apprehension of a figure - what we see in a perceived figure at first glance as determined by the unconscious integration of Gestalt laws of figural organisation (Duval, 1995). During perceptual apprehension it is possible to discriminate between component sub-figures of the perceived figure. By way of example, the visual stimulus shown in Figure 2 could be subdivided into squares and triangles. This could potentially lead to a complicated algebraic expression for the general term which would need to take into account overlapping matches: $T_n = 4n - (n - 1) + 3(2n) + 6 - (2n + 2)$.
- *Sequential apprehension*: the emergence of subfigures or elementary figural units which stem from either the construction of the perceived figure, or a description of its construction. Specific sub-figures or elementary units arise not from unconscious laws of figural organisation, but from the physical process of construction. Sequential apprehension could arise from noticing that the construction (either mental or physical) of each subsequent term requires the insertion of a 7-match additive unit. This has the potential to yield the general expression $T_n = 7n + 5$.
- *Discursive apprehension*: a process of perceptual recognition during which certain gestalt configurations gain prominence due to an association with discursive statements accompanying the geometric figure. Within a classical geometry context this relates to the limitation that it is not possible to determine the *mathematical* properties represented in a figure through perceptual apprehension alone. The provision of initial discursive criteria is also necessary. Discursive apprehension may be invoked by accompanying the visual stimulus shown in Figure 2 with the wording "for 2 squares you need a total of 19 matches", thus foregrounding the structural unit of the square. This could potentially yield the general expression $T_n = 4n - (n - 1) + 2(2n + 2)$ where, after counting squares and adjusting for overlapping units, the remaining matches are seen as V-shapes around the perimeter.

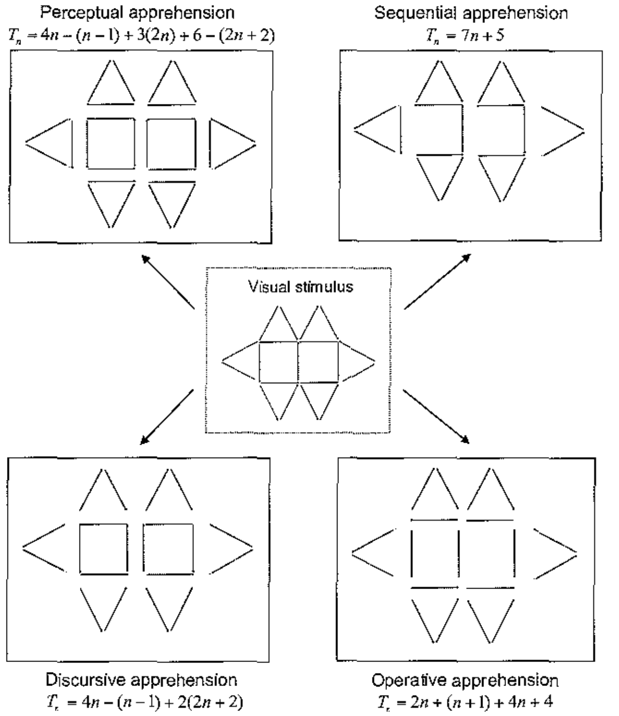


Figure 2 Possible outcomes for different modes of figural apprehension

- *Operative apprehension* relates to the various ways by which a given figure can be modified while retaining the properties of the figure, for example by a reconfiguration of the whole-part relation through a recombination of various elementary figural units. Operative apprehension may allow the visual stimulus to be seen in terms of horizontal lines, vertical lines, V-shapes above and below, and a constant 2 matches at either end. Expressed algebraically, this yields $T_n = 2n + (n + 1) + 4n + 4$.

Within the context of pattern generalisation, perceptual apprehension may on occasion be sufficient to generalise a given figural pattern. However, perceptual apprehension will not necessarily evoke gestalts which are appropriate to the generalisation process. An inability to move beyond mere perceptual apprehension of a figure can lead to what Duval (1999, p. 17) refers to as *heuristic deficiency*. In order to actualise the heuristic potential of a diagram it is necessary not only to be aware of the scope of the diagram but also to be able to use it flexibly (Rösken & Rolka, 2006). Being able to see a diagram in multiple ways thus necessitates a move beyond perceptual apprehension. This raises an important question: what are the embodied processes that either hinder or assist pupils in moving flexibly between different modes of apprehension?

Combining Three Theoretical Ideas

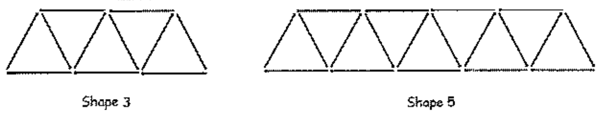
Within the context of figural pattern generalisation, the processes of visualisation and generalisation are deeply interwoven. Pattern generalisation rests on an ability to *grasp* a commonality from a few elements of a sequence, an awareness that this commonality is applicable to *all* the terms of the sequence, and finally being able to use it to articulate a direct *expression* for the general term. Inherent in this notion of generalisation are (a) a *phenomenological* element related to the grasping of the generality, and (b) a *semiotic* element related to the sign-mediated articulation of what is noticed in the phenomenological realm (Radford, 2006, p. 5)

The relationship between the embodied processes of pattern generalisation and the visualisation of accompanying pictorial cues is undoubtedly complex. However, we believe that an analysis based on the combination of enactivism, figural apprehension, and knowledge objectification has the potential to shed light on this inter-relationship, showing sensitivity to both the phenomenological and semiotic aspects of figural pattern generalisation

An important aspect of figural pattern generalisation lies in the notion that such pictorial cues possess both figural and conceptual properties, each of which resonates with a different mode of visual perception – sensory and cognitive, respectively. This position perhaps seems somewhat at odds with an enactivist view of perception as being a fully embodied process However, we are not suggesting that these two modes of perception are independent of one another, or that they are able to occur in isolation: a drawing of a triangle, for example, can only be perceived as such through a combination of both sensory and cognitive perception. Indeed, one could argue that sensory perception cannot occur without cognitive perception – a view that resonates strongly with the mind-body unity that is the core of enactivism Nonetheless, we believe that the distinction between *figural* and *conceptual* properties provides a useful framework to discuss figural pattern generalisation. Perhaps an appropriate analogy would be that of a Möbius strip, where figural and conceptual properties are simply different aspects of the same phenomenon depending on the stance of the observer

Although figural cues contain *simultaneously* both figural and conceptual properties, and while it is acknowledged that perception is *at once* both sensory and cognitive, what is important is the *nature* of the figural and conceptual properties of pictorial cues within the context of pattern generalisation. A stand-alone visual stimulus such as that given in Figure 2 can be considered *figural* to the extent that it is a geometrical figure. As such, it could be visualised in any number of ways – overlapping squares and triangles; horizontal matches, vertical matches and V-shaped units; overlapping squares and V-shaped units around the perimeter; triangles around the outside with a single vertical match in the middle. However, the moment the visual stimulus is seen to be a term in a sequence of pictorial terms with similar structures – either by the physical presentation of one or more additional terms of the sequence or by a verbal explanation of the scenario – then it immediately takes on conceptual properties. Thus, the visual stimulus given in Figure 2 can be considered to contain *conceptual* properties by virtue of it being a pictorial term in a sequence of terms containing one or more common characteristics

A visual cue such as that shown in Figure 1 thus contains simultaneously both figural and conceptual properties Each



Visually mediated expressions for the n[th] term suggested and justified by a single research participant:

$3n+n-1$	$3(n-1)+4+n-2$	$2n+n+n-1$
$4n-1$	$n+n+n+[n-1]$	$2(2[n-1])+3$
$4(n-1)+3$	$n+3(n-1)+2$	$3n+3(n-1)-[2n-2]$

Figure 3. Two non-consecutive terms of a figural pattern

"shape" is in its own right a geometrical figure, but in the given context, it is also part of a sequence of similar terms and thus contains associated conceptual qualities What is important here is that there will be interplay between the figural and conceptual properties, resulting in different *apprehensions* of the visual stimulus. While some of these apprehensions may evoke gestalts which are appropriate to the generalisation process, others may simply obfuscate or complicate the process.

What follows is a short vignette to illustrate these ideas In particular, the vignette demonstrates how the combination of enactivism, figural apprehension and knowledge objectification provides complementary multiple perspectives to the framework of analysis.

Vignette

Grant, a high-ability Grade 9 pupil, was presented with the two non-consecutive terms shown in Figure 3. In the space of one hour he managed to determine nine different visually-mediated expressions for the nth term of the pattern We were surprised by the diversity in the algebraic expressions shown in Figure 3. Although algebraically equivalent, each expression was arrived at through a different mode of visual apprehension, and Grant was able to fully justify each of his expressions through a verbal description of his visual apprehension of the pictorial pattern.

The vignette which follows describes the 3½ minutes Grant spent arriving at the expression $n + n + n + [n - 1]$ through a process of operative apprehension. This was the fifth general expression Grant was able to arrive at for the given visual stimulus and, to place the expression in context, his visual apprehensions for the preceding four expressions are shown in Figures 4 – 7, all of which are illustrated for T_4.

Grant's initial apprehension (*i e.* his *perceptual* apprehension) of the figural cue shown in Figure 3 was in terms of overlapping upward-pointing and downward-pointing tri-

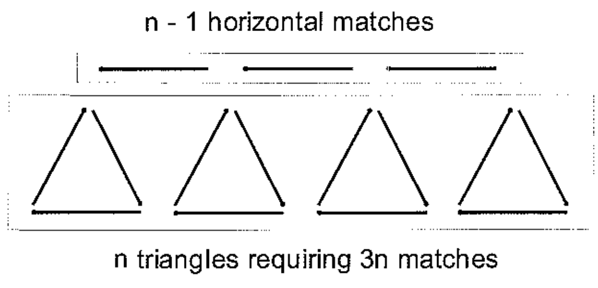


Figure 4 Grant's apprehension for the expression $T_n = 3n + n - 1$.

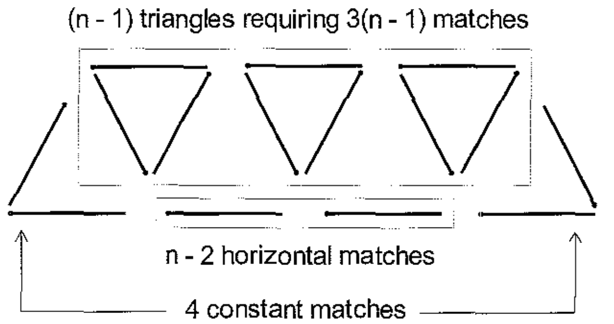


Figure 5. Grant's apprehension for the expression $T_n = 3(n-1) + 4 + n - 2$.

angles - Shape 3 containing a total of five such triangles and Shape 5 containing nine. To arrive at a general expression for the nth term from this apprehension would have required a relatively complicated correction for the overlapping matches. Rather than attempting this, Grant was able to subconsciously reconfigure the whole-part relation of the given pictorial terms into upward-pointing triangles and an upper horizontal row of $n - 1$ matches (Figure 4), arriving at a general expression $3n + n - 1$. His second general expression (Figure 5) stemmed from a further reconfiguration of the figural cue into $n - 1$ downward-pointing triangles, $n - 2$ horizontal matches along the bottom, and a constant 2 matches at either end. Figure 6 shows yet another reconfiguration into an upper row of $n - 1$ matches, a lower row of n matches, and $2n$ matches for the central zigzag. The reconfigurations of the whole-part relation of the pictorial terms as shown in Figures 4 - 6 represent *operative* apprehension in Duval's nomenclature. The fourth general expression as shown in Figure 7 was arrived at through a process of physically building the 4th term of the sequence using matchsticks and thus represents *sequential* apprehension.

In order to arrive at his 5th general expression, Grant began by counting the forward-leaning parallel matches of Shape 5 from left to right. After a brief pause he then worked his way back from right to left counting the backward-leaning parallel matches. He then counted the remaining top and bottom matches in pairs, rhythmically alternating between top and bottom as shown in Figure 8: 1, 2. 3, 4.. 5, 6. 7, 8. 9. Rhythm, whether in speech or gesture, is not merely the perception of order, it is "the demand, preparation and anticipation for something to come" (You, 1994, p. 363). There is thus an inherent sense of expectancy associated with rhythm, and it is seen as a crucial semiotic device in the process of generalisation (Radford *et al.*, 2007, p. 522). This counting procedure was thus central in alerting Grant, whether consciously or unconsciously, to the non-paired match in the bottom row. Based on this counting procedure, Grant was able to arrive at the following general expression for the nth term of the sequence: $n + n + 2n - 1$.

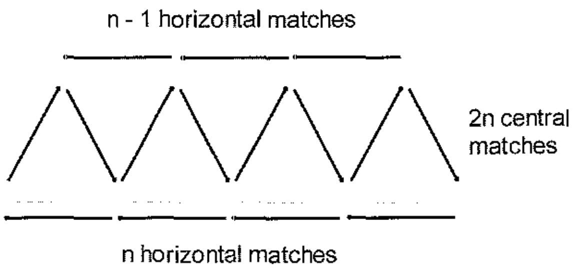


Figure 6. Grant's apprehension for the expression $T_n = 2n + n + n + n - 1$.

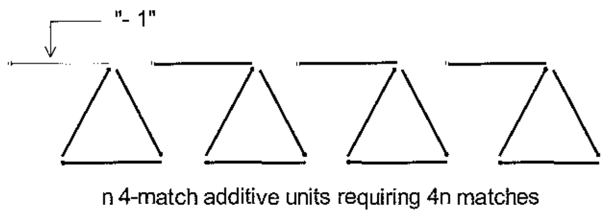


Figure 7. Grant's apprehension for the expression $T_n = 4n - 1$.

Grant was able to justify his general expression ($n + n + 2n - 1$) by relating the $n + n$ portion to two sets of "parallel central matches", while the $2n - 1$ he associated with what he referred to as the "outside matches". Just prior to writing the $2n - 1$ part of the expression, Grant made use of *indexical* gesturing - he first gestured a horizontal line across the top of Shape 5 and then a second horizontal line across the bottom of Shape 5. He also made the comment that "it'll always be one less on top", use of the word "always" performing a generative action function (Radford, 2000, p. 248) and thus aiding the notion of generality.

Interestingly, there seems to be a slight mis-match between the $2n - 1$ portion of Grant's expression and his indexical gesturing of the top and bottom rows of matches in Shape 5 - the "outside matches". As Grant wrote down the $2n - 1$ expression he commented that he was just simplifying $n + [n - 1]$. When asked to articulate how he was "seeing" it, he was insistent that he saw the structure as $n + [n - 1]$, *i.e.*, in terms of n matches along the bottom and $n - 1$ matches along the top, and that the $2n - 1$ portion of his expression was in fact an algebraic simplification of $n + [n - 1]$. Grant went further to describe the $[n - 1]$ as representing the "top gap-filling matches", a metaphorical visualisation of the spaces between the inverted V-shapes created by the two central series of parallel matches. Grant then re-wrote his

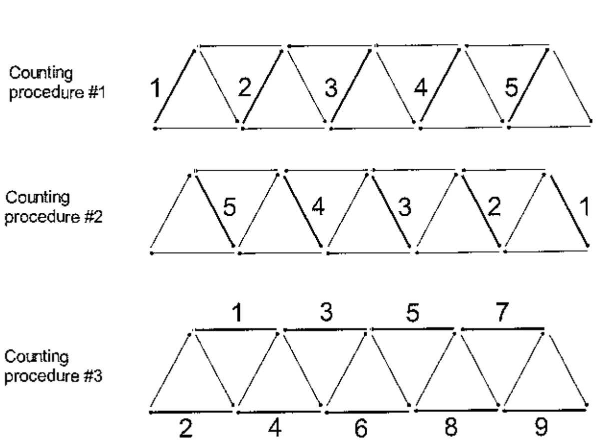


Figure 8. Grant's different counting procedures.

expression for the nth term as $n + n + n + [n - 1]$ which he explained as being a truer representation of his visual apprehension of the pictorial pattern

There seems to be an interesting tension between two different modes of operative apprehension with figural modification having been accomplished by means of a recombination of various elementary figural units in two different ways. Although Grant ultimately presented the expression $n + n + n + [n - 1]$ as being representative of his apprehension of the figural pattern under investigation, his initial formula was $n + n + 2n - 1$.

Both formulae suggest a sub-division of the "central" matches into n forward-leaning and n backward-leaning parallel matches. However, the remaining "outside" matches seem to be sub-divided differently in each of the two formulae. The initial formula $n + n + 2n - 1$ suggests that the "outside" matches have been split into pairs - one match of each pair forming part of the upper horizontal row with its paired match positioned below it in the bottom row. The $2n - 1$ "outside" matches in Grant's initial formula seem to represent n pairs of matches, making $2n$ matches in total, the "-1" being an adjustment required due to the right-most pair missing a match in the upper row. However, his final expression ($n + n + n + [n - 1]$) suggests that the "outside" matches were in fact sub-divided into two distinct horizontal rows, with n matches along the bottom and $n - 1$ matches along the top

Grant maintained that in his initial expression he had written $2n - 1$ as an algebraic simplification of $n + [n - 1]$ which in turn is likely to have been a remnant of an earlier operative apprehension, that shown in Figure 6. However, the $2n - 1$ is likely to have been inspired by his counting procedure shown in Figure 8, in which the rhythmical pairing of the top and bottom matches was central to alerting him to the non-paired match in the bottom row. Even though Grant's algebraic expression $2n - 1$ retains what Radford (2002) refers to as a *symbolic narrative*, this narrative by no means has a unique interpretation. Seen in isolation, the $2n$ portion of the expression could equally represent either 2 multiples of n matches (top and bottom horizontal rows) or n multiples of 2 matches (matches grouped in pairs) It was only through careful observation of the *activity* that mediated Grant's experience with the figural cues that the $2n$ portion of his general expression could be fully interpreted Thus, what is important here is that the visual tension inherent in Grant's fifth algebraic expression only became apparent through a combined analysis of multiple semiotic means of objectification (linguistic devices, metaphor, gestures, rhythm, symbolic expressions) using a lens of figural apprehension within the context of figural pattern generalisation being viewed as a fully embodied process.

Concluding comments

The purpose of this paper was to establish a framework of analysis for exploring the inter-relationship between the embodied processes of pattern generalisation and the visualisation of pictorial cues We believe that the combination of three key theoretical ideas - *enactivism, figural apprehension,* and *knowledge objectification* - provides a powerful framework that can be used to shed light on the embodied processes that either assist or hinder pupils' abilities to visualise figural cues in multiple ways within the context of pattern generalisation, and ultimately to provide practical strategies to support and encourage a multiple representational approach to pattern generalisation in the mathematics classroom

Literature relating to figural pattern generalisation generally falls into three broad categories of focus: (a) descriptions of solution strategies, (b) the affordances offered by technological environments, and (c) the transition between pupils' arithmetic and algebraic reasoning. These three categories tend to focus on the *product* of visualisation, while very little empirical research focusing on the *process* of visualisation seems to have been carried out. Our framework of analysis is significant in that it not only focuses specifically on the *process* of visualisation, but it does so from a fully embodied stance

Furthermore, we believe that our framework allows for an additional depth of analysis when compared with other frameworks presently employed to analyse the process of pattern generalisation This extra layer of insight arises from the complementary multiple perspectives that constitute the framework of analysis. Not only does this framework acknowledge perception as being critically related to the *manner* of one's interaction with perceptual objects, but it also remains sensitive to both the phenomenological and semiotic aspects of the generalisation process This combined lens allows the researcher access to the subtle yet powerful underlying tensions that exist as different modes of figural apprehension jostle for prominence.

The cognitive significance of the body has become one of the major topics in current psychology (Radford *et al.*, 2005, p 113). Furthermore, the use of multiple representations has been acknowledged as playing a central role in problem solving, the learning and understanding of mathematical ideas, and the development of a deeper appreciation for the interconnections between mathematical concepts (Goldin, 2002; Greeno & Hall, 1997; Kaput, 1998). As Adler (2005) comments, "at the most basic level, we have yet to understand how to make mathematics learnable by all children" (p 2). By focusing on issues of visualisation and pattern generalisation, central components of mathematical activity, this paper engages with the critical notion of mathematical accessibility

Our research has a number of immediate implications for teaching mathematics While many teachers support a multiple representational view of pattern generalisation in the classroom, there is a subtle ambiguity inherent in these expressions of generality. This ambiguity has the potential to open up interesting and powerful spaces for classroom discourse However, a necessary prerequisite to capitalising on this potential is for teachers not only to be aware that such semantic ambiguity exists, but to be vigilant in terms of identifying it in the classroom context. Teachers need to have the confidence to validate multiple visually mediated interpretations of pictorial patterns. Where students have arrived at identical general expressions through different modes of figural apprehension, teachers will also need to help the class reconcile the different interpretations in a meaningful manner. Thus, not only does the teacher need to have both the visualisation and algebraic capacity to verify students' general expressions, there is also a need for teachers to critically engage, at an embodied level, with students' explanations of their generalisation process.

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