MEDIA IN MATHEMATICAL WRITING: CAN TEACHING LEARN FROM RESEARCH PRACTICE?

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This article investigates mathematicians' use of different media throughout the mathematical writing process and discusses how, and to what extent, the findings are relevant to mathematics education.

Computers and mathematical writing

It is generally acknowledged that computers, as a new medium for writing, facilitate a new kind of writing process, and that, in the Western world at least, the computer is currently replacing pen and paper as the most dominant tool for writing. Furthermore, universal access to e-mail and the internet has made it easier to use computers to support various forms of collaboration, including collaborative writing.

Computers are highly successful as support for writing in many areas, but they are typically not adopted with ease in the mathematics classroom.

Moreover, even though mathematical research in the last decades has changed to be more collaborative in nature (Burton, 1999), computer systems that support such collaboration seem to be especially problematic to use (Guzdial, Lodovice, Realf, Morley and Carroll, 2002). One reason for the lack of success in introducing computer systems to support both writing and collaboration in mathematics teaching and learning might be that mathematical notation is difficult or even impossible to write in many computer systems.

This issue was addressed by Guzdial *et al.* (2002) with the construction of a formulae editor for a successful computer system to support collaboration among students (the *CoWeb*). The mathematics students did not adopt this improved system, which suggests that either the students did not want to collaborate or that the possibility of writing formulas was somehow insufficient to facilitate collaboration through such a system.

Since collaboration, according to (Burton, 1999), is increasingly common among mathematicians, and since computers seem to support collaborative writing well in many areas, but not in mathematics classes, it seems reasonable to look at what medias mathematicians use to support their mathematical writing process [1], in order to understand how technology can support collaborative writing among students of mathematics.

Relevant practices

It is possible to study different practices in order to approach potential problems and possibilities in using computers for writing (both personal and collaborative) in connection to the learning of mathematics. I see at least three relevant practices that might be looked at; mathematics education, studying mathematics at university, using mathematics in another subject and mathematical research.

In mathematics teaching, at most levels, computers are used only rarely for writing. They are most often used as advanced calculators (Dreyfus, 1994; Lagrange, Artigue, Laborde and Trouche, 2001). Hence, a study of the collaborative mathematical writing process, mediated by computers, is not easily conducted in an educational setting. What is possible, however, is to introduce technology for writing in an educational setting as a development project, but I believe that such an introduction would benefit from some knowledge of how computers are used for mathematical writing elsewhere.

In other subjects, where writing is important, word processors have had a big influence on the writing process (Sharples, 1992) and on the tendency among students to collaborate and share their work. Therefore, it might be relevant to study the potential of word processors in, for example, language education (Sharples and Evans, 1992). There are still a lot of questions regarding mathematics that remain unanswered by such studies, though, because the mathematical writing process differs from other writing processes at crucial points (Duval, 2000).

I have chosen to look at mathematicians' use of computers and other media to support their writing. Apart from this being interesting in its own right, I believe that this approach may yield some insight into the possibilities and problems of using computers for mathematical writing. This belief is based on the assumption that the' 'mathematical writing process' has some distinctive features that are found both in educational settings, at least at the tertiary level, and among researchers.

The assumption that one can speak of the 'mathematical writing process' at several different levels is not unproblematic. Here, I will briefly describe how research practice can be seen as a benchmark for teaching and learning, and discuss how the results of my studies of the mathematical writing process among researchers can be relevant to the teaching and learning mathematics.

The role of research practice for teaching and learning

The fact that the work of mathematicians and the work of students of mathematics are different is obvious. These two environments are different. The criteria for success differ somewhat (having papers accepted as opposed to passing examinations). The mathematical content domain can differ greatly (even though this can be the case among researchers and among students as well) and the work setting, both physically (office or classroom) and process-wise, are very different (*e.g.*, time spent on a research paper *vs* time spent on weekly homework).

Despite these differences, mathematical research activity is often seen as a benchmark for mathematics teaching in several ways:

- the work of mathematicians is used as a metaphor while introducing a more investigative type of activity (see the editorial *Children as mathematicians* in Fuys and Huinker, 2000).
- the use of the historical development of a mathematical content domain (as it has been developed by researchers) as a model for planning a guided (re-)investigation of the content domain by students (Freudenthal, 1973, p. 109)
- as a case for discussing a framework for understanding learning of mathematics (Sfard, 1991, p. 12).

Brousseau (1997) claims that

> [t]he intellectual work of the student must at times be similar to this scientific activity (p. 22)

but that this "adidactical situation" cannot be the only type of activity when learning mathematics. The adidactical situation should be accompanied by didactical situations such as instruction by a teacher. The adidactical situation, which is essential to ensure that the students learn, clearly uses the work of mathematicians as important benchmarks, whereas the didactical situation, that is equally important, does not.

Burton (1999) has, in a large interview-based study on how research mathematicians come to know mathematics, revealed that the discipline is moving towards a more collaborative nature (p. 137). She describes this as being in contrast to the practices in mathematics teaching and learning. By studying mathematical researchers, Burton tries to provide inspiration for, and reveal potential misconceptions about, the ways in which mathematics is typically taught (*ibid.* p. 121). Likewise, Burton and Morgan (2000) reveal profound differences in the use of natural language in research papers, and conclude that this diversity is not reflected in the way writing is introduced in mathematics classes.

In Burton (2001), it is argued that a model describing research mathematicians' way of learning should be applicable to any learner of mathematics. Her argument is based on the notion of *communities of practice* (Wenger, 1998), and highlights the epistemological practice over the actual knowledge objects.

In this section, I have described how mathematical research can be considered to be a benchmark for mathematics teaching and it seems to be clear that the practice of researchers can be considered in the planning of educational activities of students from a variety of educational perspectives. Burton, Brousseau and the metaphorical approach of Fuys and Huinker all point out the similarity in practice, with only little reference to the content of mathematical research. On the contrary, the guided reinvention approach focuses on how the historical development of mathematical knowledge can be used to inform the planning of didactical activities. Neither of these approaches claim that research practice should be the only source for inspiration when planning teaching.

In this article, I will investigate what media researchers use throughout their writing process. The idea behind this introduction has been to explain why it is meaningful to investigate research practice in order to inform teaching and learning. I have tried to make the case that a closer look at the writing process of researchers can be of relevance to teaching and learning.

A mathematician's writing process

In the autumn of 2002, I conducted an interview study among eleven mathematicians, who were employed in four different institutions on two different continents (Misfeldt, 2003). The aim of the study was to describe the mathematicians' writing processes, paying special attention to their use of different media throughout this process. I interviewed all eleven respondents. The interviews revolved around two main themes:

1. communication during collaborative research projects
2. the use of writing in the personal research process.

This paper is mainly concerned with the second of these questions. The interview data is supplemented by samples of the mathematicians' working papers.

To provide insight into one mathematical writing process, I will present a case of a young Danish mathematician I shall call *Peter*.

Peter used three different types of media for writing mathematics (two paper-based and one electronic). He clearly classifies his work according to the medium in use. The kinds of media being used are: blank draft paper (the back of

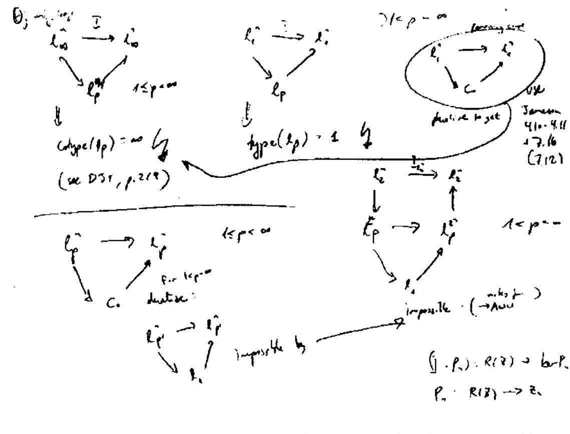
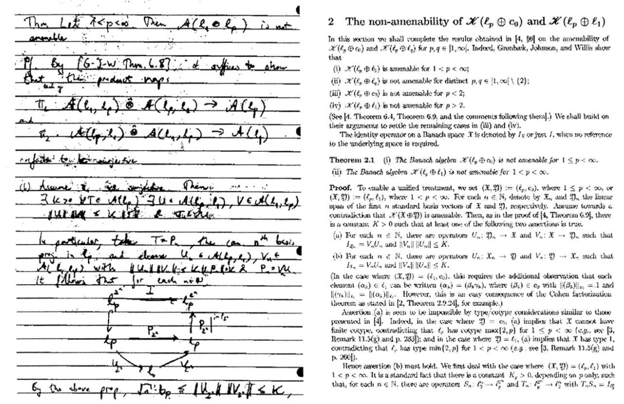


Figure 1: Draft paper: "Look at my desk, typically I have some paper that I draw diagrams and write stuff on, just very brief to see if this works at all [...]"

old printouts) for handwriting; a lined pad, also for handwriting; his computer with an email program and *LaTeX* [2].

Peter uses the draft paper for his personal scribbles, and he explains that these papers only make sense while he is in the process of working on a problem and that most of them are thrown away almost immediately. The scribblings on the draft paper have no obvious linear structure (Morgan, 1998 p. 89), consisting of scribbles scattered around the paper.



Thm Let $1<p<\infty$. Then $A(\ell_1\oplus\ell_p)$ is not amenable

Pf. By [G-J-W Thm. 6.8] it suffices to show that the product maps

$\pi_1 : A(\ell_1, \ell_p) \hat{\otimes} A(\ell_p, \ell_1) \to A(\ell_p)$

and

$\pi_2 : A(\ell_p, \ell_1) \hat{\otimes} A(\ell_1, \ell_p) \to A(\ell_1)$

[illegible] to be surjective

(1) Assume π_1 is surjective. Then

$\exists K>0\ \forall T\in A(\ell_p)\ \exists U\in A(\ell_p,\ell_1), V\in A(\ell_1,\ell_p)$ [illegible]

In particular, take $T=P_n$, the can. n^{th} basis proj. in ℓ_p, and choose $U_n\in A(\ell_p,\ell_1)$, $V_n\in A(\ell_1,\ell_p)$ with $\|U_n\|\|V_n\|\leq K\|P_n\|\leq K$ & $P_n=V_nU_n$. It follows that for each $n\in\mathbb{N}$

By the above prop, [illegible] $\leq \|U_{2^n}\|\|V_{2^n}\|\leq K$,

2 The non-amenability of $\mathscr{K}(\ell_p\oplus c_0)$ and $\mathscr{K}(\ell_p\oplus\ell_1)$

In this section we shall complete the results obtained in [4, §6] on the amenability of $\mathscr{K}(\ell_p\oplus c_0)$ and $\mathscr{K}(\ell_p\oplus\ell_q)$ for $p,q\in\,]1,\infty[$. Indeed, Grønbæk, Johnson, and Willis show that

(i) $\mathscr{K}(\ell_p\oplus\ell_p)$ is amenable for $1<p<\infty$;
(ii) $\mathscr{K}(\ell_p\oplus\ell_q)$ is not amenable for distinct $p,q\in\,]1,\infty[\,\setminus\{2\}$;
(iii) $\mathscr{K}(\ell_p\oplus c_0)$ is not amenable for $p<2$;
(iv) $\mathscr{K}(\ell_p\oplus\ell_1)$ is not amenable for $p>2$.

(See [4, Theorem 6.4, Theorem 6.9, and the comments following them].) We shall build on their arguments to settle the remaining cases in (iii) and (iv).

The identity operator on a Banach space $\mathfrak{X}$ is denoted by $I_{\mathfrak{X}}$ or just I, when no reference to the underlying space is required.

Theorem 2.1 (i) *The Banach algebra* $\mathscr{K}(\ell_p\oplus c_0)$ *is not amenable for* $1\leq p<\infty$.
(ii) *The Banach algebra* $\mathscr{K}(\ell_p\oplus\ell_1)$ *is not amenable for* $1<p<\infty$.

Proof. To enable a unified treatment, we set $(\mathfrak{X},\mathfrak{Y}) := (\ell_p, c_0)$, where $1\leq p<\infty$, or $(\mathfrak{X},\mathfrak{Y}) := (\ell_p,\ell_1)$, where $1<p<\infty$. For each $n\in\mathbb{N}$, denote by $\mathfrak{X}_n$ and $\mathfrak{Y}_n$ the linear span of the first n standard basis vectors of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Assume towards a contradiction that $\mathscr{K}(\mathfrak{X}\oplus\mathfrak{Y})$ is amenable. Then, as in the proof of [4, Theorem 6.9], there is a constant $K>0$ such that at least one of the following two assertions is true.

(a) For each $n\in\mathbb{N}$, there are operators $U_n\colon\mathfrak{Y}_n\to\mathfrak{X}$ and $V_n\colon\mathfrak{X}\to\mathfrak{Y}_n$ such that $I_{\mathfrak{Y}_n}=V_nU_n$ and $\|V_n\|\,\|U_n\|\leq K$.
(b) For each $n\in\mathbb{N}$, there are operators $U_n\colon\mathfrak{X}_n\to\mathfrak{Y}$ and $V_n\colon\mathfrak{Y}\to\mathfrak{X}_n$ such that $I_{\mathfrak{X}_n}=V_nU_n$ and $\|V_n\|\,\|U_n\|\leq K$.

(In the case where $(\mathfrak{X},\mathfrak{Y})=(\ell_1,c_0)$, this requires the additional observation that each element $(\alpha_k)\in\ell_1$ can be written $(\alpha_k)=(\beta_k\gamma_k)$, where $(\beta_k)\in c_0$ with $\|(\beta_k)\|_\infty=1$ and $\|(\gamma_k)\|_{\ell_1}=\|(\alpha_k)\|_{\ell_1}$. However, this is an easy consequence of the Cohen factorization theorem as stated in [2, Theorem 2.9.24], for example.)

Assertion (a) is seen to be impossible by type/cotype considerations similar to those presented in [4]. Indeed, in the case where $\mathfrak{Y}=c_0$, (a) implies that $\mathfrak{X}$ cannot have finite cotype, contradicting that ℓ_p has cotype $\max\{2,p\}$ for $1\leq p<\infty$ (*e.g.*, see [3, Remark 11.5(g) and p. 283]); and in the case where $\mathfrak{Y}=\ell_1$, (a) implies that $\mathfrak{X}$ has type 1, contradicting that ℓ_p has type $\min\{2,p\}$ for $1<p<\infty$ (*e.g.*, see [3, Remark 11.5(g) and p. 260]).

Hence assertion (b) must hold. We first deal with the case where $(\mathfrak{X},\mathfrak{Y})=(\ell_p,\ell_1)$ with $1<p<\infty$. It is a standard fact that there is a constant $K_p>0$, depending on p only, such that, for each $n\in\mathbb{N}$, there are operators $S_n\colon\ell_2^n\to\ell_p^{2^n}$ and $T_n\colon\ell_p^{2^n}\to\ell_2^n$ with $T_nS_n=I_{\ell_2^n}$

Figure 2: Lined pad and LaTeX versions.

Lined pad: “If it seems to work out right, I will take another piece of paper and write it out to see if the details are right.”

LaTeX version: “Then, while I am trying to write as many details as possible in order to make sure that I got it right, then when I am satisfied I put it into the computer, but much shorter in order to fit a paper.”

Peter explained that if something on the draft paper seemed to work out, he would take the lined paper pad and try to write down as many details as possible. When that is done, the draft papers are thrown away, but the lined paper, with all its details, is carefully stored in a system of binders. Peter explained that the purpose of writing it out in detail was twofold: to make the work accessible later and to check, in detail, if his ideas were correct.

When Peter thinks he has enough work for a paper, or a part of one, he will write a *LaTeX* version of his work. He describes that version as “much shorter, to fit a paper”. The *LaTeX* version is sent to fellow collaborators for comments and proofreading, but the main reason for creating the *LaTeX* version is to produce a paper.

Peter communicates with fellow mathematicians mainly through e-mail. He explains that the content is often very close to the content of his notes on the lined paper that he keeps in his binders. To be able to express mathematics in an e-mail, he will often use *LaTeX* code [3] in that e-mail and that leads to some extra typing. When Peter communicates longer arguments and developed ideas, he will sometimes type it out in *LaTeX* because it is easier to send, but he only does that if there is a good chance that he will publish it later. To avoid unnecessary typing, Peter has tried to fax the notes from the binder, but has had difficulties with this. Peter explained that if, when writing, he had to think about how it could be faxed (*i.e.*, choose a dark pen; avoid using the margin; think about a potential reader) then this would disturb his thinking and he would be less able to concentrate on checking the mathematical details of his idea.

Five functions in the mathematical writing process

The interviews have led me to consider the following five different functions of writing in mathematics. By distinguishing between these five functions, I do not claim that they are completely independent of each other or that they completely cover what mathematical writing is. Rather, I try to establish a framework for further analysis of the mathematical writing process. The five functions are:

1. *Heuristic treatment* consisting of getting and trying out ideas and seeing connections.

2. *Control treatment* is a deeper investigation of the heuristic ideas. It can have the form of pure control of a proposition or be a more open-ended investigation (*e.g.*, a large calculation to find *x*). It is characterised by precision.

3. *Information storage* to save information for later retrieval. This can be done as electronic files or papers.

4. *Communication* with fellow collaborators: such communication can have various forms ranging from annotation of an existing text, comments or ideas concerning a collaborative project, to suggestions of sections that are to be included in a paper.

5. *Production* of a paper, where writing is used to deliver a finished product intended for publication and aimed at a specific audience.

I will now use these five functions to describe and analyse the interviews.

Media to support the functions

By distinguishing the media each of the 11 respondents used to support the five functions described above we get the following table (see Figure 3).

Peter used pen and paper for heuristic treatment and Figure 3 clearly shows that this was normal among the participants. The only exceptions to that pattern were; R10 who did not use any media at all and R4 who sometimes used a CAS [4] to support heuristic treatment.

The respondents (R1 (Peter), R3, R4, R7, R9 and R10) held a clear distinction between whether a piece of paper was used for heuristic or control treatment. For the rest of the respondents, the distinction was less clear, but both functions came into play at the beginning of their work and were mainly supported by handwriting.

Peter used papers in binders for saving information, and Figure 3 shows that several of the respondents used both *LaTeX* files and handwritten notes to save information. R6 and R11 always used computers to save information. On the other hand, several respondents (R2, R8, R9) claimed that the amount of ideas that did not end up developing into a publication was large and therefore it was not worth saving everything on the computer.

Respondent	Heuristic treatment	Control treatment	Information storage	Communication	Production
R1 (Peter)	Paper	Paper	Paper	E-mail, *LaTeX*, fax and letter	*LaTeX*
R2	Paper	Paper or printouts	*LaTeX* and paper	E-mails	*LaTeX*
R3	Paper	Paper	Paper	E-mails and paper	Paper draft for secretary who types in *LaTeX*
R4	Paper and CAS [4]	MATH-TYPE (WORD), CAS and C++	Paper, MATH-TYPE (WORD) or *LaTeX*	MATH-TYPE (WORD), fax and *LaTeX*,	MATH-TYPE (WORD) or *LaTeX*
R5	Paper	Paper and scientific WORD	*LaTeX* or paper	E-mail and *LaTeX*	*LaTeX*
R6	Paper	Paper	*LaTeX*	E-mail and *LaTeX*	*LaTeX*
R7	Paper	*LaTeX*	*LaTeX* and paper	*LaTeX* and e-mail	*LaTeX*
R8	Paper	Paper	Paper	*LaTeX*, e-mail and fax	Paper draft, typed in *LaTeX*
R9	Paper	Paper or paper and *LaTeX* (synchronously)	Paper and *LaTeX*	E-mail, *LaTeX*, phone, and fax	*LaTeX*
R10	No media	Paper	Paper	Paper, fax and e-mail	Paper draft for secretary, who types in *LaTeX*
R11	Paper	Paper and printouts	*LaTeX*	E-mail and *LaTeX*	*LaTeX*

Figure 3: Types of media supporting the five functions in mathematical writing. The table shows that the closer you get to publication, the more computers are used.

All the respondents communicated with their collaborators through e-mail, sometimes with *LaTeX* files attached. All but R4 used *LaTeX* commands directly in the e-mail in order to include mathematics. There was a tendency to send longer arguments and fully developed ideas as attached *LaTeX* files. Questions and ideas that were not quite finished were generally written in the body of the email using *LaTeX* commands. All but R7 preferred to perform annotations and reviewing by pen on a printout (this preference was also found by Kim and Erklundh (2001), investigating collaborative writing among academics in different topics), and several of the respondents (R1, R4, R8, and R10) frequently faxed handwritten comments or annotations of manuscripts. All of the respondents occasionally e-mailed annotations as a list of corrections, which were written using *LaTeX* code to write mathematics.

To produce papers, all of the respondents used *LaTeX*, except for R3 and R10 who both wrote handwritten manuscripts, which they would hand over to a secretary who typed it up in *LaTeX*. In general, the respondents explained that the *LaTeX* files were important in collaborative projects, both for communicating ideas and to keep track of progression, but for many of the respondents, *LaTeX* files such as these were, from the beginning, thought of as potential publications and they were therefore thought of in a more formal sense than the other media used (*e.g.*, e-mails, fax, phone).

Peter had a very linear writing process where he moved from heuristic treatment to control treatment (both supported by handwriting), followed by production supported by a computer. The respondents R3, R8, and R10 all had similar 'linear' approaches, with the other respondents having less linear approaches. In fact, R2, R6, R7, R9, and R11 had successive writing processes where they continuously reviewed and changed their work. This procedure typically started after the heuristic treatment (supported by pen and paper) and was supported, at least in part, by a computer. The respondents R4 and R5 were difficult to classify in this way. They both used a WYSIWYG (what you see is what you get) editor to write mathematics on a computer (R4 used MATH-TYPE and R5 used SCIENTIFIC WORD). They both used the computer early in their writing process and in a successive way, but they both switched to *LaTeX* when they started producing a paper, and this change of program marked a complete change in working style. From now on the mathematics was considered finished and the purpose of their work was to produce a paper. Peter e-mailed his fellow collaborators during the control treatment and production phase of his work but never before. In general, it seems that the eleven respondents only communicated with fellow collaborators during heuristic treatment in face-to-face situations.

How are the five functions relevant to education?

It seems fair to question whether and to what extent the empirical results from a few researchers' use of media in their mathematical writing process are of any relevance to mathematics education. Both mathematicians and learners of mathematics are concerned with mathematics and write mathematically, but their writing activities also differ, both in goals and intended audience (*e.g.*, publication *vs* homework assignment) and process-wise (*e.g.*, timeframe). Apart from these differences, mathematicians are, obviously, more experienced writers of mathematics than students.

Even though the practices of researchers and students are different, the discussion earlier shows that there is, or should be, a process-wise similarity between the practices of students and researchers. Therefore I find it reasonable to attempt to interpret the findings from the study educationally.

A mathematical writing process

Pimm (1987, p. 198) explores the structural metaphor of "mathematics as a language". If we take the structural aspect of the metaphor seriously, it makes sense to talk about mathematical conversations, mathematical argumentation, mathematical vocabulary and mathematical writing. Furthermore, I have argued that the activities of researchers can serve as a guiding principle or inspiration for planning students' learning activities. This does not mean that insights concerning mathematical writing among researchers are immediately transferable to educational settings, but nevertheless these insights might be able to shed light on problems and potentials with various media in mathematical writing among students.

The control treatment function seems to be closely connected with the communication and saving of information. Most of the respondents mainly used pen and paper to support control treatment, while they used computers for communication and sometimes for saving information. Therefore, the development of better computer based tools to support writing as a control treatment function seems to be relevant in order to support collaboration in mathematical research. It seems *a priori* reasonable to think that this would be the case among students as well, but it is also important to note that the distinction between heuristic treatment and control treatment is not reflected in the literature (see Duval, 2000, who only talks of treatment). It could be interesting to investigate whether the distinction is less clear for mathematics learners than for the professional mathematician.

An important question for this whole discussion is to what extent we can talk about "a mathematical writing process" while ignoring the expertise of author and audience, the particular topic, the natural language used (is the writing in English or Chinese?), and many other variables. Of course such variables influence, often significantly, the way that writing is used in connection to mathematics. But nevertheless it seems fair to raise the question whether there are some unique features of mathematical writing that cut across levels and cultures and hence defines a mathematical writing process.

Even though the data show some similarities in the mathematical writing process, they definitely also show diversity. Students are also different and have different writing processes. I find it particularly important to note that this framework does not suggest one canonical 'best sequence of phases' in mathematical research. Rather than using technology to encourage or force students to approach problems in one specific way, I would suggest that the students, while working with mathematics, should have access to the best possible support for their writing at all times and learn (implicitly or explicitly) to choose the kind of support that best suits the task at hand, acknowledging that this support will be different for different persons.

Media and tools

I believe an awareness of what types of mathematical activities electronic writing tools are suitable to support is important, and this article reflects my attempt to investigate this question: what types of mathematical activities electronic writing tools are suitable to support? Furthermore, an increased awareness of the mathematical writing process could be relevant for teaching and learning mathematics in general.

Pen and paper played a central role in the heuristic phase of almost all the eleven mathematicians' work; none of them used a computer in that phase. This suggests that the computer systems that mathematicians use do not support their heuristic mathematical writing. Since none of the mathematicians used the heuristic phase directly for saving information or for communication it can be argued that computer support for heuristic treatment is not essential for collaboration.

As already mentioned, the mathematical activities of students, as well as their use of computers, are of course different from those of researchers but one can at least question whether students will benefit from using the existing technology for heuristic writing when researchers do not. A tendency to support collaboration by having all documents and working papers stored digitally might force students into using a computer for activities that are better supported by conventional means such as a pen and a piece of paper. On the other hand, the shorter timeframe of most educational activities suggests that successful use of computers for collaborative writing activities in connection to teaching and learning is stronger when connected to development of digital support for heuristics. Furthermore, I can imagine that the dependence on natural language is different in many educational settings than among researchers. Of course, this is highly context-dependent, but in teaching situations, where natural language is the dominant representation form, as, for instance, in some kinds of mathematical projects, it is obvious that the potential of using computers for writing will be significant.

What we can say is that as long as pen and paper are the primary support for one or more of the functions in mathematical writing among researchers (as seems to be the case at the moment), it makes sense for students to have access to this medium at all times. Doing mathematics in computer labs without any desk space can be a problem if some essential work in mathematics is best done with a pen and a piece of paper. On the other hand, technology such as digital ink (Golovchinsky and Denoue, 2002) could maybe make the computer the preferred writing tool for both heuristic and control treatment. Such a development could, in the long run, change the mathematical writing process entirely.

But why is it that, on the one hand, it is difficult for students to use collaborative platforms in mathematics courses, while on the other hand it seems that mathematicians collaborate increasingly and actually use electronic media for their collaboration?

One reason could be that mathematicians, knowing *LaTeX*, have a code to use for writing mathematics in e-mails and this 'code' is unavailable to most students. Another reason, that might seem more pertinent, is that the actual communication in collaborative research is sometimes a time-consuming activity involving typing of mathematical formulas (either in *LaTeX* or *Pseudo LaTeX*), and almost all of the respondents used fax or letter from time to time to avoid this typing. The time-consuming activities are worthwhile in research projects running over long periods of time. But, as an augmentation of normal class activities, posting solutions on an electronic bulleting board or asking questions to the class mailing list could very well seem like a lot of unnecessary work. These activities will involve a lot of work in all topics, but as long as this work mainly has to do with composing a solution or formulating a question, it will make sense for the learner to do it. Transcribing a completely finished solution from paper to a digital medium will rarely be a rewarding activity, and I believe this is the main reason that collaborative platforms have a stronger tendency to fail in mathematics than in other topics.

Concluding remark

Looking ahead, at least two aspects of the challenge of increasing the use of digital support for writing and collaboration in mathematics teaching and learning appear on the horizon, each having to do with improving the tools that the students are offered and with developing the working style and evaluation criteria we confront students with.

Development of improved forms of interaction for mathematical writing and research-based evaluations of existing writing systems are obvious paths to take. A surprisingly small amount of research in mathematics education concerns how digital tools can support the writing of mathematics. This is particularly striking considering the amount of research on computer support for other aspects of mathematical work (for instance symbolic manipulation and interactive geometry). I believe that the very basic issue of simplifying the means of expressing mathematical ideas in a written digital form poses an important challenge.

It is one thing to improve the technology for writing mathematics, and another to make it worthwhile to use, and learn to use, such technologies. A change in form of teaching strategy and evaluation can be a powerful vehicle for such a change. If, for instance, the evaluation form encourages students to develop a text over a period of time, they will most likely use electronic media. Grønbæk and Winsløw (2003) provide one example of such an evaluation form.

Finally, formal training in the use of a specific computer system for writing mathematics is a part of an individual's mathematics education that is not usually systematically planned and developed. A comparison with the culture of training students in the use of graphical calculators as a general tool, whilst critically discussing how this tool affects the working process and the development of students' conceptual understandings, (Guin and Trouche, 1999), would seem to be a promising path to take.

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Notes

[1] Writing is defined broadly, including formulas and drawings.

[2] *LaTeX* is a typesetting system widely used within the mathematical community. The author writes a source code in a text editor and processes it to a print file. So, for example, the author would write:

`$/int_a^b f(t)dt=F(b)-F(a)$`

in a text editor, to typset:

$$\int_a^b f(t)dt = F(b) - F(a)$$

in the print file.

[3] A special use of *LaTeX* commands has developed in e-mails. Since e-mail does not support use of mathematical symbols, it is common to use the *LaTeX* command whenever a symbol is needed, and this style seems to be fully accepted in e-mail communication among mathematicians.

[4] *Computer Algebra Systems* (CAS) are systems like *Maple*, *Mathcad* and *Mathematica*. These are all able to perform calculations both numerically and symbolically.

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