MODES OF ALGEBRAIC COMMUNICATION: MOVING FROM SPREADSHEETS TO STANDARD NOTATION

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Many students seem to find it difficult to learn to express generalisations in standard algebraic notations. One approach to making the acquisition of these skills easier for students is to use the environment of the spreadsheet as a means of semiotic mediation for them in forming meanings for algebraic symbolism.

In this article, we consider the responses of a number of students to a task that set out to offer this opportunity. Drawing on video and audio recordings of one pair of twelve- year-old students as they worked in the lessons and on the written work produced by the whole class, we examine some of the differences in the way students move to the use of standard notation to express generality. Our analysis sets out ways in which we see the purpose of the tasks and the interventions of the teacher influencing the meanings constructed by the students.

The need for algebra

The recent history of research and development in early algebra teaching can be seen as a struggle to create pedagogic situations in which students can construct meaning for standard algebraic notation.

There is a large body of work from around a decade ago focusing on individual cognition and students' cognitive difficulties rather than on aspects of task design and task setting to facilitate meaning (*e.g.*, MacGregor and Stacey, 1993; Rico *et al.*, 1996; Sasman *et al.*, 1999). The common feature of the analysis in these three studies is that the authors recognise that the task of expressing the generalisation in algebraic symbols is not meaningful to the students.

Arzarello *et al.* (1994) point to a similar gap in meaning when they talk about the importance, in solving more advanced problems, of choosing a suitable representation for the variable(s) in the problem. For example, if we seek to prove that the sum of two consecutive odd integers is a multiple of four, then a helpful choice is to represent the two integers as $2n + 1$ and $2n + 3$, rather than, say, x and y. Using a semiotic analysis, the authors account for teachers' recognition of a choice of representation, whilst students often do not experience such a choice. Rather, they see the single letters as "rigid designators" (Arzarello *et al.*, 1994). For these students, the representation of a variable by a letter has become meaningful, but they are not yet ready to attach a meaning to the manipulation of symbols to produce expressions as representations.

These studies point to the need for students first to be able to construct a meaning for variable, in these cases variable as denoting the position of a term in a sequence. Radford (2003) begins to focus more clearly on the way in which meaning can be constructed for algebraic symbols in the socio-cultural setting of the mathematics classroom. He suggests how such construction can take place by processes of objectification, through gesture and words. He distinguishes between *factual generalisations* and *contextual generalisations*. The former is a generalisation of actions but "remains bound to the concrete level" (p. 47), whereas the latter deals with objects that are abstract rather than concrete, but contextually bound. Radford's analysis shows how the development of the factual generalisation relies on the use of language of spatial positioning ("the next") and of temporal language ("always") in conjunction with gesture and rhythm. The move to contextual generalisation is driven by the need to communicate beyond the group:

> Implicit and mutual agreements of face-to-face interaction (e.g., gestures, clue words) need to be replaced by objective elements of social understanding demanding a deeper degree of clarity. (p. 50)

The fourteen-year-old students in Radford's example finally use standard algebraic symbols to express a general term as $(n + n) + 1$, and also as $n + (n + 1)$. However, the students do not recognise these two expressions as denoting the same mathematical object. Radford argues that the students are using the symbols as indexes, and that these expressions objectify two different sets of actions. The symbols are therefore not yet manipulable. The idea of two expressions with the same denotation but different senses is not yet available. In other words, the students are using n and $n + 1$ as *rigid designators* in very much the same way as Arzarello's students.

We will examine how the combination of the task setting and the spreadsheet environment offered opportunities for pupils to move to expressions of generality, and to give meaning to the manipulation of these expressions. In doing so, we will look for what the spreadsheet environment can add to the set of tools which Radford sees as available to students.

Meanings for variable in the spreadsheet environment

The potential of the spreadsheet to provide a medium for introduction of algebraic symbolism has been recognised for some time. Research approaches to students' use of spreadsheets broadly fall into two categories. On the one hand we might see spreadsheets as a pedagogic tool for teaching alge-

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braic ideas (Dettori *et al.*, 2001), or we might ask simply what meanings students are constructing for their work on spreadsheets. Although we have analysed data from our project in both of these ways, the emphasis here is on the contribution of spreadsheet work to the understanding of standard notation. We draw on Ursini and Trigueros' (2001) categorization as a way of articulating one distinction which has become apparent within our analysis.

The spreadsheet offers a strong visual image of the cell as a container into which numbers can be placed. The meaning for variable this image seems likely to support is that of a *placeholder for general number* (Ursini and Trigueros, 2001), implying that students are able to:

> interpret a symbol as representing a general, indeterminate entity that can assume any value, and symbolise general statements, rules or methods. (p. 336)

Haspekian (2003) refers to this use of the spreadsheet as a "variable cell" and claims that the cell argument (*e.g.*, A2) has four important features, only the first of which is shared with the algebraic use of a letter to stand for a variable. The cell argument is

- *an abstract, general reference*: it represents the variable (indeed, the formula does refer to it, making it play the role of variable)
- *a particular concrete reference*: it is here a number (even if nothing is edited there, some spreadsheets attribute the value 0)
- *a geographic reference* (it is a spatial address on the sheet)
- *a material reference* (it is a compartment of the grid, some pupils can see it as a box) (p. 6)

These aspects of the cell reference give it potential as a semiotic mediator beyond those of the standard notation. They offer possibilities of deixis (verbal pointing using, for example, 'that' or 'this') and gesture, which, according to Radford's analysis, can enable objectification to take place. The different meanings of the cell reference also offer a powerful ambiguity in which the cell reference is used to name both the physical location of a cell in a column and row, and the information that the cell may contain. The spreadsheet thus offers a strong visual image of the cell as a number container whose contents can be changed.

The spreadsheet also supports another visual metaphor by allowing the 'filling down' of a column using a formula, giving rise to a "variable column" (Haspekian, 2003, p. 6). This image is likely to support the idea of variable as a *range of numbers in functional relationships* (Ursini and Trigueros, 2001).

The Purposeful Algebraic Activity Project

The data reported on here was collected as part of the *Purposeful Algebraic Activity Project* [1], a three year longitudinal study in which we designed and used a series of algebraic tasks, based on the use of spreadsheets, with 11-13 year olds in two secondary schools.

Design of tasks

This project takes up the challenge set by Sutherland (1991) to create "a school algebra culture in which pupils find a need for algebraic symbolism" (p. 46) through the use of a framework of five design principles. The tasks are designed to provide opportunities for students to use algebraic notation and ideas in purposeful ways. The five design principles are

> *Balance between different types of algebraic activity*: Kieran (1996) describes three kinds of activities within the scope of school algebra; generational activities, transformational activities and global, meta-level activities. Within our task design we have attempted to achieve a balance of all three sorts of activities within meaningful contexts.
>
> *The continuum from arithmetic to algebra*: in designing tasks we have aimed specifically to exploit students' fluency and familiarity with arithmetic structures in expressing generality through algebraic notation.
>
> *The spreadsheet as an algebraic environment*: in the tasks we have designed, we aimed to deliberately exploit the algebraic features of the spreadsheet as we have described them above.
>
> *Purpose*: Ainley *et al.* (2006) define a *purposeful* task as one which has a meaningful outcome for the learner, in terms of an actual or virtual product, the solution of an engaging problem, or an argument or justification for a point of view. The authors explain how, as a principle for the design of pedagogic tasks, the idea of *purpose* responds to the literature on situated cognition, and how it relates to the constructionist movement and to Realistic Mathematics Education (RME), which places an emphasis on offering students a context problem: a "problem situation that is experientially real to [them]" (Gravemeijer and Doorman, 1999, p. 111). While RME places emphasis on the problem context, the situated cognition literature has drawn attention to the sense of purpose characterising out-of-school mathematical activities in contrast with typical classroom mathematics activity, because of the importance of the outcome.
>
> However, task design that attempts to reproduce authentic mathematical activity in the classroom is beset by many problems. Our principle of purpose suggests rather that tasks should have the potential to be purposeful *within the classroom*. In this respect, it has something in common with the constructionist movement, which proposes that tasks in which students make products (usually virtual products such as computer programmes or routines) are particularly conducive to learning (Harel and Papert, 1991).
>
> *Utility*: alongside purpose, we also use Ainley *et al.*'s (2006) second construct of *utility* as a design principle. Understanding the utility of a mathematical idea is defined as knowing how, when and why that idea is useful. Whilst engaged in a purposeful task, learners may learn to use a particular mathematical idea in ways that allow them to appreciate its utility by applying it in

that purposeful context. We believe that opportunities to understand utility can be provided through purposeful tasks. In particular, we aim to provide opportunities for understanding the utility of algebraic notation for generating data, finding the value of an unknown, showing structure, and explaining particular results or relationships.

The fairground game

We looked at lessons that took place as part of the teaching programme through field notes, audio-, video- and screen-recordings for a targeted pair of students and students' written work. Here we focus on one high-attaining group whilst they were working on a task which we call *Fairground Game*. The context is a game for a school fair in which players try to make a target number (61 in the solution in Figure 1) by placing the numbers 1 to 5 in the cells in the left-hand column:

	A	B	C	D	E
1	1				
2	3	4			
3	5	8	12		
4	4	9	17	29	
5	2	6	15	32	61

Figure 1: A completed game.

The target number is set by the stall holder. Once any player has achieved the target, the stall holder must change the set of five starting numbers and set a new target. The overall purpose of the task is for students to be able to advise the stall holder on what the target number should be for any particular set of five starting numbers. The task involves students in identifying the 'rule' that is used to calculate each number in the grid, setting up their own spreadsheet to play the game and finding a strategy for arranging the numbers to get the highest total.

Students use spreadsheet notation to express the rule for calculating the grid but the nature of the spreadsheet formulae disguises the underlying structure of the problem. The need to identify this underlying structure is driven by the purpose of providing advice for the stall holder. Students are therefore encouraged to move away from the computer and use a more standard algebraic notation to represent the numbers, combining and simplifying expressions to make this structure more transparent.

Jordan and Jacob

The pair of students observed we call Jordan and Jacob. They had more difficulty with the task than most of their peers and needed to ask their teacher to explain the context of the task to them again. In common with the vast majority of their classmates, Jordan and Jacob had no difficulty in entering appropriate formulae into the spreadsheet (see Figure 2) once they had realised that it would be helpful:

	A	B	C	D	E
1	1				
2	2	=A1+A2			
3	3	=A2+A3	=B2+B3		
4	4	=A3+A4	=B3+B4	=C3+C4	
5	5	=A4+A5	=B4+B5	=C4+C5	=D4+D5

Figure 2: Jordan and Jacob's first spreadsheet.

Once these formulae were set up, Jordan used the grid to experiment with placing the numbers 7, 9, 10, 12 and 15 (a set of numbers chosen by the class at the teacher's invitation) in a variety of orders. He managed to match the highest total found by other members of the class, but there is no evidence from his work that he was being systematic in his choice of how to place the numbers or was making conjectures about why the successful ordering gave him the highest possible total.

The second lesson on this activity took place in a traditional classroom where computers were not available. Earlier trials of the activity had suggested that time away from the computers was a useful way of focusing the students' attention on structure rather than further generation of examples.

Early in the lesson Jordan explained to Jacob a general strategy for getting the highest total from five numbers using 8, 15, 20, 38 and 91 as an example (see [2] for transcript conventions):

> So look, you put the "↑ three biggest ones in the middle like this, twenty, look (.......) Ninety-one, thirty-eight, that's the three biggest numbers then you put the smallest numbers which are fifteen and eight at the bottom and the top, that will make a big number.

A little later the researcher (R) asked the boys to explain why this strategy will always produce the highest total.

Jordan (Jo): Because if they're in the middle then they go into two blocks there (.) no (..) If the smallest ones are there then that will just go into that one

R: What do you mean that will "go into that?

Jo: No, that and that will go into that (*points to the 1 and the 2, and then to the cell adjacent to the 2; see Figure 3*)

R: Right

Jo: But if it's in the middle then that, "that will go into that "and that (.)

At this point there was a general encouragement from the teacher for students to "try using letters" to explain why this

	A	B	C	D	E
1	1				
2	2	3			
3	3	5	8		
4	4	7	12	20	
5	5	9	16	28	48

Figure 3: Jordan and Jacob's spreadsheet as it stood during the conversation with the researcher

ordering strategy always gives the highest total. Jordan and Jacob followed the suggestion of placing the letters *a*, *b*, *c*, *d* and *e* in order to represent the five numbers and after some initial reassurance (Jo: Is it *a* plus *b*?; R: Yeah because you add that number plus that number so it would be a plus *b*) they completed the grid, making just one mistake in the final cell, which they give as $a + 4b + 4c + 4d + e$, as shown in Figure 4.

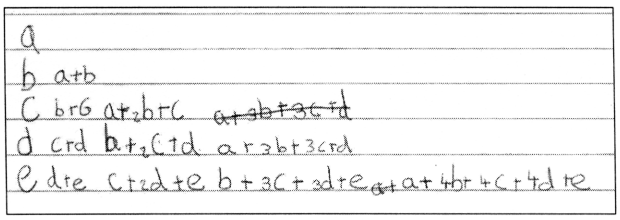

Figure 4: Jacob's written work.

Challenged to extend their result to a six number competition the boys chose six specific numbers and completed the grid using these numbers but without completing any calculations (see Figure 5).

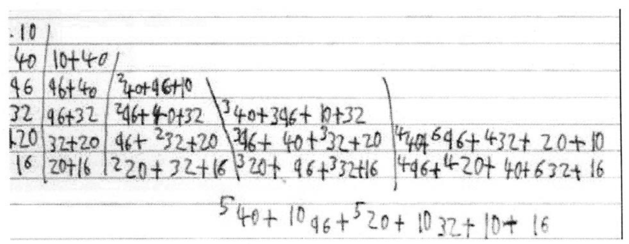

Figure 5: Jordan's written work.

At the end of their conversation with the researcher the two students seem to have some sense that there is something significant in the number of times that each of the five starting numbers contributes to the final total. However, their explanation, and possibly their understanding, is very much shaped by the physical structure of the problem as it was presented to them and constrained by the spreadsheet structure of cells in rows and columns. In their explanation (as opposed to their statement of the solution) they remain at Radford's factual-generalisation level, using verbal and spatial deixis (*e.g.*, 'that goes into that').

The next stage of the work, to "use letters" as suggested by their teacher, has the potential to free them from the geometrical structure within which the task is defined. Expressing the variables as letters offers the opportunity to produce and interpret an expression for the final total that does not depend, for its structure, on the geometrical structure of the problem. The first stages of using standard notation to represent the situation (for example, filling in $a + b$ and $b + c$ in the second column of the table) was frequently seen by pupils as linking directly to the visual structure of the problem, and to their verbalisations of that structure in terms of numbers "going into" others or "travelling". However, the expression for the final total was some way removed from this structure, since as they simplified their working, the series of additions that went to make it up became invisible in the final form.

It is not clear from the information we have how much Jordan and Jacob were able to appreciate this potential. They did not notice the mistake in their expression for the final total ($a + 4b + 4c + 4d + e$), and chose to return to using specific numbers in order to address the extension to six numbers. Both of these actions suggest that they have not fully seen the utility of producing the algebraic expression. However, in their work on the six-number problem they do not revert to their first strategy of calculating the totals for a variety of arrangements, but move straight to demonstrating the structure of the final total by leaving it expressed in terms of its addends. The geometrical structure of the problem provides a powerful structuring for both their thinking and talking about the justification for their strategy, and for their interpretation of the production of the expression for the final total.

Other students

The general invitation to the class to "use letters" left open the scope to choose to use them in a variety of ways. A majority of students chose to use the letters in a way similar to Jacob's.

For example, Darren used letters at an early stage, filling the left hand column in the order *a*, *b*, *c*, *d*, *e*. He wrote "to get the biggest number the order must be *c* biggest, *b* and *d* next biggest, *a* and *e* smallest". After demonstrating his point with two examples, he wrote, "Using algebra helped me because we know which number appeared the most times". Darren's partner Edward added,

> To get the answer you add together the first number, add 4 of the second, add 6 of the third, add 4 of the fourth and add the fifth. The most numbers that turn up is *c* so the biggest number has to be *c*.

Faith began with a general description of how to place five numbers to get the highest total. She used numerical examples to illustrate, and wrote:

> The reason my strategy works is with the largest numbers together they add up to a bigger number than smaller numbers would, and smaller numbers get added with larger numbers which is bigger than the small and the small added together.

Challenged to find a strategy for six numbers, she went immediately to algebra, setting out *a*, *b*, *c*, *d*, *e*, *f* in the first column and completing the grid without error to finish with $10c + 10d + a + 5e + 5b + f$ in the final cell. She wrote, "*c* and *d* appear most often so that is where the largest numbers should be placed" and finally gave a numerical example.

By contrast, Elizabeth filled the first column of her grid in the order *a*, *c*, *e*, *d*, *b*, and afterwards wrote beside it $1 = a$, $2 = b$, $3 = c$, $4 = d$, $5 = e$.

Karen began by giving several numerical examples, then wrote

> I have worked out a strategy to get the biggest number for a sequence. I have put the biggest number in the middle because it is used the most. (…) I will back this up by trying it with A-E.

She then completed a grid similar to those drawn by other students, starting with *a*, *b*, *c*, *d*, *e* in the first column and ending with $a + 4b + 6c + 4d + e$ in the final cell. Afterwards she added a column to the left of the first one, headed v = value, and containing the entries 1, 2, 3, 4, 5. Underneath her final cell she wrote 1 + 8 + 18 + 16 + 5 = value = 48. Following this she set up another grid, again with a column headed v on the left, but this time with the letters in the order a, c, e, d, b. Underneath her final cell entry of $a + 4c + 6e + 4d + b$, she wrote 1 + 12 + 30 + 16 + 2 = value = 61 and concluded, "Algebra confirms that my strategy is correct. Algebra helps us".

We want to suggest that the different ways in which students used letters to pursue their thinking about this problem represent, for different students, quite different relationships between the semiotic systems available to them and used by them. Students, who like Darren and Faith use the letters a, b, c, d, e to stand for numbers in the first, second, third, fourth and fifth cells, are using the notion of location to guide their representation. For them, a represents the number in a particular location, or possibly the location itself. This is the interpretation that the task designers and teachers intended, and one making use of the ambiguous nature of the cell reference, offering metaphoric possibilities for the spreadsheet cell to stand for a variable in a way that can later be represented by a letter.

On the other hand, students who, like Elizabeth and (in her later work) Karen, order the letters used according to size rather than location (for example, by placing them in the order a, c, e, d, b) are linking each letter with a number rather than a location. In this case, there is evidence that most students were not linking the letters to particular numbers, but understood them as standing for, for example, "the biggest number of the five, whatever that is". For these students the links established by ordering of size (a smallest to e biggest or *vice versa*) were stronger than the metaphor of location offered by the spreadsheet.

The task itself, as presented on this occasion, offered opportunities for students to establish both types of correspondence between letters as signifiers and the signified. In the first place, the spreadsheet was used to represent a physical situation and the language of spatial metaphors was used in describing the task itself. Competitors in the fairground game were invited to "place the numbers in the cells". This context encouraged students to understand the spreadsheet cells as number containers. As students moved from constructing the spreadsheet for themselves and experimenting with placing five given numbers in different orders in the starting cells to using different sets of five numbers to establish a 'rule', attention focused on the relative size of the five numbers, and on placing the largest number in the right location. Hence, students are encouraged to attend to the positioning of a number within the set (whether it is, for example, the largest or smallest) rather than its absolute value or its physical location in a particular cell.

It was then suggested to students that they might use letters to stand for numbers in order to help them understand and explain why their 'rule' for positioning the numbers would give the highest total. It seems that a focus on location (which cell) and on position (largest or smallest) weighed differently for different students at the moments when they set out to use letters in this way.

The purpose of using letters in this task was, for most students, one of describing, explaining or confirming what they already knew about why the numbers had to be positioned in a particular way to achieve the highest total. For these purposes, both of the main systems chosen by students to represent the problem using letters were adequate. The students undertaking this task were not, in the majority, sufficiently fluent in using algebra as a tool for thinking to choose to use it in order to decide on the strategy necessary to achieve the highest total. Rather they were confirming the strategy they had already established empirically and by informal reasoning (as, for example, in Jordan and Jacob's explanation above). This is important in the context of looking at students' chosen representations. We have argued that it was the process of establishing the 'rule' by investigation of different numerical cases, which offered the possibility of using letters to represent numbers in certain positions (largest, smallest) rather than certain locations (top cell, bottom cell). Without the knowledge that position was important in this problem, such a representation would not be available.

Reflections

We have argued elsewhere (Ainley *et al.*, 2004) that the relationship between the 'purpose' of the activity for students and the meanings of variable available within the task is crucial in enabling students to access the utility of the algebraic process involved. In the *Fairground Game* task the purpose of justifying their method for producing the highest possible final total, and of understanding how to calculate what this total will be in the general case, provides the framework within which students can access the utility of the general expression for the final total. What emerges from this analysis is the subtle relationship between task design, including purpose, and the meanings for variable that students actually construct. Although nearly all of the students whose work is reported above (and also nearly all of the students in the class) were able to access this utility and to describe it in words, the precise meanings that they constructed for the variables varied considerably.

In the task reported by Radford (2003) (as well as the majority of those considered in the other studies cited in the first section of this article), the expression of the general term in standard notation was the end point of the task. Students generalised their findings verbally before expressing essentially the 'same' generalisation in standard notation. In *Fairground Game* the purpose of using standard notation was not to express a generalisation which had already been established, but to provide an alternative means of expressing relationships, which would enable the students to justify their informal rules and establish the arithmetic connection between the five starting numbers and the final total. It was often the case, then, that students' *contextual generalisations* came after their use of standard notation, for example when Edward finishes his work by stating,

> To get the answer you add together the first number, add 4 of the second, add 6 of the third, add 4 of the fourth and add the fifth.

Written responses such as Edward's, Faith's and Karen's, which make interpretations of the final expression in terms of the task of setting a target number, give a clear indication that students had established a meaning for the simple transformational activity they had undertaken. They offer the beginnings of evidence that, for these students, the letters they have used are not just "indexes" (Radford, 2003), or "rigid indicators"' (Arzarello, *et al.*, 1994) but are available for use in operations with meaningful outcomes.

In making this claim, we recognise that the students working on *Fairground Game* are using letters as placeholders for general numbers (Ursini and Trigeros, 2001), rather than as a range of numbers in functional relationships (*ibid.*), as is the case for Radford's students. We also acknowledge that our evidence falls a long way short of suggesting that students made a conscious choice of the best form of representation, as did the teachers in Arzarello *et al.*'s study. Nevertheless, we argue that the design of the task offered the opportunity for students to appreciate the utility of transforming an algebraic expression, and that it is this aspect of the task setting that provides access to a meaning for the transformation of the algebraic expressions.

The strongly visual structure for the setting of this task provided a framework for students' thinking and explanations and enabled them to move both to generalising in natural language and to using transformational algebra to express the structure of the final total in standard notation. However, there is some evidence that the strength of this structure inhibited students' ability to see the utility of the algebraic expression that did not rely on the structure.

We began this article by talking about the research agenda for creating a "need for algebra" and described our project as responding to Sutherland's (1991) call for "a school algebra culture in which pupils find a need for algebraic symbolism" (p. 46). Brown and Coles (1999; Coles and Brown, 2001) have also responded to this challenge aiming to "develop a school algebra culture in which students find a need for algebraic symbolism to express and explore their mathematical ideas" (Coles and Brown, 2001, p. 122). Their approach focuses on the role of the teacher in establishing this culture where students have a need to express their "awarenesses within complex situations" (1999, p. 154). The choice of tasks was seen as secondary. In both cases, however, the role of purpose in the students' work was seen as vital, not just as a form of motivation but as material for the construction of meaning for variables and for algebraic notation and expression.

In conclusion, we suggest that:

- use of the principles of 'purpose' and 'utility' in task design can provide learners with opportunities to construct meanings for variables, for their representation by letters and for manipulation of these letters.
- the spreadsheet environment offers facilities (specifically here the variable cell) that learners can use to develop generalisations.
- details of task design and structure can be highly influential in the kind of meanings that students construct.

Notes

[1] The *Purposeful Algebraic Activity Project* is funded by the Economic and Social Research Council, R000239375.

[2] In transcripts the following conventions are used:
(laughs) brackets enclose description of relevant non-linguistic communication
[ ] encloses transcriber or situational comment
(…) timed pause (number of dots corresponds approximately to number of seconds
" precedes emphatically-stressed syllable
↑↓ pitch deviation

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