# WHAT DOES 'INSIDE OUT' MEAN IN PROBLEM SOLVING?

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What are the mental processes set in motion when a mathematician faces (and solves) a problem? What cognitive functions come into play during these processes? And what are the interactions connecting these functions?

In the 2005 film 'Inside Out', the core emotions living in the head of Riley, an eleven-year-old girl, are personified by five main characters: Joy (who guarantees happiness to the girl), Disgust (who protects Riley from physical and social poisoning), Fear (who defends Riley against danger), Anger (who prevents Riley from suffering injustice) and, finally, Sadness (an emotion that we often suppress but is fundamental to the achievement of psychic balance). The director's aim was to take the movie's audience "to a place that everyone knows, but no one has ever seen: the world inside the human mind" [2]. This article transposes this aim to mathematics education. We describe our efforts to guide students along a journey inside the mind of a mathematician engaged in a process of problem solving.

## The rationale

Problem solving can be described as a journey through an opaque environment, like a fog, in which the mathematician searches for viable ideas to exit the fog. As this journey nears a successful completion, a dual need—articulated through validation and the imperative to present the outcome of the problem-solving process—arises and facilitates the actual completion of the journey (Liljedahl, 2007). In rethinking what mathematicians do in such a situation, we have identified a number of phases, presented here in a non-ordered, and certainly amendable, list:

a) Problem solvers look for paths, trying to find a way;

b) They organize themselves; they choose strategies;

c) If they become frustrated or stuck, they try to reassure themselves and regain their calm;

d) They look for an insight or ask for help from an expert or some other source (books, articles, *etc*.);

e) They question themselves: does the solution path work or not?

f) They look for evidence for their conjectures;

g) When they think they have found something convincing, they try to present their solution in a 'fair copy'.

This process is not necessarily linear; it can cycle through phases multiple times, omit phases, or take them out of order. We can identify in these phases interactions among different 'voices', *i.e.,* forms of thought and discourse that represent the point of view of a character (Bartolini Bussi, 1996, p. 16, citing Wertsch & Baktin) in this individual process. Each voice represents a cognitive function needed by a successful problem solver. This process is naturally collaborative among the various voices existing and operating inside the person. While communicating with ourselves, 'various parts of us' (that is, various cognitive functions), coming into play in problem solving, are actually communicating with each other. Therefore, we modeled it as a collaboration between characters, each taking on the role of a cognitive function, as in 'Inside Out'.

## The design

Parallel to the *'Inside Out'* storyline, the cognitive functions in the mathematical argumentative process become characters of a story-problem. Through personification, these functions acquire gravity and importance.

Our idea is in tune with the persona-based framework (Liljedahl, 2007), where pulling different personas apart improves students' journaling and facilitates their access to the very real problem solving process they are engaged in. Moreover, the influence it exerts upon such a process takes place at a metacognitive level, crucially improving the students' problem solving competency. In our case, the personas are inside the process (they are the story's characters), defined according to cognitive functions occurring in problem solving (the roles represented by the story's characters) and their actions are steered by how these functions come into play in problem solving. We believe this approach can encourage students to reflect on the problem solving process and to be conscious of all its internal facets.

### Characters and roles

We selected four roles corresponding to the key functions needed to manage, and be successful in, collaborative problem solving activity, and assigned characters to each role.

*Boss* performs the function of organizing the whole solving process, corresponding to actions *b* and *c* in the list above. Boss keeps the thread of what is going on, chooses strategies and moves towards the goal. Boss's contribution is also affective: to be reassuring and restore calm in case the process is derailed by frustration or other negative emotions. Boss's role also has a regulatory element, being the one who takes care of the group and makes sure that everyone participates in the process.

*Promoter* is the explorer, needed to initiate the problem

solving process, looking for paths, trying to find a way (*a* in the above list) and, in case of an impasse, looking for an insight or for help from an expert or another source (*d*). A certain amount of creativity is present in all the roles, but the specific problem solving function of looking for the idea that unlocks the situation ultimately belongs to Promoter. By providing insights, Promoter instigates a reconfiguration of the perceptual field, through which the data at stake are seen in a different light and a new creative image is produced.

*Pest* is the critical mind, who introduces doubts, makes judgements, and questions what has already been produced by others. Pest also questions and looks for evidence for conjectures (*e* and *f*). Ultimately, Pest makes objections or responds to the objections of others.

*Blogger* is in charge of producing a publishable text, the outcome of the conjecturing and proving, which can be informal, as in a presentation to colleagues, or more formal, according to current mathematical standards (Boero, 1999). This function corresponds to arranging in 'fair copy' the solution path, once a convincing solution has been found (*g*). Blogger's role is important not only at the communication level but also at the metacognitive level. Writing promotes awareness and also supports control and reviewing processes.

In our model these four roles are taken on by student actors. Two roles (Boss and Blogger) are linked to the effectiveness of the collaboration and the two others (Promoter and Pest) guarantee the progress and the validity of the problem solving process. Boss, Pest and Blogger are focused on individual steps; Promoter pays attention to the wider process, in order to assess it holistically and facilitate its advancement.

There are some other issues to be taken into account. There may be valuable insights made in the discussion that are not recognised as valuable and so not followed up. The validity of the mathematical argumentation cannot be left only to Pest, who may not be aware of the criteria for valid arguments in mathematics. Similarly, there are criteria for the expression of arguments that may not be known to Blogger. There is, therefore, the need for a mediator who applies criteria for argumentation and expression and who points out valuable insights. We became aware of the need for this role in a pilot study, and expanded our initial model to include another character, *Guru,* who performs this mediating function. Guru is the expert, intervening during the interactions, encouraging students to better clarify what they have said and to improve communication. Guru asks, "What did you mean?", "Please explain, clarify, complete the sentence, … did you mean this or that?". Guru also supports creativity by mirroring the ideas and insights coming from Promoter or from the others. Moreover, Guru can intervene in moments of impasse, when none of the other characters is able make further progress. In addition to being a mediator, Guru also takes on the role of 'wisdom', turned to when encountering difficulties.

As the problem solving process itself is non-linear, so are the actions to be performed by the different characters outlined above. Nonetheless, an outline of the process can be described. The onset of the process usually involves the need to investigate ideas related to solving the problem in question. Here, Promoter should start offering insights, which can be extemporaneous or supported by inspiration based on the background culture. That is, Promoter may ask Guru for some hints. As the investigation plan is outlined, Boss takes charge of the organization of the work. Should a stalemate occur, new insights may come up (Promoter may offer new ideas) or expert advice could be sought (Promoter may turn to Guru). Here and earlier, Pest questions whether the proposed ideas will work. Boss takes charge of verifying and validating and Blogger proceeds to systematize the work in a publishable text.

**The story**

There is more to a movie than its characters, and there is more to problem solving than the cognitive processes involved. In our case, the story is outlined in a script that reflects the important distinction between the conjecturing process and its product, a statement to be proved, and also between the proving process and its product, a mathematical proof (Boero, 1999, 2007). Taking into account the argumentative process, the script moves through a sequence of episodes, each of them corresponding to a specific activity. In each episode the students follow the script, playing the roles corresponding to the cognitive functions described above. The episodes are:

1. *Exploration:* producing a brief description of what has been observed;
2. *Conjecture:* refining the description in order to obtain a statement;
3. *Formalization:* manipulating the initial statement, often produced in verbal language, to produce a properly formalized one;
4. *Proof:* organizing the arguments in an appropriate deductive chain, justifying each step of the deduction;
5. *Reflection:* looking back at the story from the mathematical point of view (as an evaluation on the cognitive level) and thinking about the roles played (as a self-assessment, on the metacognitive and affective level).

Unlike a movie, however, the script leaves openings for the students to 'ad lib' the actions of the characters. Only the starting points of each episode are scripted.

**Actors and Onlookers**

Our challenge is to produce an engagement model that could promote the appropriation of the attitude of a mathematician struggling with a problem solving situation. The educational goal of such a model is that social practice fosters appropriation of the script's roles, moving over time towards internalization. This involves not only in reflecting on the actions appropriate to each role, but also to reflecting on those actions.

The model is designed for students working in groups of four, participating in the story in two different engagement roles. For each episode one group acts out the script as

'Actors': each student in the group takes the role of one of the characters, that is they try to say things the characters would say, building on the story in their own way. Other groups of students are 'Onlookers' of the work of the group of Actors: each student takes charge of observing a specific character in the story and reflects, by means of a guided Logbook, on how the character being observed acts with respect to both the mathematical problem and the role of the character.

With each new episode, students change roles, with some Onlookers becoming Actors, and others observing different characters. This means they experience all the functions, which is essential for the emergence and the development of the students' resources (Pesci, 2009). As every student experiences all the roles, this experience can lead to the appropriation of the corresponding cognitive functions. It is important that students are required to change roles, as in unstructured collaborative problem solving students often remain in the role with which they are most comfortable.

If collaborative problem solving works well, the role of Guru should progressively fade away and peers should take on the responsibility of the Guru. The students have the opportunity of observing how this role is played and so, at a certain point, they could take it on. For instance, a student could ask a peer for more clearness in communication, reproducing Guru's attitude. Guru's function should be appropriated by everyone, in addition to the role they are playing at the time.

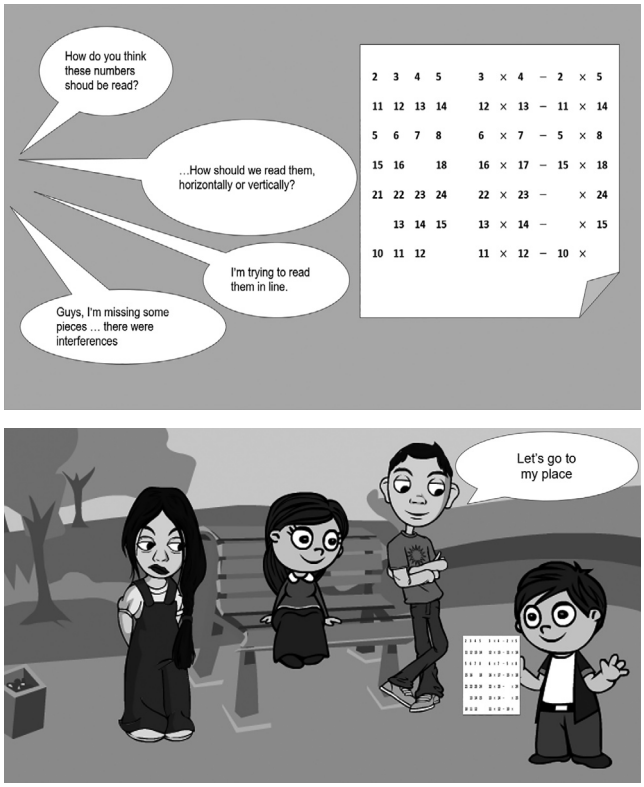


*Figure 1. The Comic.*

## The first online implementation

We have embedded our model in an online platform within a digital storytelling framework, and here we will briefly describe one implementation, to illustrate the design principles outlined above (see, *e.g.,* Albano, Coppola, Dello Iacono, Fiorentino, Pierri & Polo, 2020, for more details). The task was presented to the students by means of comics (see Figure 1) that provide the script for the beginning of each episode. In this case, the task was:

> Choose four consecutive natural numbers, multiply the two intermediate numbers, multiply the two extremes, and subtract the results. What do you get? (adapted from Mellone & Tortora, 2015)

We selected this task as it has been used to introduce students to algebraic proof.

### The digital environments

We devised two digital environments: one for the Actors, and one for the Onlookers.

Within the Actors' environment, each student has three windows at their disposal: one consisting of the comic strips outlining the story (Figure 1), another composed of a Group Chat where the students can interact informally (Figure 2), and finally the Forum, dedicated to the communication between Guru and the group (Figure 3). The actor playing the Promoter character also has access to a Private Chat with Guru (Figure 4) [3]. *Guru's* role was played by the mathematics teacher in the classroom involved, sometimes supported by the researcher.

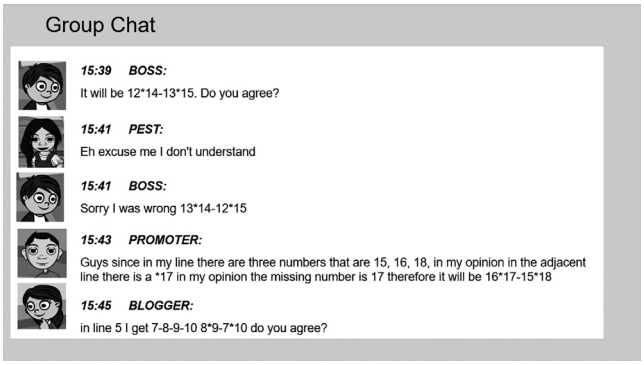


*Figure 2. The Group Chat.*

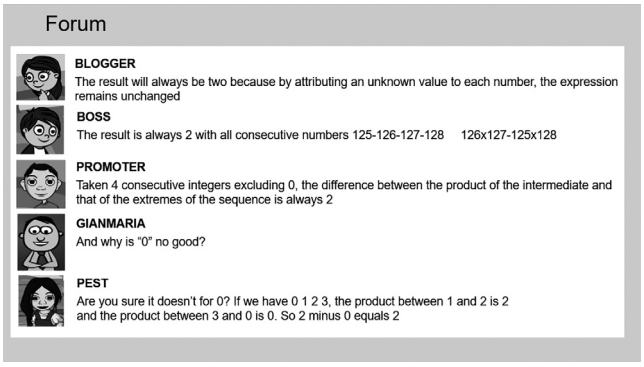


*Figure 3. The Forum.*

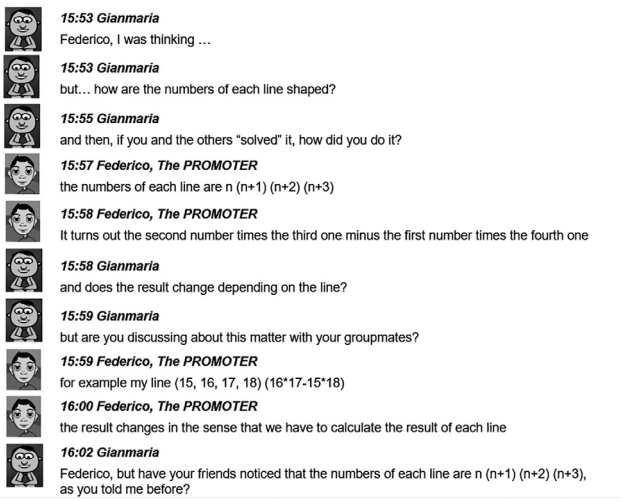


*Figure 4. The Private Chat between Promoter and Guru.*

The Onlookers have at their disposal the same three windows as the actors only in a read-only mode, and another window consisting of a group Logbook, where each one is required to write notes (Figure 5).

The chats, both Group Chat and Private Chat between Promoter and Guru, the Forum and the Onlookers' Logbook keep track of the whole problem solving and argumentative process. Therefore, they show outside readers, *e.g.,* the teacher, not only the final product, but also the mistakes and efforts to obtain it.

### The Actors

The student Actors play the story's characters. To make them more accessible, the characters are given names, as is the Guru, who is described as Promoter's uncle, Gianmaria. In the Forum, Guru can intervene from two points of view. Mathematically, Guru can help to solve possible deadlocks (for example, in Figure 3 Guru asks "Why is 0 no good?" and this seems to activate Pest in the discussion). From a social point of view, Guru's interventions can help to keep each actor 'in character', playing the role in the problem solving process they are meant to.

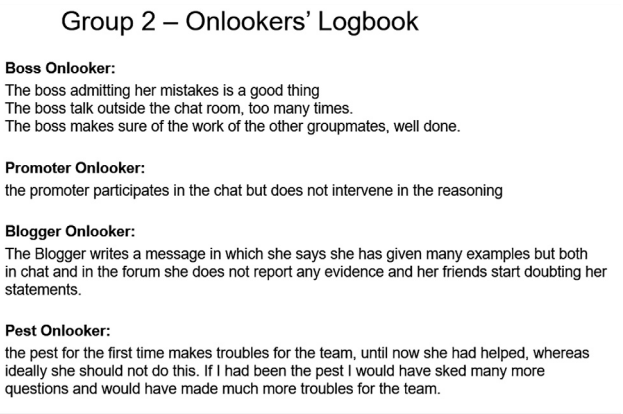


*Figure 5. The Onlooker's Logbook.*

Previous experiments seemed to show that it is not enough that Guru's intervention and mediation work only occur in the Forum. Hence, we added the Private Chat between Promoter and Guru. This gives Promoter a privileged relationship with Guru, helping Promoter to play the role more effectively, for example, by giving suggestions or highlighting some key points neglected by the characters. Promoter can also turn to Guru in moments of impasse. This pattern replicates what happens in the mental model, where the problem solver draws on resources of the culture, ancient wisdom, and previous experiences.

In the Group Chat and the Forum we can see the student playing Pest asking for clarification and questioning assumptions, in keeping with the role of the critic. Boss is taking responsibility for keeping the work going, and keeping the other group members involved. Promoter is promoting innovation, making conjectures for the others to react to. In the Forum Guru assists in this, as well as assisting Pest in raising doubts. Blogger can be seen in the Forum expressing a conjecture in a coherent form, along with a justification.

### The Onlookers

The student Onlookers are actively engaged, since they are required to look critically at the ongoing episode from the outside, focusing on one Actor, reporting on how that student has been playing the assigned character, and giving recommendations on how to improve. For example, in the Logbook (Figure 5), the Onlookers give both positive and negative feedback on each role.

The value of the Onlookers is twofold. First, they enhance the collective dimension of the mathematical story. When former Onlookers enter as Actors in a new episode of solving a task, they already know the mathematical background of what occurred in the previous episodes, thanks to their observations. Second, the Onlookers compare what the Actor group is doing with what they would do if they were in the Actor's shoes, so the students become aware of 'being a mathematician' by observing what others are doing. This encourages meaningful and conscious learning, in which guided reflection plays a fundamental role.

## Final remarks

In this article, we identified cognitive functions that come into play when a mathematician is engaged in argumentative and problem solving activities. In order to help students develop their problem solving competencies, we take them into the mind of a mathematician engaged in problem solving, by personifying cognitive functions as characters in a story, and engaging students in playing the characters in a partially scripted story. Moreover, in order to spur the appropriation of an attitude of a mathematician struggling with a problem solving situation, we provide for every student to experience all the roles corresponding to cognitive functions.

We extend Liljedhal's idea of the persona-framework by means of defining personas according to cognitive functions in a problem solving process. We made the personas characters in a story and allowed the students to play and observe

them, with a goal of facilitating the improvement of argumentative and problem solving competencies. In the overall activity, the main focus is placed on the Onlookers, since each student plays once as an Actor and all the other times as an Onlooker. The Onlooker groups are important in the story context, as students acquire the experience of previous processes through observation. The Onlookers experience the processes from the outside, thus they can grasp them better than the Actors, thanks to the reflective feature of their engagement in the story.

It is worthwhile to note that, as an added value of the metaphor of drama and characters, our model is also aligned to research on Mathematics Learner Identities. In a recent literature review, Radovic, Black, Williams and Salas (2018) recognized subjective/social, representational/enacted and change/stability dimensions characterizing the defining features of identity. The first dimension refers to "identity as performed in social practice and recognized in these social spaces by others" (p. 26). The second, in turn, concerns the mediation of discourse or language as well as the role performed during activities to define identity. The third dimension, finally, calls attention on the "identity as being constructed through a process, and consequently, learnt and open to change" (p. 26). We are exploring if our script and engagement model could take into account all the three dimensions and allow the design of educational activities able to engage students in an evolving process, to be performed in a social practice, making use of discourse and language as mediators. In order to reach autonomy, each student should identify herself not only with a specific character, but with the community of all of them, together with their interactions while working.

The implementation of the drama aims to replicate what happens with emotions in the movie *'Inside Out'*. Each time that an emotion acts, a memory is produced. The most important ones, called basic memories, remain in headquarters, where they act on the mind of the protagonist defining her personality. Similarly, our assumption is that each time a student plays a role of a character, a memory is produced. The most important ones, called 'basic memories', are internalized and act on the mind of the students defining their Mathematics Learner Identities. It is worthwhile to note that such identities should not arise only from the reproduction of the roles the students have played, but also from the students' own synthesis of them, building on the Onlooker's role. However, an open question remains. Since the students work in groups, *can* they build a synthesis that is more than the sum of the individual roles, that is, a new collective entity with its own intelligence?


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## Notes

[1] The authors contributed equally to the development of the manuscript and all read and approved the final version.
[2] https://www.slashfilm.com/pixar-pete-docters-movie-mind/
[3] The online platform stores all the data.

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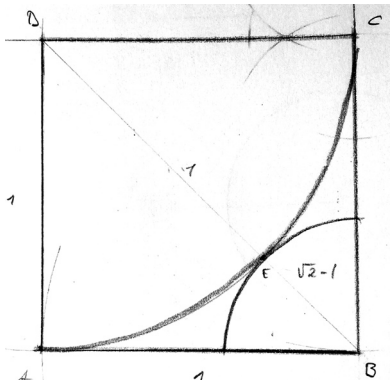


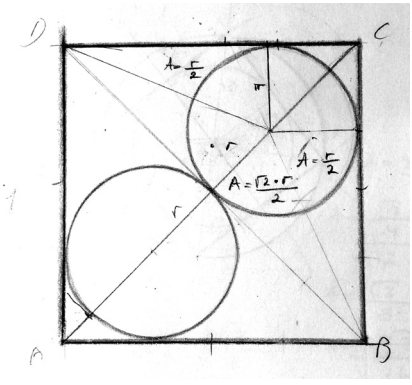


Two constructions by Benjamin, age 14.