Division Problems: Meanings and Procedures in the Transition to a Written Algorithm

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The problem of learning calculation methods, especially algorithms for division, has been considered in the literature on account of its cognitive and didactical importance. The spreading use of calculators has reduced the social importance of the goal of learning a written algorithm to be carried out automatically In relation to the educational needs urged by the growth of computerized processes in many human activities, it appears instead more significant to take a reasoned approach towards algorithms, regarding them as procedures whose universality and profitability can be evaluated and compared The 1980 French school syllabuses were already oriented in this direction [Teule-Sensacq and Vinrich, 1982] The new Italian syllabuses for the primary school (1985) regard the written methods for calculating arithmetical operations as opportunities to become familiar with algorithms, and require that pupils not only learn to perform written computations but also understand the procedure they have used

As will be illustrated in this paper, the development and comparison of written strategies for division may have an important effect in terms of the deeper mastery of the meanings of this operation.

We believe that these facts justify the development of research, from a curricular point of view, aimed at producing teaching proposals oriented towards a conscious approach to the written algorithms for arithmetical operations (in particular for division)

This report briefly reviews pupils' behaviours and conceptual achievements in the transition from informal calculation strategies to a written division algorithm (section 3) This transition is managed by the teachers within a 5-year project.

With respect to several other research studies [Treffers, 1987; Teule-Sensacq and Vinrich, 1982; Treffers and Goffree, 1985], the following aspects are in common:

- The strict relationship between the analysis of the learning processes and the long term design of the practical teaching.
- The attention paid to children's "spontaneous" strategies (i e , in our case, freely expressed in the framework of a curricular project which, nevertheless, deliberately orients the children and forces suitable strategies to surface) The guided comparison of these strategies (see 1 4) allows the children to exceed the intrinsic limits of "empirical" learning [Teule-Sensacq and Vinrich, 1982]
- The emphasis on making the children aware of the procedure they have learnt

The following aspects distinguish our research from other studies on similar subjects. We shall discuss these aspects in more detail in this report, trying to ascertain their effects on the learning results achieved by our pupils.

- The importance of the "fields of experience" and, in them, of the sequence of the situations chosen, in order to make the children grasp the meanings of the operations and develop calculation strategies
- The attention we pay to the relationship between the development of the calculation strategies and the acquisition of the meanings of division, convinced of their possible synergic effects (see 1 2, 3 4).
- The inclusion of the research in a complete curricular framework in which great emphasis is put on the mastery by the children of the physical variables they usually find in standard arithmetical problems.

As regards the learning results attained, we can conclude (on the basis of both the analysis of the work carried out in class since 1984 and of the assessment performed in grade 5 in 1989, with over 400 children) that our approach to a written division algorithm for children aged from 8 to 10 has been successful for over 90% of the pupils (within the framework of the teaching choices sketched above and detailed in section 1) Moreover, over two thirds of the pupils are able to take part, in an independent and conscious way, in the discussion on the fitness of the different strategies produced in class from the point of view of both profitability and universality.

The methods of observation and assessment adopted to ascertain these results and their limitations are discussed in section 2

1. The framework

We operate within the framework of a complete teaching project covering, usually with the same teacher, grades 1 to 5 (6 to 11 year olds). It concerns all the main contents planned for the various disciplines in the new syllabuses for the Italian primary school [Boero, 1988] In this paragraph we describe the teaching choices which seem to us to directly influence the behaviour of pupils and their acquisition of concepts (which we shall analyse in section 3) on the basis of:

- an analysis *a priori* of the relationships between the teaching choices and the expected results
- an analysis of pupils' reactions to the teaching choices
- a comparative analysis with groups of pupils, exposed to partially or completely different teaching paths.

1.1 Choice of problems: the importance of the "fields of experience"

The problems proposed to the pupils are, as a rule, included in long-term "subject paths" (one year or more) which relate to important aspects of the life and experience of the pupil (for example: "economic exchange," "production of goods," "shadows," ...). We feel that this contributes (together with a correct management of problems from a didactic point of view) to the building of important conceptual and procedural skills, essential to the resolution of problems and the acquisition of productive attitudes by pupils.

The importance of the "external context" for the development of mathematical skills has been widely acknowledged, in more recent years, in research on the psychology of learning in general and on the learning of mathematics in particular [French, 1985; Lesh, 1985]. However, in our opinion, there are no plain evaluations nor analyses supported by sufficient experimental data on the specific effects of the "external context" on the process of learning mathematics. We will use (instead of the term "context") the expression "field of experience" to refer to those environments, identified by the child as unitary and recognizable on the basis of their own characteristics, in which the teacher proposes activities such as mathematical modelling, arithmetical and geometrical problem solving, etc.

Let us now see several examples of our "fields of experience," pointing out, in each case, the aspects which concern the acquisition of meanings and the development of division strategies.

— Productions in the classroom (grades 2 to 5): divisions occur in production cost analysis, especially with regard to the sharing of the cost of the instruments among the number of units of product, the sharing of the total expense among the number of the children, and the analysis of the number of units of product that can be purchased with a given amount.

Both "measurement" and "partition" senses of division are already present. The latter usually arises with a variable number of portions, which often induce the children to reason in terms of inverse proportionality. The numerical values are usually rather high, but right from grade 1 our pupils are accustomed to operate with money in real or simulated purchase situations.

— Fields of experience regarding time (the seasons, the life of the child, his parents and grandparents): a lot of division problems are proposed to children aged from 8 to 10 through the representation on the walls of the classroom of the "time line" associated with a given period ("how much space must we leave for one year?" or "if each sheet represents a century, how many centuries can we represent on the wall?") and the positioning of dates and particular periods thereon. Other problems concern comparisons ("how many times greater is the price of a newspaper than 50 years ago?"). Both measurement and partition senses of division also appear in these problems but with a different degree of difficulty compared with the problems on production. Children are requested to relate "times" to "spaces" analogically and to compare situations occurring at different times.

— The field of experience of shadows (8 to 10 years): often problems arise from the scale-reduction of the length of the shadows of a pole, recorded during the day, to be drawn in a copybook, the division of the page so as to draw up a table or a graph, the division of the space in the courtyard in order to play games with the shadows.

Every field of experience in the above-mentioned examples has (as we have seen) its own characteristics in terms of the variables involved, the meanings of division, the order of magnitude of the numbers and the children's perception of the situations. We have observed that many children are able enough to perceive changes in any of these variables, as is widely reported by research [Lesh, 1985].

This sort of "sensitivity" of the children influences the mastery of the meanings of division in each field as well as the particular calculations strategies adopted every time. Several examples following this argument are given below:

- with similar numerical values and with the same variables it is not the same thing for the children to state "how many times this is bigger than that" or "how many times this is bigger than before," and the strategies are indeed different; in the latter case an additional difficulty is represented by the need to think of quantities associated to different moments.
- the calculation strategies adopted to obtain the numerical results associated to partition problems concerning production substantially differ from those performed with similar numerical values for structurally analogous problems in other areas (for example, division of lengths). Money, regarded as structured materials, and its widespread social use suggests, for example, when performing division (as well as multiplication) strategies, the decomposition of the amounts that make up a price or a cost, thus starting up reasonings based on distribution according to the model $(a + b) : c = a : c + b : c$. The same strategy is less frequent with problems concerning lengths or times.
- the division of lengths into equal parts gives rise to different solution strategies depending on whether it involves spaces available for the children to play with the shadows or spaces to allocate periods on the "time-line," when working on history. For example, the strategy "I measure and divide the number I have found

into equal parts" is very much more frequent in the former case than in the latter (possibly as an heritage of strategies employed in children's games when fair sharing is requested).

In conclusion, the careful choice of fields of experience and, within them, of the problem situations not only avoids the risk of repetitiveness [Teule-Sensacq and Vinrich, 1982] and starts up advantageous dialectics between the different meanings (see 3.4), but also builds up and discusses a large variety of reasoning patterns that help to complete gradually (under the careful guide of the teacher) the mosaic of division problems.

1.2 Management of problem-solving activities

Initially, the problems are related to situations the children have directly experienced. If a problem deals with the cost of a plum-cake, this means that the children have, at least once, really produced a plum-cake in class, with the help of the teacher or any other adult, and that they have really divided it up into the required number of portions. Later, the children have to deal with problems which also refer to similar situations they have not directly experienced, or that they may have experienced outside the school. Generally, the problems situations proposed in our project are chosen and run in such a way to allow the children to progressively and autonomously find the available data and determine the necessary steps to reach the required solutions. For this reason fictitious problems which do not allow children to identify themselves in the problem situations and so employ useful strategies to explore gradually the nature of the problem and the means to solve it, are excluded (except, sometimes, in assessment activities).

In this respect, several problems are proposed for which, starting from actual numerical data (and therefore usually rather "awkward" ...), it is necessary to arrive at the expected numerical results without having mastered written division algorithms. In particular, the written subtraction algorithm is introduced only at 8-9 years, and the division one only at 10 years. For both subtraction and division, the children must, for over a year, build suitable naive calculation strategies in order to face the numerous subtraction and division problems they are given.

Arithmetical problems without numerical data for which the pupil must determine and provide the strategy to solve them, apart from carrying out the calculations, are becoming increasingly important in our project as well.

We are convinced, in the case of division, that by supporting the solution procedures and calculation strategies with these activities it is possible to build skills that interact synergically and allow the child to achieve a unified insight of the division operation as a mathematical model for miscellaneous problem situations (see 3.4). Besides written algorithms, the standard formalization of each arithmetic operation is also postponed until the attainment by the children (through a great number of problems in different fields of experience) of a positive and broad mastery of the meanings of the operation. This is related to another important feature of our work in class on these problems: the considerable development of verbal activities, both when building the solution procedure and when reporting it. Systematic work on verbal language over the years during which the children learn to deal with mathematical problems has several useful functions:

- it encourages the children continually to relate their reasonings to problem situations;
- it supports the children's solution-design processes in which it is necessary to manage hypothetical thinking [see Ferrari, 1989], link different operations together, etc.;
- it simplifies the comparison of the strategies employed by different children in order to recognize and stabilize the acquisition of concepts and procedures and encourage their transfer;
- it allows the teacher to have an accurate picture of each child's reasonings and an objective record of their evolution with time.

1.3 General characteristics of the Project which influence the work on problems

The above reference to the importance of verbalising in problem-solving is related to a general feature of our project. It concerns the strong growth of the mastery of spoken and written language, right from the age of 6 years, through activities which, in our opinion, can "force" [see French, 1985]) the acquisition of expressive forms adequate for representing the complexity of processes when subject to constraints which are outside the individual who reports them (in particular, the productive processes carried out in the classroom, the use and working of common machines, the paths followed, etc.).

Another general characteristic of the project is the widespread practice of thinking about the meaning of the work carried out, of comparing the reports of different children on the same subject (also related to the expressive forms they have used), of making explicit and clarifying the mathematical concepts built in different "fields of experience" (according to the "tool-object" dialectics of R. Douady [1985])

We believe that such activities can encourage, when working on division problems, the comparison of the different strategies and the necessary metacognitive "detachment" in order to determine the procedures most economic and best generalizable to other numerical values.

Last, the considerable development of the work on mental arithmetic (explicitly declared by the teacher and acknowledged by pupils as "training in mathematics" and not "making mathematics") that follows the "meaningful" work on the various arithmetic operations (particularly, addition and multiplication) can explain the self-assurance that most children display in the development of their calculation strategies in order to attain the result of a division.

1.4 The role of the teacher

When managing the work in class on problems, the teachers of our group adopt behaviours which we consider relevant for this research: on the one hand they oversee (above all, with the pupils who have considerable learning difficulties) the (necessarily individual) choice of the linguistic expressions through which each child represents his thoughts; on the other hand they guide the work in class as regards the comparison of the different strategies adopted by each child, the gradual choice of the more effective and universal strategies, etc.

All this is consistent with the role played by the teacher in running our project in class. The child's adaptation to the problem situations (not only in a mathematical setting!) requires, in our opinion, individualized interventions by the teacher in order to support him and help him to reach, at least in part, explicit "products" which he and his fellow-pupils can acknowledge. On these "products" comparison and selective reflection activities must then be carried out; such activities must never be left to the internal dynamics of the class, but must be directed by the teacher towards the planned results.

2. Observation and test methods

The analyses of the learning processes on division problems that are illustrated in the following paragraphs, require that observation methods and specific test instruments should be adopted that also relate to the context (the complete teaching project) in which the teaching work on the problems is included.

The research involves, in various ways, the 120 classes that follow our project each year. Some classes (about ten overall, on average two for each age group) are "observation classes" for 6 to 11 year olds, often under the guidance of the same teacher (as is usual in Italy). Many protocols are gathered (individual and independent solutions to problems, for about thirty problems every year for each class) and individual teacher-pupil interactions and collective discussions are recorded in these classes. Research staff also partake in these classes as "observers" and/or as "additional teachers." The other classes in the project perform the same activities as the observation classes, but only provide protocols concerning individual solutions to a fixed set of word problems (from 5 to 8 each year) for the "observation classes." This internal check is very important for us because it provides several guarantees that the indications obtained by analysing the "observation classes" do not depend upon the peculiar characteristics of the classes (psychological "pressure" of the research staff which follow them or directly operate in them, characteristics of the teachers working in them, etc.)

The "external" comparison is based above all on the over 800 children who every year enter the comprehensive school classes to follow the project designed by our group for the comprehensive school. The "external" comparison is carried out on word problems similar to those which serve as a final test in our primary school classes (in general, since it involves primary and secondary schools of different places, no child comes from "our" primary school classes). Other interesting elements for comparison derive from the analysis of the performances of the children who enter our "observation classes" from classes which do not work with our project.

We believe that in certain cases data or quantitative analytical comparisons may be unreliable for the subjects dealt with in this article. We have, in fact, gradually noticed that by operating with various groups of pupils, assigned to different teachers employing different methods, the percentage differences (for example) found in the adopted solution strategies or in the mastery of the written calculation methods can be generated (or concealed) by teaching choices which have little or nothing to do with the objectives being tested. For example, the degree of accuracy the teachers have worked with or the correctness of the use of the written algorithm can conceal the degree of awareness acquired by the pupils and the independence reached in building efficient calculation strategies in non-standard problems.

We generally believe that the following points appear to be most significant for the particular questions dealt with in this article:

- an analysis of the manner by which the pupils work (degree of autonomy, ability to think . . .)
- an analysis of the kind of strategies adopted for a vast and diversified range of problems
- an analysis of individual learning approaches of the pupils over a sufficiently long time span (at least one year, in several cases from 3 to 4 years)

For these reasons, we will provide some general quantitative indications (above all on the final results attained on standard problems and also on the different calculation strategies adopted in the context of a substantially homogeneous teaching course followed in different classes). However the main stress will be laid on the qualitative analysis of the pupils' performances.

3. Description of activities and behaviours

3.1 Teaching path on division problems for the 8 to 10 years old

The partition meaning of division is strictly linked to the ordinary usage of the verb "to divide." In fact, the children easily realize that in order to share a given amount of money (or a given number of objects) into equal parts among a given number of persons, it is necessary to "divide" the sum (or the number of objects) into as many equal portions as there are people. The difficulty that arises immediately when the numbers involved are not particularly easy is the building of a calculation strategy in order really to perform the division into equal parts. For example, a deadlock is usually reached when 1100 Lire have to be divided among the 18 children in class (9 year olds), (assuming, as in the rest of the paper, that the children still do not know any written algorithm for division).

In spite of this difficulty, the traditional choice of operating on simple numerical values, with the repeated execution of numerous similar problems and the subsequent introduction of the written algorithm (usually, starting with one-digit divisors), justifying it only approximately and consolidating it through repeated drills, does not allow the pupils to gain a sufficient understanding of the meanings of division (particularly relating it to inverse ratio) nor to become aware of the motivations and of the nature of the written algorithm adopted [see Treffers, 1987; Teule-Sensacq and Vinrich, 1982])

Our teaching sequence has been structured to have the following main features, with the aim of achieving those objectives:

— Proposal of partition problems after and in parallel with measurement problems (which stimulate the children to adopt trial-and-error strategies tied to the numerical values involved). Overall there are at least 40 problems over the span of about a year and a half which concern, in general, problem situations chosen from the "fields of experience" presented in 1.1

— Rather limited work on division problems with "easy" numerical values, that is, those that allow the "fair sharing" strategies to surface. Although these strategies are adopted "spontaneously" by the children (probably because related to their everyday-life experiences), an excessive emphasis on situations like these does not encourage the production of new strategies

— Guide to the construction of a universal algorithm (Laing and Meyer, 1982] and to the algorithm described in [Teule-Sensacq and Vinrich, 1982, p. 193]. This is attained (in the context of what has been illustrated in 1.2) through the analysis and discussion of the different calculation strategies adopted by each child, gradually determining their inadequacies (for example, in relation to the different numerical values) and steadily enhancing the value of the calculation strategies that can be more widely generalized. This work of comparison and reflection is performed, among the 8 to 10 year olds, at least 15 times out of a total of over 40 proposed problems.

3.2 Examples of division problems

Right from the end of grade 2 (8 year olds) the children solve several division problems which they find in a natural way during the various class activities, as for example the following:

> (P1) (end of grade 2) "How many cakes costing 300 Lire can I buy with a 1000 Lire note?"
>
> (P2) (end of grade 2) "In order to take part in the swimming lessons each child must pay 40000 Lire. The 40000 Lire are needed for all 13 times we will go to the swimming pool. Find the cost of one swimming lesson."

Problems of the type P1 are dealt with firstly in a concrete way, using, for example, actual money and handling it in order to solve the problem. Activities of this kind force the children, when they find themselves successively in front of similar problems but without the possibility of representing them in a concrete way, to operate through arithmetic calculations of repeated additions or subtractions. In the case of the problem P1 the children operate in either of these ways:

		+300		+300		+100	
	300		600		900		1.000
		-300		-300		-300	
or	1000		700		400		100
		1 cake		1 cake		1 cake	remainder

The first way is a procedure of successive approximations to the dividend. The second one is a procedure by removal, in which at each step the remainder can be directly checked. The first approach is chosen by over three-quarters of the children. These approaches broadly correspond to those indicated by A_2 and S_1 respectively in [Teule-Sensacq and Vinrich, 1982]. The teachers, anyway, play the important role of helping each child to accomplish the strategy he has chosen.

As regards problem P2, many children usually operate in the following way. *"First I tried with 1000; I repeated it 13 times, it added up to 13000, which was not enough. Then I tried with 2000; I repeated it 13 times, it added up to 26000, which was not enough. Then I tried with 3000, I repeated it 13 times, it added up to 39000, which was nearly enough."* Note that the answer *"a little more than 3000 Lire"* was considered acceptable. The approach by "trial and error" described here seems to have elementary reference models which differ from P1 and to be related to some sort of "fair sharing" procedures which are present in the experiences of the children. Other children proceed instead by representing the 40000 Lire in a drawing and dividing them among the thirteen lessons.

In grade 3 (8 to 9 years) the children deal with division problems which are generally related to actual problem situations and whose numerical complexity is then not strictly determined by the teachers, but who exploit, however, all the occasions which arise to encourage the passage to more efficient strategies.

> (P3) (grade 3, end of November) "To make a pizza for 20 children we have spent 18000 Lire. How much must each child pay?"

To solve problem P3 the children first try with 1000 and realize that it is too much and, in most cases, it is not too difficult for them to find that 900 Lire is right at the second attempt. The numbers involved do not stimulate the child, in this case, to find a more efficient approach to the problem.

In the following problem we can see how a particular situation, even in its relation to the experiences of the children, can affect the choice of a strategy:

> (P4) (grade 3, December) "I would like to invite some fellow-pupils home. I would like to buy some tarts costing 400 Lire each. How many tarts can I buy for 8500 Lire? If there are eight of us, how many tarts can each child eat?"

On the first question of problem P4 we have noted both the trial-and-error approach, as in P1, and a sort of "fill-up" strategy on the basis of which numerous trials are made to be then added up. In this latter case the children realize that 10 tarts cost 4000 Lire and therefore that 20 (= 10 + 10) tarts cost 8000 (= 4000 + 4000) and that if one further tart is added the total number becomes 21, and at this point they note they have no further possibility of buying more tarts Compared with the previous approach, the relation between the different trials in this strategy is more explicit, and for this reason it is encouraged, at this stage, by the teachers.

As regards the second question, a clear preference for the drawing-based distribution strategy was observed. In several classes, even, all the children, including those with a good mastery of mental arithmetic and possessing more efficient techniques, made use of this approach. (see **4**)

The examples that follow show how the numerical data affect the ability of the children to accomplish trial-and-error strategies and contribute to stimulating them to find more efficient approaches.

> (P5) (grade 3, November) "To make a cake we have spent 24800 Lire on ingredients. How much must each pay if there are 17 of us?"
> (P6) (grade 3, February) "We have made a plum-cake and calculated that the total expense is 13200 Lire. If we cut the plum-cake into 20 slices, how much will each slice cost us?"

Both problems require a quotient to be calculated with more than one significant figure.

This creates difficulties for the lower-level children. To solve P6, for example, proceeding by the trial-and-error method, they can calculate 600 as partial quotient, and thereafter, very often, they come to a stop In this case the teacher directs them towards the explicit calculation of the remainder. A slight difference was observed between the two problems in the development of the trial-and-error strategies performed by these children To solve P6 the divisor 20 allows trials to be carried out which, even though related to the mastery of multiplication, still preserve a strong tie with hierarchical approximation procedures In P5, since the divisor has two significant figures, this relationship appears less important in the sequence of multiplications

> 1000 * 17 = 17000; 2000 * 17 = 34000;
> 1500 * 17 = 25500; . . .

With respect to both of these problems, filling-up strategies such as the one already described for problem P4 were also noted. Moreover, some children proceed to gradually "empty" the dividend by calculating the remainder This approach is reinforced by the teachers

After 3-4 months of work along the lines already indicated, a considerable number of children are able to deal with problems by employing strategies which refer to meanings that may differ from those suggested by the text of the problem. Let us examine some examples in this sense:

> (P7) "A magazine which today costs 3300 Lire cost 445 Lire in 1979. How many times higher is the price than in 1979?"
> (P8) "I measured my shadow, which is 603 cm long I am 132 cm high. How many times longer is my shadow than my height?"
> (P9) "I would like to depict 80 equal parts on a wall 430 cm long, and use all the available space. How long will each part be?"

P7 is a classic measurement problem with the characteristic, which cannot be ignored from the point of view of children's mental representations, of asking for a comparison of "measurements" associated to different periods. Many children are able to use a multiplication type model:

> "445 * 10 = 4450, too much; 445 * 5 = 2225, too little; . . ."

Some brighter children were already able to use a similar model two months before. Initially this model was found only in partition-type situations.

As regards P9, which is a partition problem, many children reason in this way: *"80 by 1 cm equals 80 cm, plus another 80 equals 160 . . ."* until they determine that *". . 5 cm represents each year and 30 cm remain to be divided."* They use a "fill-up" approach typical of measurement problems, even if for the remainder they again seem to refer to a partition model A similar strategy was used by the same children almost one month before to solve P8, which is a measurement problem

3.2 Summary of the children's behaviour

We can briefly list the different strategies observed in the solution of division problems:

(1) *manipulative distribution strategies* (only in partition problems). The dividend is shared, with the help of concrete materials or a drawing, among the units of the divisor (which is always an integral number), one or more elements at a time. This strategy broadly corresponds to the first approach listed in [Weiland, 1985]

(2) *distinct trial-and-error strategies* (in partition and measurement problems): the child carries out numerous attempts to get steadily closer to the required quotient. It should be noted that each attempt depends only qualitatively on the previous ones for the child realizes whether the attempt gives rise to a number greater or smaller than the dividend, but does not calculate the remainder, at least initially However it very often seems that the choice of the successive attempts takes into consideration the approximation that has been obtained In a second stage, especially confronted with numerically complex problems, like P5 and P6, several children integrate this approach with the explicit calculation of the remainder (above all if the teacher fosters strategies which are oriented in the right direction). This strategy presents two possibilities according to the situation which arises. In measurement problems (2a) the children build, by repeated addition, a sequence of multiples of the divisor by a procedure in which the approximation by defect seems to play a major role. In these cases the strategy is widely performed also by children who have little mastery of multiplication

In partition problems (2b) some children seem to relate to a multiplication-type model, attempting to find the solution to an equation of the type $d * x = D$. In some particular partition situations there are children who do not seem to interpret multiplication as repeated addition. In general they have the mastery of multiplication as an independent algebraic operation, while the children who still identify it with repeated addition find more difficulties than in measurement situations.

It should be observed that this latter use of trial-and-error strategies in partition type problems has been found especially in those "observation classes" in which multiplication has been introduced (in terms of methods of writing down, etc.) as an independent operation.

(3) *convergent series of decreasing trials* (as in the second approach illustrated for problems P4 and P5): the child "fills-up" the dividend by repeatedly summing up the multiples of the divisor (for powers of 10 or other numbers that can be easily handled). For each step, the child calculates (in some cases in the mind) the partial sum, but does not explicitly calculate the remainder. It is obviously an addition-type strategy, even if procedurally it differs from 2a. This strategy is influenced by the skills built up with the work on money, and, in particular, by the mastery of the distributive law for * over +.

(4) *repeated subtractions* (third strategy illustrated for problem P5): the child operates on a trial-and-error basis, but calculates the remainder each time, utilises it as the new dividend for the subsequent trial, and sums up the various partial quotients. This strategy approximates more closely the division algorithms which organize the progressive and explicit emptying of the dividend [Teule-Sensacq and Vinrich, 1982; Laing and Meyer, 1982].

Concerning these strategies we have observed what follows (even if the children's behaviour is almost never the same):

— The children gradually abandon strategy 1 as not very efficient and, anyway, only applicable to problems with very small numerical data. It is employed for a longer period by those children who do not master well the base-ten structure and mental arithmetic. Many children however can only apply it, initially, to problems with very simple data, whilst in other problems some intervention by the teacher is required to fill the gap between their embryonic intuitions and the specific difficulties of the situation.

In particular situations (distribution of foodstuffs, cakes or pizzas, among the children, as in the second question of P5) many children return to adopting this strategy. It is probable that the particular situation (including the numerical values involved) suggests a fair-sharing procedure (maybe related to children's games). We have observed as well how this strategy is used even by children clever at calculating and able to employ more advanced strategies. (see 4)

— We have not observed any convincing correlation between the children's level of understanding and their choice among strategies 2, 3 and 4. The only information which emerges is a slight preference of lower-level children for strategy 2a in measurement problems: nevertheless, during the year, many children will pass from one strategy to another.

— The progressive (but not complete) abandonment of the type 2 strategies was noted in the course of grade 3 (8 to 9 year olds), above all when dealing with problems involving calculations with many significant figures (as P5), thus requiring an accurate management of the various attempts. The abandonment of such strategies was more manifest in lower-level children. Some children clever at mental arithmetic went on using them occasionally until the end of the school year. The abandonment was supported and encouraged by the teachers, both through the presentation of "critical" problems such as P5 or P6, and the construction of the means (for example, linguistic means) or of the attitudes (for example, reflection on the adopted strategies and their complexity) which could favour the use of different strategies. Strategy 3 can spread if fostered by the teacher.

— The presentation of problems with more complex numerical data (for example, with 3 significant figures in the quotient) leads to a wider use of strategy 4. This increased use was encouraged by the teachers who pointed out the extended efficiency of this strategy. The spontaneous display of this strategy was made easier by the particular numerical values involved. It can gradually spread through the class if the teacher fosters its comparison with other strategies and makes its characteristics of profitability and universality gradually emerge through numerous problems and specific reasonings oriented by questions such as "which way rapidly provides the results in all the cases considered up until now?" On the basis of systematic observations made in recent years in our "observation classes" it seems that over 65% of the children at the age of 9 are able, in the context of our project, to participate actively in and be aware of such comparisons and reflections.

— The specific problem situation, and the numerical values involved, in the framework of both measurement and partition division, seems anyway to influence the algebraic models to which the children refer. Moreover, in all the trial-and-error strategies in problems related to economics, a considerable influence of the work carried out in grades 1 and 2 with money has been noticed: the trials are almost always performed using numbers (like 100, 200 or 500) which correspond to the coins actually existing in Italy. These skills are frequently transferred also to problems of other kinds.

— As regards the final results of the work, we have performed the same assessment at the beginning of grade 5 (10 year olds) in our classes, and for 11 year olds (in the first year of the comprehensive school) in classes which had not participated in our project. The assessment consisted in finding the solution to the problem of sharing out an expense of 9520 Lire between 17 persons, choosing the operation and performing the calculation. In "our" classes (480 pupils) 96% of the children chose division and only 5% of these obtained an incorrect result. Out of 768 children at the age of 11 unaware of our project, only 84% chose division (the others, in general, either did not reply or chose multiplication) and, out of those who chose division, about 17% of the total performed the calculation incorrectly.

3.4 Some considerations on children's behaviour

Some general conclusions can be drawn, in our opinion, from the results exemplified in the previous paragraph.

First of all, the children do not initially recognize the relationship between measurement and partition problems; on the other hand, trial-and-error procedures, typical of measurement problems when we have to calculate how many times a measure A goes into another B where B is much bigger than A, accustoms children to approach the dividend (which they regard as a sort of container) with successive approximations (which gradually familiarizes them with the idea that, by increasing the number of times one adds up the divisor, one approaches the dividend and then exceeds it ...).

Furthermore, the proposal of partition problems without pre-arranged numerical values but which refer to fields of experience which can be mastered by the children (problems of the sharing of costs, ...) contributes to forcing the adoption of trial-and-error strategies which can support the "fair sharing" strategies.

As remarked in a previous paragraph, the trial-and-error strategies used in partition problems do not correspond to a unique conceptual model. Multiplication seems to be used by a few children as the inverse operation of division and by others as repeated addition (according to Teule-Sensacq and Vinrich [1982], the former involves an M-type strategy, and the latter an A-type strategy). In measurement situations the corresponding strategy 2a is used above all by children with a low level of understanding.

We have already seen that the children tend to mix, in an interesting way (in cognitive terms), calculation strategies elaborated in different problem situations. This happens, in particular, when in the same problem different division models are used at different stages (as in P8), or when some pupils recognize as "doing division" procedures carried out by themselves using other operations. For example, in grade 4 (9-10 years), when dealing with a problem which involved the calculation of "how many times does 38 cm go into 420 cm?" most lower-level children, after having answered by using an additive-type strategy, recognized without hesitation that they had performed a division. This phenomenon is surely of considerable importance because it is a relevant step towards a full awareness of the relationship between the different meanings of division and the algorithms. The children with higher autonomy can early on master different conceptual models.

Our work shows, moreover, that our choice of not introducing written algorithms early and directly, but of starting from the (informal) strategies built by the children and orienting them progressively towards more efficient but always meaningful strategies, has the following results:

— the pupils are put in the condition of penetrating even the less accessible meanings (to common sense) of the operations and, in particular, of division. Furthermore, the lower-level children, even if they go on using addition-type strategies, can relate with self-assurance these strategies to the idea of division.
— the discovery that the same mathematical model may fit substantially different situations can be achieved before reaching the complete mastery of that model through the finding of analogies and invariant procedures, the progressive explicitation of algebraic relations between the four arithmetical operations and the construction of the necessary linguistic means
— the pupils grasp with relative ease the "sicilian division algorithm," which is the natural conclusion of the activities illustrated above, endowing it with the meanings they have called up. This method, unlike the standard ones, allows the children, when performing it, to keep an awareness of the problem situation, that is, to operate by taking meanings into consideration rather than in a mechanical way. A comparative analysis between our classes and classes where the standard algorithm has been directly taught has shown that the effectiveness of the "sicilian algorithm" is not much lower than the standard one as regards the speed and the accuracy of the execution, and, on the contrary, much higher as regards the checking of results and the percentage of correct executions
— the pupils have the possibility of discovering many number facts and regularities (also related to operations) by referring to the meanings as they occur. For example, the relationships between the four arithmetical operations, which are particularly highlighted in our approach, the mastery of the order of magnitude and of multiplications by powers of 10, and the transition to division of decimal numbers achieved in the final grade of primary school, at 11 years).

4. Open questions

We would like to briefly sketch some questions which the experiments conducted in the classroom have raised.

A first question concerns the role of mental arithmetic strtategies in the development of the written calculation methods. It seems to us that the children who are more confident in carrying out mental arithmetic calculations of average complexity (as, for example, 13 * 21 = 13 * 20 + 13 = ...) are those who are able to produce effective written calculation strategies even for problems of considerable complexity. Nevertheless, it is still unclear whether the development of mental arithmetic is an effect of or a prerequisite for the development of the skill of planning written calculation strategies.

A second question concerns the fact that the development of hypothetical reasoning in (mathematical and non-mathematical) problem-solving seems to go hand-in-hand with the pupil's increasing self-confidence in the development of calculation strategies for division problems. The link between the two kinds of ability is still unclear to us [see Ferrari, 1989].

Finally, the fact that most children return to particular, and seemingly less evolved strategies than the ones they have already mastered, when facing division problems of particular conceptual difficulty, seems to us very interesting. It is not easy, nonetheless, to interpret this phenomenon, which could, at first sight, make us believe in the existence of deep primitive models for calculation strategies, strictly related to the meanings of the operations and therefore called up when these meanings are fully utilized in "difficult" problems.

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