outrage and low risk-high outrage situations may help create the credibility to get the appropriate messages across. Research should explore both aspects and may contribute insights to inform curriculum planners and teachers.

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Divisors and Quotients: Acknowledging Polysemy

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Durkin and Shire (1991) discuss several different types of lexical ambiguity, and suggest that by attending to lexical ambiguity:

> we can identify the basis for particular misinterpretation by pupils, and hence develop teaching strategies that circumvent or exploit such tendencies. (p. 73)

With respect to the mathematics classroom, they mention *polysemy* as one of the principal concerns, namely certain words having different but related meanings. Words such as 'combinations', 'similar', 'diagonal' or 'product' are examples of polysemous terms. The lexical ambiguity in these words is between their basic, everyday meanings and their meanings in the 'mathematics register' of the language, that is, their specialized use in a mathematical context. When a word means different things in different contexts, the intended meaning is usually specified by the context, including when different meanings of the word are presumed in everyday context and the mathematics register.

When the intended meaning of the word in the mathematics register is not available to a student, a sense is often 'borrowed' from an everyday situation. The word 'diagonal', for example, used by a teacher in a geometry class, would probably refer to a segment connecting two vertices of a polygon. An insightful dialog about diagonals is presented by Pimm (1987, pp. 84-85), where a 13-year-old child interprets 'diagonal' as a "sloping side of a figure relative to the natural orientation of the page".

Polysemy can also occur *within* the mathematics register itself, and context usually provides the primary identifier here as well. For example, the word 'operation' means one of the four - addition, subtraction, multiplication or division - in an elementary school classroom, while the same word means a function of two variables in a group theory course. We talk about the number 'zero' in elementary or middle school, and 'zeros of a polynomial' in an analysis course. 'Congruence' has different meanings in geometry and number theory. The meaning of a 'graph' depends on whether one is thinking about graphing functions in grade 10 algebra or graph theory.

'Divisor' and 'quotient'

I would like to focus here on two instances of polysemous terms: 'divisor' and 'quotient'. In my view, they deserve special attention because the lexical ambiguity presented by these terms is not *between* their everyday and mathematical usages, but arises *within* the mathematical context, within the mathematics register itself. In addition, as you will see, the context does not help in assigning meanings in this case. Both meanings for these words appear within the same 'sub-register' - the elementary mathematics classroom and a mathematics course for elementary school teachers.

'Divisor' has two meanings in the context of elementary arithmetic:

(i) a *divisor* is the number we 'divide by';

(ii) from a perspective of introductory number theory, for any two whole numbers a and b, where b is non-zero, b is a *divisor* (or *factor*) of a if and only if there exists a whole number c such that $bc = a$.

The latter is a formal definition of a divisor in terms of multiplication. Formulated in terms of division, b is a divisor of a in this sense if and only if the division of a by b results in a whole number, with no remainder.

'Quotient' means:

(i) the result of division;

(ii) in the context of the division algorithm, the integral part of this result.

Examples of discord 1: the classroom

What is the quotient in the division of 12 by 5? There was no consensus about its value among my students in a 'Foundations of mathematics' course for pre-service elementary teachers, and suggestions included 2 (with a remainder of 2), 2 2/5, 2.4 and 12/5. Trying to determine the solution in a democratic way, we took a vote: 19 students voted for 2, 37 students voted for 2.4, 2 2/5 or 12/5 (a combined count after agreeing that these were essentially different representations of the same number) and 9 students abstained. I purposefully ignored the lonely, uncertain voice from the audience who claimed: "It matters what you mean by a quotient, doesn't it?"

However, democracy is not the best description of the regime in my classroom, so the majority vote was not accepted. The students were asked to reconsider their decision and to bring to their next class meeting justifications for their decisions and to be prepared to defend them.

Rita was strongly convinced that the quotient in the divison of 12 by 5 was 2.4. She justified her decision by bringing her peers' attention to the lines from the course textbook (Gerber, 1982):

> We say that 6 divided by 3 is equal to 2. The number 3 is called the divisor, the number 6 the dividend and the number 2 the quotient (p 78)

Based on this example, she concluded that the quotient was what you got when you did division, and found further validation for her conclusion in the Random House Webster's College dictionary (1992), which defines a quotient as a "result of division" (p 1109).

Wanda used the same textbook as the first reliable information source:

> Suppose that we have an array of a elements having b columns. Then the quotient of a divided by b, written $a \div b$, is the number of rows in the array. (p. 79)

Arranging the elements in the following 5 column array,

```
* * * * *
* * * * *
* *
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Wanda was convinced that 2 was the "only possible quotient". The following definition on the subsequent pages of the textbook reassured her:

> Let a and b be two whole numbers, b non-zero. Then b divides a with remainder r (where $0 \leq r < b$), if and only if there is a solution of the sentence $a = (b \times \square) + r$. The number that satisfies this sentence is called the quotient. (p. 84)

Renate and William extended their search for the meaning of 'quotient' in the library. Mathematics dictionaries, however, did not resolve their disagreement. According to the James and James (1968) mathematics dictionary:

> Quotient is a quality resulting from the division of one quality by another. Division may have been actually performed or merely indicated; e.g. 2 is the quotient of 6 divided by 3, as also 6/3. (p. 297)

Following this suggestion. Renate claimed that 12/5 would be the quotient in division of 12 by 5, which is "obviously equal to 2.4 and definitely not equal to 2"

William brought to the attention of his group that the Concise Oxford Dictionary of Mathematics (Clapham, 1996) does not define a quotient; rather, it refers the user to "see Division Algorithm", where William found the following:

> For integers a and b, $b>0$, there exist unique integers q and r such that $a=bq+r$, $0 \leq r<b$. In the division of a by b, the number q is the quotient and r is the remainder. (p. 78)

This was put forward as William's justification for his claim that the quotient was 2 and his disagreement with Renate.

Several other reference sources were brought to class. However, the further discussion reinforced the disagreement, rather than resolving it. The students looked at me in anticipation of resolution, which they referred to as 'the right answer'.

Examples of discord 2: interviews

These interviews were conducted as a part of a research project that investigated students' learning and understanding of concepts related to introductory number theory. (I = Interviewer, S = Sonya)

I: Let's take another number, 6x147+1. If we divide this number by 6, what would be the remainder and what would be the quotient?
S: The remainder is 1.
I: Can you explain?
S: This number, you take something and add 1, so you had 6x147, but then you add 1, so it is no longer a multiple of 6, because you've added 1 here, you have the remainder of 1
I: And what about the quotient?
S: Do you want me to figure this out?
I: Yes, please
S: Am I allowed to use the calculator?
I: You may, if you wish.
S: It's 147 1666667

In this excerpt, the interview's purpose was to examine whether the student could determine the quotient and the remainder from the form in which the number was presented. As it turned out, Sonya's meaning assigned to the term 'quotient' was different from the interviewer's

I: Consider the following equation:
A÷B = C(D)
which means, if we write numbers instead of letters, that if we divide A by B, we get a quotient C and remainder D. OK? What happens to the quotient if we increase A, say increase A by 1?
C: The result, I mean the quotient, you like me using these words, ah, will be bigger, if A is bigger, the result is bigger.
I: Is this always the case?
C: I think so. It's kind of reasonable, if you have more amount to divide, and you're dividing by the same number, so your divisor's not changed, you end up with more.

In this excerpt, Cindy was invited to explore the relationship between B and D. Depending on the difference between these components, the increment in A may increase the quotient C or have no effect. However, the student interpreted the quotient differently, ignoring the interviewer's attempt to establish a 'shared meaning' at the beginning of the conversation.

I: Here you've listed all the factors of 117, which are 1, 3, 9, 13, 39 and 117, right?
K: Uhum.

I: Do you think there is a divisor of 117, which is not a factor of 117?
K: Uhum.
I: Can you please give an example.
K: It can be, anything you want, like 2, 3, 4, 5, anything you want can be divided into this number, but you will get a remainder, or a decimal, whatever.

The interviewer's purpose in this excerpt was to establish the equivalence between the terms 'factor' and 'divisor' of a number. It is clear from Karen's response that the meaning she assigned to the word 'divisor' was different from the meaning intended by the interviewer. In the following excerpt, there was an explicit attempt to clarify the meaning.

I: When I say 'divisor', what does this mean to you?
K: It is hard to explain, it's like this number.
I: Can you give an example?
K: Yes, you say that 3 is a divisor, because 12 divided by 3 equals 4
I: So, in this case would you say that 3 is a divisor of 12?
K: Uhum
I: Can you give another example?
K: 4, 4 is a divisor of 12.
I: OK And if I have 12 divided by 5 equals 2 4, would you say in this case that 5 is a divisor of 12?
K: I guess so.
I: Can you explain?
K: But here your result isn't integer. You can always divide by your divisor, but sometimes you'll get fractions or remainders.

These interviews took place after a chapter on number theory had been studied in class, which included factors, divisors and multiples, prime and composite numbers, greatest common divisor and least common multiple, the division algorithm and prime decomposition, among others. The definition of a quotient and a divisor that was introduced in class and used by the instructor and the interviewer was consistent with both meanings (ii) given above. However, the meaning assigned to these concepts by Cindy, Karen, Sonya and many of their classmates was consistent with both meanings (i) For these students, the terminology acquired at the elementary school appeared to be more robust that put to use in their mathematics course.

The above exerpts show an obvious discrepancy between the meaning intended by the interviewer and the meaning understood by the student. The remainder of this piece focuses on the sources of this discrepancy

Polysemy of 'division'

'Quotient' in the mathematics classroom often appears in concert with 'sum', difference' and 'product', in naming the result of each of the four basic arithmetic operations. Why is the meaning of 'quotient' ambiguous when the other result-of-operation words appear to be unproblematic? The meaning attached to 'quotient' depends on the meaning attached to the term 'division'.

There is divison of rational numbers, in which the result is the quotient. There is division of whole numbers, resulting in quotient and remainder. Therefore, a simple request to identify the quotient in the division of 12 by 5 can be confusing. Assuming whole number division, the quotient is 2, and the remainder is 2. Assuming rational number division, the quotient is 2 4 (For an exhaustive discussion on rational vs. whole number division, see Campbell, 1998.)

The set of whole numbers is closed under addition and multiplication: performing addition or multiplication on a pair of whole numbers results in a whole number. In this sense, division is different: the set of whole numbers is not closed under division – the outcome of this property is the introduction of rational numbers. However, the desire to restrict division to whole numbers introduces 'division with remainder'. When the remainder is zero, the whole number quotient and the rational number quotient give the same number. Moreover, in this case, the number we divide by is a divisor (sense ii) of the dividend. Most of the students' introductory experiences with division involve whole number results (as do many of the textbook examples). When this happens, the conflicting interpretations of 'quotient' and 'divisor' are not apparent

Variations in grammatical usage

Pimm (1987) has pointed out that using common English words as specialized terms in the mathematics register may lead to shifts in their grammatical category. For example, the adjective 'diagonal (line)' has undergone a "syntactic category shift" to the noun 'a diagonal'.

Although there is no category shift, the intended meaning of the word 'divisor' can be recognized by its grammatical usage.

> The number 3 is *the divisor* in the number sentence $12 \div 3 = 4$.
>
> Also, 3 is *a divisor* of 12
>
> The number 5 is *the divisor* in the number sentence $12 \div 5 = 2\ 4$
>
> However, 5 is not *a divisor* of 12

One interpreation of 'divisor' attends to the number's role in a binary operation. Another interpretation of 'divisor' attends to a relation among numbers: 3 is a divisor of 12 even if no one ever intends to perform the divison. In fact, this is an (incomplete) relation of order: it is asymmetric, reflexive and transitive, an intrinsic property of numbers, independent of their representation.

In the case of 'divisor', one grammatical construction hints at a specific interpretation. When one talks about *a divisor of* (another number), the indefinite article *a* together with the preposition *of* hint at a 'relation' interpretation of divisor (meaning ii). When mentioning *the divisor* or *the divisor in* (a division number sentence), the definite article 'the', at times together with the preposition 'in', hint at a 'role-player' interpretation (meaning i).

The polysemy of 'quotient' is more problematic since the grammatical form offers no hint. If one talks about the 'quotient' and the 'remainder' in the situation of division with remainder, the meaning is specified by the context. If only the quotient is mentioned in the discussion, the meaning is ambiguous. However, as the above examples show, students often do not attend to the explicit specifications suggested by the context or by the grammatical form

Extending the meaning

In a mathematical context, we are used to the idea of 'extending the meaning' for certain words. Multiplication, for example, is originally introduced as 'repeated addition', where the word 'times' (2 times 7) has identical meaning to the words 'multiplied by' (2 multiplied by 7). Later, this meaning of multiplication is extended beyond the whole numbers, to rational, real or complex numbers, to matrices and functions, where we still talk about multiplication but do not mean 'times' any more. This extended meaning can be seen as a metaphorical usage of the word 'multiplication'

Other examples of broadening the meaning or metaphorical usage are in extending exponentiation to negative or fractional exponents or extending 'sine' from the ratio in a right-angle triangle to a function. These metaphors appear naturally as we build our mathematical knowledge and students are often not aware of them

There is an embedded 'luxury' in these extensions: the operations resulting from the extended meaning are consistent with the original meaning. For example, we all think of the 'sum' as the result of the addition operation, without explicitly specifying whether the addends are integers, real numbers or matrices. However, the original definition of the sum (of *a* and *b*) emerged as the number of elements in the union set of two disjoint sets (where *a* and *b* refer to the number of elements in the two disjoint sets respectively). Extending the meaning of 'sum' creates no conflict: 17 is the sum of 12 and 5 regardless of whether the addends are viewed as whole, integer or real

The case of divison is different: extending the meaning of division from whole to rational numbers does not provide the 'luxury' of consistency. I suggest that one solution to this difficulty is through becoming aware ourselves and raising the awareness of our students to this potential discrepancy

Conclusion

This communication focused on the lexical ambiguity of two words – 'divisor' and 'quotient' – that frequently appear in elementary mathematics classrooms. Excerpts were presented from class discussions and interviews with individual students that outlined the confusion about the meanings of these terms. I suggested that the regular 'tools' to determine meaning, such as context or grammatical form, are not always sufficient.

Durkin and Shire (1992, p. 82) list several ideas for practical implications in the situations of polysemy. They suggest monitoring lexical ambiguity, enriching contextual clues, exploiting ambiguity to advantage and confronting ambiguity. Even though their suggestions refer to the ambiguity across registers, I find them completely applicable in case of divisors and quotients, where the ambiguity is within the mathematics register and even within the elementary classroom sub-register

Most textbooks for pre-service elementary school teachers provide the whole-number-division definition for quotients and remainders and ignore a potential conflict with students' prior knowledge. Such mathematical precision is not in the best interest from a pedagogical perspective. An appropriate didactical activity could awaken the conflict and then resolve it.

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Univalence: A Critical or Non-Critical Characteristic of Functions?

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There is a long history of attempts to use the function concept as an organising theme for the secondary mathematics curriculum. The appeal of the concept as a central idea is fairly clear. It can serve to unify seemingly different and unrelated topics; it can give meaning to algebraic procedures and notation which, in many other cases, are restricted to acquiring manipulative facility without leading to understanding; it easily lends itself to several different representations (an aspect much appreciated nowadays by the mathematics education community); and today, with the availability of advanced technological tools, its visual aspects are within everyone's reach.

With the current intense focus, both in curriculum research and the development of the function concept, it seems to us that there is one aspect that deserves more careful attention than it receives: the *univalence* requirement, that corresponding to each element in the domain there be only one element (image) in the range. We propose a re-examination of the present place of univalence in school mathematics and its implications, as there may be didactical advantages to postponing its introduction. The following is intended as a contribution to this re-examination