The Autonomy of Mental Models

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Traditionally, concepts and images are considered the main components of our symbolic behavior As a matter of fact, in a productive reasoning process, these two aspects do not act separately. They are deeply and intricately involved with each other and collaborate in various ways in learning, understanding and problem solving. The main product of such an interaction is represented by what may be called *mental models*. The terms *model* and *mental model* are not clearly and univocally defined in the present literature.

In a paper published in 1983 Norman distinguishes: ". . the *target system*, the *conceptual model* of that *target system*, the user's *mental model* of the target system and the *scientist's conceptualization* of that mental mode" [Norman, 1983, p. 7] Mental models are naturally evolving systems, says Norman, "That is, through interaction with a target system people formulate mental models of that system " ... Mental models will be constrained by such things as the user's technical background, previous experiences with similar systems and the structure of human information processing" [Norman, 1983, pp. 7-81]

Johnson-Laird's use of the term mental model encompasses a very large variety of cognitive activities. "It is, now, plausible to suppose that mental models play a central and unifying role in representing objects, states of affairs, sequences of events, the way the world is and the social and psychological actions of daily life They enable individuals to make inferences and predictions, to understand phenomena, to decide what action to take and to control its execution " [Johnson-Laird, 1983, p. 397] Johnson-Laird assigns three immediate constraints on mental models: (a) Mental models and the machinery for constructing them are computable, (b) A mental model must be finite in size and cannot directly represent an infinite domain, and (c) A mental model is constructed from tokens arranged in a particular structure to represent a state of affairs [*ibid.* p. 398] Johnson-Laird mentions also the principle of structural identity: "The structures of mental models are identical to the structures of the state of affairs, whether perceived or conceived, that the models represent" [*ibid* p 419]

Carrol describes mental models in the following way: "Mental models are structures and processes inputed to a person's mind in order to account for that person's behavior and experience Ideally speaking, a mental model is a psychological theory that could give answers . . " [Carrol, 1984, *cf.* Streitz, 1988 (p 169-170)]

Summarizing the above quotations, one may retain the following aspects: A mental model is meant "to give the answers", that is, it has a heuristic function. Mental models are theories about certain realities. They allow us to make inferences and predictions. Mental models are structured entities and their structure must correspond to that of the reality they are supposed to represent. With regard to the finitist character of mental models, mentioned by Johnson-Laird, one has to distinguish between dynamic (or potential) infinity and actual infinity. As has been proven, even young adolescents are ready to accept an infinite process (referring for instance to the endless extension of a line) and even adults do not possess the appropriate mental models for dealing adequately with actual infinity. [See Fischbein, Tirosh and Hess, 1979]

Let us now try to define the meanings of the terms model and mental model as we understand them. "Given two systems A and B, B may be considered a model of A if, on the basis of a certain isomorphism between A and B, a description or a solution produced in terms of A, may be reflected, consistently, in terms of B and vice versa". [Fischbein, 1989, p. 9] This definition emphasizes a certain number of aspects First, that a model should be able to act as a *substitute* in a reasoning process. For instance, instead of operating, mentally, with pure abstractions, like point, line, plane, etc , which have no physical reality and are, then, not representable as such, we use pictorial models of them

Secondly, the relationship between the original and the model must be based on a certain structural correspondence, otherwise the heuristic task cannot be fulfilled. Superficial similarities may be misleading.

Thirdly, the distinction between original and model is not an absolute one: the two systems may change their roles A computer program and a solving strategy used by a person may be considered, each of them, to represent a model for the other. However, there is no absolute symmetry. Usually, one of the terms more adequately plays the role of model than the other Because a model has, usually, a heuristic function it should be more accessible cognitively than the original in the given circumstances It should be more adequate for a pictorial, behavioral or mechanical representation. If one intends to prove the consistency of an axiomatic system, one resorts to what is called a model of it For instance, in order to prove the consistency of a non-Euclidian geometry one gives a model of it in terms of Euclidian geometry. Theoretically, the two systems are interchangeable but practically, psychologically, they are not. Euclidian geometry is more accessible because its concepts and theorems are intuitively representable, acceptable, while those of a non-Euclidian geometry are not.

Various types of models may be described. Models may be mental constructs (mental models) or objective, material constructions. Mathematical models are mental models, while the material model of a building has an objective existence. The distinction is relative because a material model expresses, usually, a mental representation and a mental model is related to some kind of objective representation. The mental, pictorial representation of a line corresponds to a drawn image of a line.

Models may also be intuitive (figural) and abstract. The function $y = ax^2 + bx + c$ is an abstract model of a parabola, while the corresponding graph is a figural model of the same function

Mental models may be tacit or explicit. Our mental operations, our solving strategies may be influenced by interpretations of which we are not aware. We are, usually, not aware of the fact that our time concepts are based on spatial representations because time in itself (duration), can only be lived, not directly represented [Bergson, 1889]. We speak of the *length* of a certain time interval, we *divide* a time interval, or *extend* it, etc. But that terminology is inspired by spatial representations of time and every attempt to imagine time as measured, divided, extended, etc., is intimately associated with space representations which act, very often tacitly, as substitutes. On the other hand, Venn and tree diagrams, graphs representing functions, oriented segments representing pictorial magnitudes, etc are generally used explicitly, intentionally. These may be termed *explicit models*.

Our reasoning processes — learning, understanding, problem solving — are largely dependent on our mental modelling mechanisms. Many of the aspects we are dealing with are not directly accessible to our cognitive activity. Some are too big, some are too small, some are too far away in time or in space, some are too complex. Many scientific concepts are abstractions not directly representable. Spaces with more than three dimensions, imaginary numbers, infinite sets, various levels of infinity, etc., are intuitively meaningless. We do not have a direct representation even of what is going on in the minds of other individuals. All these inaccessible aspects may be integrated meaningfully in a reasoning process only by modelling them. So the abstract is represented pictorially, infinity is represented by finite means, the too-big or too-small represented by objects and features belonging to the activities of our everyday life

Language is essentially metaphorical. An individual is *strong*, physically, intellectually, affectively; a feeling may be strong, a string is strong, an argument may also be strong. Our reasoning processes resort tacitly or explicitly to an immense variety of analogies — terrestrial, anthropomorphical, mechanical. (For reviews concerning models and metaphors and their role in scientific and mathematical reasoning, see Duit [1988] and Steiner [1988].)

The autonomy of mental models

A model is not a mere simplified version of a certain reality. A main characteristic of a model is that *it is a reality on its own*. A model is not simply subordinated to the dictates of the original — to the constraints it is meant to obey and to the needs which it is supposed to serve. A model is never a simple property or a conglomerate of properties related among themselves by the particularities of the original.

What characterizes a model is, first of all, its *structurality*, its autarchy — so to speak — its autonomy with regard to the original. A concept is not by itself a model of a certain class of objects, because a concept is a general idea reflecting that class of objects. Being structurally unitary and autonomous, the model very often imposes its constraints on the original and not vice versa! Consequently a model is not simply a substitute, an auxiliary device (more simple, more familiar, more accessible). It is *another* reality, jealous of its independence, and often insufficiently permeable to genuine intimations of the original

Thinking through models does not necessarily imply that we think about the corresponding originals. Thinking through models may simply mean, very often, thinking *about* the models *themselves*.

The autonomy of models is a condition of their heuristic efficiency. A model which could not function according to its own constraints and would be obliged to obey, at each step, the dictates and suggestions of the original, is of no use. On the other hand a model must be faithful to the original, otherwise it is not able to yield solutions which are meaningfully translatable in terms of the original. These two, somehow contradictory requirements — autonomy and truthfulness to the original — make the use of mental models so problematic and very often so full of dangers for the accuracy of the reasoning process.

Williams, Hollan and Stevens consider that: "Central to the conception of mental models is the notion of autonomous objects. An autonomous object is a mental object with an explicit representation of state, an explicit representation of its topological connections to other objects and a set of internal parameters. Associated with each autonomous object is a set of rules which modify its parameters and thus specify its behavior". [Williams, Hollan and Stevens, 1983 p. 133] The authors go on to describe how such autonomous mental objects act. The basic idea, as in our conception, is that a mental model is not a mere reflection of "a state of affairs" but a structure governed by its own parameters and rules. By identifying these characteristics of a mental model one may explain the subject's inferences and even the uncertainties he expresses. [Williams, Hollan and Stevens, p. 133] The autonomy of a model is then a condition of its heuristic productivity. At the same time, as an effect of its autonomy, the model may cease to communicate with the original, may become opaque, imposing on the reasoning process its own features

The object of the present research

The above ideas represent descriptions of phenomena based on current observations but, as far as we know, no investigations have been undertaken to analyze them systematically (except the attempt of Williams *et al*, which does not check specifically the concept of autonomy)

The central issue of the present research refers to the autonomy of mental models. It has been assumed that models act, in fact, as mental objects on their own and not as mere reflections of an original. A model *with all its properties* may then *replace* the original in a reasoning process instead of remaining an auxiliary of it. We consider that this assumption may be proven in the following way:

Let us assume two structurally different entities that have some superficial properties in common. A model adequate for solving a problem related to one of them cannot, in principle, be useful for the other. Nevertheless, if the model chosen by a subject really possesses the property of autonomy, thus obstructing the vision of the original, the subject may use, indiscriminately, the same model in both cases

The method

THE SUBJECTS

Two hundred students, of middle class background, from the Sharon area in Israel participated in the study. The following grade levels participated: 7, 8, 10 and 12 — fifty students at each grade level. The seventh and eighth graders belonged to two heterogeneous classes. The tenth and twelfth graders studied mathematics as their major subject. The seventh grade students had some experience with geometry concepts gained from their previous mathematical instruction, and some elementary knowledge concerning the structure of matter. The tenth grade students had studied Euclidian geometry, including a systematic presentation of basic concepts like point, straight line, plane, etc., but they had not received any additional instruction concerning the structure of matter. The twelfth grade students had taken an introductory course in calculus and received further information concerning the nature of the atom from their chemistry courses

THE PROBLEMS

Initially two, apparently similar, but structurally different questions were asked:

1. *The division of a line segment*
"Let us consider a line segment AB. Let us divide this segment into two equal parts. Let us divide each of the two halves in the same way. Let us suppose that we repeat the operation of division again and again. Will the process of division come to an end? Explain your answer."

2. *The division of a copper wire*
The same question was asked with respect to a length of copper wire

The two problems seem similar but, in fact, there is a structural difference. The segment is an ideal, mathematical object. The process of division is an ideal, conceptual operation. In the realm of geometry, a line segment is infinitely divisible. This is a conceptual operation and no material obstacles have to be considered. But it is possible that the subject does not consider the ideality of the line segment and of its division. It is possible that the subject has in mind some material model (the inked image of a line, or a fine thread, etc.) In this case the reaction will be different, disclosing the real nature of the model used. It will be assumed that the process of division will come to an end for various practical reasons. (The segments obtained become too small to be divided, the division instrument is not fine enough, etc.)

With respect to the wire, the situation is different. The adequate model is that of its atomic structure. The process of division must, physically, stop somewhere. If the subject has in mind the material nature of the wire, he must assume that the division process will have to stop. Beyond the atom the copper ceases to exist. Or — in more primitive terms — at a certain moment the obtained fragments become too small to be divided. Anyhow, the physical nature of the wire, contrary to the geometrical segment, must raise some physical problems concerning the division process. If, on the contrary, the geometrical model is dominant, the subject has no reason to hesitate in choosing the infinitist answer. We also assume that conflictual attitudes may arise, leading to some sort of compromise. For instance, that points are divisible!

According to our basic hypothesis, most of the subjects would adopt *one* model — the geometrical or the physical one — because of the superficial similarity of the situations, and would use it in order to decide their answers *in both* situations, despite the fundamental differences. If our hypothesis is correct, this would support the more general idea that a model preserves a sort of autonomy with regard to the situation to which it is applied. Instead of being inspired, moulded, by the situation it has to serve, the model imposes its own structural constraints *indiscriminately* on the original.

3. After the above questions had been answered, the sheets were taken away and a third question administered:

> *During a lesson, one of the students asked the following question: "I understand that when dividing a copper wire time and time again, the process will end when reaching the atomic level. In contrast, the successive division of a line segment is an endless process. Why is this so?" Do you think that the student understood the situations correctly? What explanation would you offer him?*

The main aim of this last question was to check the robustness of the model adopted by the students when a different interpretation is suggested

The questions were administered collectively to the students during their regular classroom activity. Half of the students got the mathematical question first and half got the question related to the wire first.

Results

The division of a line segment

Table 1 presents the percentages of the two main categories of answers to the segment problem, i.e. infinitist and finitist answers.

The percentages of infinitist answers increase with age, from 26% in grade 7 to 78% in grade 12. In contrast, the finitist

	GRADE			
	7 (N=50)	8 (N=50)	10 (N=50)	12 (N=50)
The process is endless: Total	26	50	78	78
One can always divide by two	24	40	48	58
There is an infinite number of points in a line segment	—	2	22	20
The segment is infinite	—	2	2	—
We shall reach a point, but a point can be divided	2	6	4	—
There is an infinite number of atoms in a line segment	—	—	2	—
The process will come to an end: Total	74	50	22	22
We will not be able to divide any more because the segment will become extremely small	44	18	6	4
There is a finite number of points in a line segment	—	4	—	4
The segment is finite	8	18	2	2
We shall reach a point	4	4	12	10
We shall reach an "atom"	18	6	2	2

Table 1
The division of a line segment before intervention (in %)

answers decrease with age, from 74% in grade 7 to 27% in grade 11. A certain equilibrium is reached in grade 8 (50%-50%).

The main type of infinitist answer is represented by the claim: "One can always divide by two" Another frequent type of infinitist interpretation says: "There is an infinite number of points in a line segment". (This occurs mainly in the higher grades, 10 and 12.) There are also subjects who adopt a compromise attitude: "We will reach a point, but a point can be divided too" That is, the segment is composed of points (the abstract, geometrical interpretation) but the points can also be divided (the point is a material entity, a kind of atom).

The main type of finitist answers (frequent in the younger subjects) is represented by the claim: "We will not be able to divide any more because the segment will become extremely small" (44% in grade 7 and only 4% in grade 12.) Another type of answer is dominated by the idea of the finiteness of the segment: The division process is limited because "the segment is finite". Some subjects claim that the process is limited because "we will reach a point" or "we will reach an atom". Evidently these two last categories transfer to the line segment the model of the atomic structure of matter.

The division of the copper wire

Table 2 presents the various types of answers to the wire problem. Here too, we distinguish two main classes: the finitist and the infinitist.

	GRADE 7 (N=50)	8 (N=50)	10 (N=50)	12 (N=50)
The process will come to an end: Total	76	74	50	50
We will not be able to divide any more because the wire will become extremely small	36	28	22	26
There is a finite number of atoms in the wire	2	—	2	—
The wire is finite	8	6	—	—
We shall reach a point	—	2	—	—
We shall reach an "atom"	30	38	26	24
The process is endless: Total	24	26	50	50
One can always divide by two	20	22	42	40
The wire is infinite	—	2	—	—
We shall reach an atom, but it can be divided too	4	2	8	8
There is an infinite number of atoms in a wire	—	—	—	2

Table 2

The division of a copper wire before intervention (in %)

Contrary to what might have been expected, in the case of the copper wire as well as in the case of the geometrical segment, there is an increment with age in the infinitist answers and a decrement with age in the finitist answers. While in grade 7 there were 24% of infinitist answers, in grades 10 and 12 50% of the answers were of this type. With regard to the finitist answers, there were 76% in grade 7 and 50% in grades 10 and 12

How do the subjects justify their answers concerning the wire? There are two main types of finitist answers: "We will not be able to divide any more because the wire will become extremely small" and: "We will reach an atom" The main justification to the infinitist answers — as in the case of the segment — is: "One can always divide by two". This last type of answer reaches a frequency of 42% at grade 10 and 40% at grade 12 There were also some answers of the type: "We will reach an atom but it can also be divided".

Let us now look at Table 3 in which we consider, simultaneously, the reactions to the segment question and those to the wire question

Segment	Wire	GRADE 7	8	10	12
Infinite	Finite	8	34	30	36
Finite	Infinite	6	8	2	8
Infinite	Infinite	18	16	48	42
Finite	Finite	68	42	20	14
Total of concordant reactions		86	58	68	56
Total of discordant reactions		14	42	32	44

Table 3

Comparative data. The answers to the segment and the wire questions are considered simultaneously (in percentages). N=50 at each grade level

At each grade level, there were more subjects adopting *the same* model for solving *the two different* problems than subjects who adapted their reactions to the specificity of the problems This situation is striking in the 7th grade: 86% gave concordant answers. These were either infinitist (18%) or finitist (68%) The number of concordant reactions decreased with age, but not regularly.

It is also evident that while in the younger students the finitist answers were dominant (for both the segment and the wire), in the older students the infinitist model is the common one. Among the discordant reactions, the most frequent, as expected, were those which adapted to the nature of the problem, but these still represented no more than a third of the answers even of the older subjects (34%, 30% and 36% respectively in 8th, 10th and 12th grades).

Let us summarize the findings described so far: First, as predicted, most of the subjects adopt the *same* model (finitist or infinitist) for the two essentially different problems This tendency is very high in younger subjects and decreases in older ones. Secondly, while the finitist (concrete) model is dominant in the younger subjects, the infinitist (geometric, abstract) model is dominant in the older ones. *That is to say, the nature of the chosen model depends less on the particular structure of the original (which it is assumed to serve) than on the mental characteristics of the individual (as determined, in this case by age and/or learning).*

In principle, a mental model should represent a bridge between a certain given reality and the individual's intellectual schemata. In order to be a heuristic device, a mental model —

although remaining autonomous — should be consistent, on the one side, with the original and, on the other side, with the individual's own mental constraints. As a matter of fact, at least in certain circumstances, the model seems to reflect more the individual's own mental features (related to his personal experience, knowledge, etc.) than the object which it is supposed to interpret!

In order to check the stability, the robustness of the models, a third question was asked, suggesting, this time, two different interpretations: one, geometrical, for the line segment and a different, physical, for the wire. The problem we addressed ourselves was: Will the student change his mind and, in a second series of answers resort to different models for the essentially different questions?

Table 4 presents the results obtained for the line segment problem after the intervention. As one can see, there is a certain modification of percentages (if compared with Table 1) but not a dramatic one.

	GRADE 7 (N=50)	8 (N=50)	10 (N=50)	12 (N=50)
The process is endless: Total	28	54	80	86
One can always divide by two	24	30	38	56
There is an infinite number of points in a line segment	2	4	22	20
The segment is infinite	2	4	2	—
We shall reach a point, but a point can be divided too	—	16	18	10
There is an infinite number of atoms in a line segment	—	—	—	—
The process will come to an end: Total	62	46	18	14
We will not be able to divide any more, because the segment will become extremely small	14	30	8	6
There is a finite number of points in a line segment	2	4	4	—
The segment is finite	4	4	—	2
We shall reach a point	14	6	2	—
We shall reach an "atom"	28	2	4	—
I don't know	10	—	2	—

Table 4
The division of a line segment after intervention (Question 3) (in %)

Before the intervention, we got, in grade 7, 26% infinitist answers and 74% finitist (incorrect) answers, while after the intervention there were 28% infinitist answers and 62% finitist answers respectively (with 10% of the subjects answering that they don't know). In the older subjects (grade 12) the proportion of answers is also changed, but not spectacularly. Before intervention there were 78% infinitist (adequate) answers and 28% finitist (inadequate) answers, while after intervention there were respectively 86% infinitist and 14% finitist answers. Question 3 produced a shift of attitudes, in the sense of better-adapted reactions, but there were still many subjects, especially the younger ones, who stuck to their initial interpretation.

Turning to the "wire" question (Table 5) one finds — comparing the reactions before and after the intervention — that, this time, the effect of the intervention is more important.

	GRADE 7 (N=50)	8 (N=50)	10 (N=50)	12 (N=50)
The process will come to an end: Total	74	88	78	66
We will not be able to divide any more, because the wire will become extremely small	26	28	22	30
There is a finite number of atoms in the wire	2	2	10	12
The wire is finite	6	4	—	—
We shall reach a point	—	—	—	—
We shall reach an "atom"	40	54	46	24
The process is endless: Total	16	12	20	34
One can always divide by two	10	6	8	20
The wire is infinite	2	2	—	—
We shall reach an atom, but it can be divided too	4	4	12	14
There is an infinite number of atoms in a wire	—	—	2	—

Table 5
The division of a copper wire after intervention (in %)

Comparing Table 5 with Table 2 one sees the following: While before the intervention 24% of subjects in grade 7 used the infinitist (inadequate) model, after the intervention only 16% were influenced by this model. In grade 12 50% of the subjects used the infinitist model before intervention and only 34% used it after the intervention. With regard to the finitist answers (the adequate ones), one gets the following data: In the younger subjects (grade 7) the situation remains almost unchanged. Starting from grade 8 one may see a more dramatic change in attitude. While before the intervention, the percentages of adequate answers were 74%, 50% and 50% respectively for grades 8, 10 and 12, after the intervention the proportions of adequate (finitist) answers were 88%, 78% and 66% respectively. In other terms, the subjects were more ready to abandon an abstract (inadequate) model when dealing with a physical reality (and to adopt a concretely oriented one) than they were ready to abandon an initial concrete model when dealing with an abstract reality.

Table 6 presents, this time comparatively, the reactions to the segment and the wire problem as an effect of question 3. At all grade levels, a shift takes place towards more differentiated answers, but *in grade 7 the majority of the subjects (74%) still resort to concordant reactions*, witnessing the influence of the same underlying mental model. Most of the subjects still present "concretist" interpretations.

Starting from grade 8, the situation is changed. About half of the students present distinct interpretations adapted to the specificity of the problem — but there are still 46% in grade 8, 38% in grade 10 and 44% in grade 12 who continue to give concordant answers. Considering that these subjects were exposed

		GRADE			
Segment	Wire	7	8	10	12
Infinite	Finite	12	48	60	54
Finite	Infinite	—	6	—	2
Infinite	Infinite	16	6	20	32
Finite	Finite	62	40	18	12
I don't know		10	—	2	—
Total of concordant reactions		78	46	38	44
Total of discordant reactions		12	54	60	56

Table 6
Comparative data (after intervention).
The answers to the segment and the wire questions are considered simultaneously (in percentages)
N=50 at each grade level.

to the impact of a corrective suggestion (although indirectly) which should have induced a change in attitude, one may conclude that the model initially adopted was very robust.

EXAMPLES OF INDIVIDUAL REACTIONS

In order to get a more complete picture of the students' representations let us compare some of their answers to the segment and the wire question

Concordant reactions

Gali (grade 8): The segment: "It will become extremely small and, at the end, we will not be able to divide any more. Therefore the process will come to an end." The wire: "The process will come to an end because we will not be able to continue to divide. It is the same as in the case of the segment."

Sara (grade 11): The segment: "The process will come to an end when one reaches all the points and it will be impossible to continue to divide." The wire: "The process will come to an end, and this is exactly as with the segment. The same principle; it will happen when we reach the smallest part — the atom."

Oded (grade 11): The segment: "The process is endless because the geometrical segment is constituted of points." The wire: "The process is endless, and this is exactly as with the segment."

Guy (grade 12): The segment: "The process is infinite because one can always divide into two parts because the sequence of numbers is infinite." The wire: "Technically the process is limited, but conceptually something always remains and therefore the process is infinite."

Discordant reactions

Einav (grade 7): The segment: "The process is endless since there is always space — that we can only see with a magnifier — which we can divide." The wire: "The process will come to an end when there is no more copper."

Hillel (grade 11): The segment: "The process is endless because there is an infinite number of points in a segment." The wire: "The process will come to an end when we reach an atom."

A small number of subjects gave disconcordant answers but apparently reversed the expected interpretations.

Michael (grade 7): The segment: "The process will come to an end when there is nothing more to cut." The wire: "The process of cutting the wire is endless because we will not be able to cut the last piece."

Generally speaking there is not a great variety of justifications for either finitist or infinitist answers

For finitist answers: (a) Dividing, one reaches a moment when the segment (the wire) becomes extremely small; (b) One reaches a point (an atom).

For infinitist answers: (a) One can always divide by two; (b) We will reach a point (an atom) but it can also be divided

Discussion

The main objective of the present research was to prove the hypothesis that mental models tend to preserve their autonomy with regard to the originals they are meant to represent. This implies that mental models may remain impermeable to specific, even important, properties of the originals, and instead of serving them, may generate inaccurate or distorted representations. In order to prove this hypothesis two distinct entities were chosen with structurally different properties but presenting, superficially, similar aspects: a line segment which is only a conceptual, ideal abstract entity, and a copper wire which is a material reality presenting a number of specific material properties. Considering only the spatial, figural properties, the two objects are similar but they represent, in fact, two totally different realities. The problem put to the subjects referred to the operation of division. If the mental models inspiring the answers are distinct for the two objects, and subordinated to their particular structure, the reactions should be different. An ideal segment is endlessly divisible while the division of a piece of matter has to consider its atomic structure or, in more primitive terms, the limits imposed physically by the cutting process.

We found that the majority of subjects rely, in their reactions to *both* questions, on *one* common model: either the *finitist-concrete* one, or the *infinitist-abstract* one, thus confirming our hypothesis. Secondly, it was found that while young subjects adopt, most frequently, the concrete-finitist model, the older subjects adopt, mostly, the infinitist-abstract model.

It follows that the specificity of the model adopted depends, at least in certain circumstances, on the internal constraints of the reasoning process (in this case, as determined by age and/or learning) rather than on the constraints of the original. This conclusion has large didactical implications. We refer to both categories of mental models: those built intentionally, systematically, by an instructional process and those generated automatically, spontaneously (and, very often, tacitly), For instance, it is difficult even for adults to imagine the discontinuity, the quantum structure, of matter and energy, the idea that even energy is not a continuously divisible form of reality (i.e. that the radiation and absorption of energy by the atom always takes place in integral multiples of a certain quantum of energy). Elementary particles are often considered to be endlessly divisible in the same manner in which a conceptual line segment is interpreted as being continuously divisible. On the other hand, young subjects seem to face a certain difficulty in interpreting the division of a line segment as a pure conceptual operation to which no obstacle exists.

Many examples may be mentioned concerning the autonomous, relatively impermeable, nature of mental models which makes the learning process so difficult. Even after becoming familiar with decimals, with irrational and imaginary numbers, the student continues to stick, tacitly, to the natural number model which remains impermeable to the particularities of other categories of numbers. For instance, students tend to believe that the decimal which has a bigger number of digits at the right of the decimal point is bigger [Nesher and Peled, 1986] or that complex numbers may always be ordered like real numbers [Tirosh and Almog, 1989].

It is now well established that even after the student gets used to the multiplication and division of decimals, his basic tacit models remain repeated addition for multiplication and the partitive and quotitive models for division [Fischbein et al. 1985; Graeber, Tirosh and Glover, 1989]. Consequently many students, even college students, keep on believing, in accordance with the above models, that multiplication always makes bigger and division always makes smaller.

Even if one knows very well the differences between a point in the visual sense and a geometrical point, or between a drawn line and a geometrical line, the pictorial model continues, usually, to impose its constraints on both concepts. The autonomy of mental models with respect to the originals they are meant to represent is then a source of conflict which may distort or even block the reasoning process.

Can one make mental models more adaptable, more flexible, more open to the influences of the corresponding realities when there is no agreement between them? This is a great psychological and didactical problem. The autonomy and stability of mental models seem to suggest that they are not mere products, mere reflections, of the originals. They belong to the mental structure of the individual, well integrated into this structure, reflecting its requirements, its particularities, its schemata, its laws. The internal coherence of mental models — which is a basic condition of their autonomy — is guaranteed by this strong integration in the entire structure of the individual's mind. A model which was subject to many adverse tensions would not be viable. Moreover a model lacking autonomy, as we have already mentioned, would be unable to fulfill its heuristic task. It would not be able to *lead by itself* to consistent solutions of problems, raised in terms of the original, and would then be useless. We are facing here a profound dilemma: A good model must be *both* autonomous and faithful to the original. But while mental models tend usually to keep their autonomy, in accordance with the structure of the mind, they are not flexible enough to adapt themselves to all the structural constraints of the original.

How, then, would it be possible to make mental models more flexible, more penetrable, without destroying their autonomy? As far as we know there is no experimentally-based answer to this problem. We can only make some general suggestions. The first is that the student should become aware of the mental models which inspire his interpretations and solutions. This, in itself, is a complex problem implying preliminary specific cognitive investigations and the use of meta-cognitive analyses performed by the students themselves under the teacher's guidance. Secondly, we believe that a better integration of the formal, algorithmic and intuitive aspects of knowledge in the students' minds would increase the flexibility of the mental models used without destroying their autonomy. We assume that such an integration would increase the students' conscious control of their mental models and would then make these models more flexible, more adapted to the constraints of the original.

For instance, a student who is aware of the formal definitions and theorems underlying operations on fractions and decimals and, at the same time, of his corresponding intuitive beliefs, is better equipped to monitor the respective models (paradigmatic, like natural numbers, or pictorial, like diagrams) and to adapt them, or at least control them, when they are used in particular circumstances. A student who understands the rationale behind a tree diagram for solving combinatorial problems is expected to be able to adapt the model adequately when shifting from one type of combinatorial problem to another. A student who is able to visualize correctly the intuitive correspondent of an abstract mathematical relationship is better equipped for understanding it and adapting it to particular circumstances. Mundy, quoted by Eisenberg and Dreyfus [1989], asked 973 calculus graduate students to evaluate:

$$\int_{-3}^{3} | x + 2 | \, dx$$

Only 5.4% of the students answered correctly. Analyzing the various erroneous answers, Mundy concluded that the students did not have a visual understanding of the fact that integrals of positive-valued functions can be thought of in terms of the area under a curve [*cf.* Eisenberg and Dreyfus, 1989, p. 4-5]. The students' formal solving models would have been more adaptable to this particular problem if their formal knowledge had been better integrated with coherent formal-visual representations.

Our general assumption, concerning the adaptability of mental models (despite their natural autonomy), is that this requirement can be accomplished through a better integration of the formal, algorithmic and intuitive components of the reasoning process.

Finally, returning to our experimental design, one has to remember that the flexibility of a mental model has its limits. If a model is structurally different from the reality which it is meant to serve, the model should be replaced by an adequate one. A metal wire is structurally different from an abstract line segment as defined within Euclidian geometry, therefore no common model is possible and the student should be aware of this difference. A child who has in mind only collections of discrete objects for the concept of number cannot adapt this model to the more formal concepts of rational and real numbers. The number line and the measurement operation together possess much richer modelling resources. As a matter of fact, we still have — as we have already said — very little information about the psychological processes related to the act of modelling, of adapting or changing a mental model in a cognitive activity, in accordance with the genuine properties of the original. Much more research is needed if we intend to make a real contribution to the instructional process.

References

Bergson, H. [1889] *Essai sur les données immédiates de la conscience*. Paris. Alcan

Duit, R. [1988] On the role of analogies, similies and metaphors in learning science. Unpublished manuscript

Eisenberg, T., and Dreyfus, T. [in press] On the reluctance to visualize in mathematics. In W. Zimmermann and S. Cunningham (Eds.) *Visualization*

in mathematics. Mathematical Association of America Notes Series
Fischbein, E [1989] Tacit models and mathematical reasoning *For the Learning of Mathematics* 9, 2: 9-14
Fischbein, E , Deri, M , Nello, M S and Marino, M S [1985] The role of implicit models in solving verbal problems in multiplication and division. *Journal of Research in Mathematics Education* 16, 1: 3-17
Fischbein, E , Tirosh, D and Hess, P [1979] The intuition of infinity. *Educational Studies in Mathematics* 12: 491-512
Graeber, A O , Tirosh, D , and Glover, R [1989] Preservice teachers' misconceptions in solving verbal problems in multiplication and division *Journal for Research in Mathematics Education* 20, 1: 95-102
Johnson-Laird, P N. [1983] *Mental models* Cambridge. Massachusetts. Harvard University Press
Nesher, P and Peled. J. [1986] Shifts in reasoning *Educational Studies in Mathematics* 17, 1: 67-80
Norman, D.A [1983] Some observations on mental models. In D Grentner and A L. Stevens (Eds) *Mental models* Hillsdale, N Y . Lawrence Erlbaum Associates
Steiner, H G [1988] Über metaphern, modelle und mathematik, In P Bender (Ed) *Mathematik Didaktik: Theorie und Praxis* I D.M , Berlin: 190-201
Streitz, N A [1989] Mental models and metaphors: implications for the design of adaptive user-system interfaces In H Mandl, A. Lesgold (Eds) *Learning issues for intelligent tutoring systems* London. Springer: 164-186
Tirosh, D and Almog, N [1989] Conceptual adjustments in progressing from real to complex numbers In G Vergnaud (Ed) *Proceedings of the 14th International Conference of the International Group for the Psychology of Mathematics Education* (Vol 3, pp. 221-227) Paris: Laboratoire PSYDEE
Williams, M D , Hollan J D , and Stevens, A L . [1983] Human reasoning about a simple physical system. In D Gentner and A L Stevens (Eds) *Mental models* Hillsdale N Y . Lawrence Erlbaum Associates

Gödel's Proof is a double bind and so is the square root of minus one We are consequently led to remark that Piaget — like all those outside the poetry of mathematics who use "mathematics" as a tool to avoid having to think about what they are actually doing — does not understand that most if not all of the paradoxes of mathematics are the result of making the discourse of mathematics into a closed system by EXCLUDING THE MATHEMATICIAN In the same way academic linguistics and information science have turned themselves into engineering games by excluding the real senders and receivers of the messages they study No properly valid metamathematical theory can be constructed which does not include the DESIRE of the mathematician ( ) In a proper metamathematics the paradox of $\sqrt{(-1)}$ ceases to be a paradox and becomes what it has always been: a "word" in the discourse of the mathematical subject. That *i* remains as a sign of higher logical type than perhaps all of the others in a given equation is of small importance In the first place it works (mathematics is after all the greatest of the arts of compromise) and in the second without exactly the same process of vacillation between logical types in the discourse of our daily lives all HUMAN discourse would simply cease

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