TRANSITIONAL DEVICES

ALF COLES

Martin Hughes offered the following suggestion in the conclusion to *Children and Number*:

> our understanding of learning and teaching mathematics might well be enhanced if we can identify those images and analogies which are particularly useful in connecting the formal and the concrete—and conversely those which are not. (Hughes, 1986, p. 171)

In this article, I make a suggestion of one "useful" image and analyse some student work arising from activities using the image. Before getting to this, however, it is necessary to unpick some of what might be meant by the "formal" and the "concrete", and connections, or "translations" (Hughes, 1986, p. 169), between them. Hughes refers to images that are particularly powerful in supporting translations between the formal and the concrete, as "transitional devices" (p. 172), surely alluding to Winnicott's (1971) notion of transitional objects in child development. I return briefly to possible links between psycho-analysis and teaching and learning mathematics in the conclusion.

Formal and concrete

Lakoff and Nunez (2000) suggest a bodily basis for all mathematical meaning. On such a view, our understanding of number begins with bodily experiences (for example rhythmic clapping, walking up steps) and we progressively abstract from these experiences, re-organising our perception and culminating in symbolic representations of our actions. An example of a re-organisation or re-presentation of perception would be the use of a tally system for number. A tally transforms the performance of a certain number of actions, or perception of a certain number of objects, into an equivalent number of dash marks. Symbolic representations of number are achieved when we are able to work with number symbols themselves (in a "formal" manner), without needing to invoke the actions we previously used to represent them. Von Glasersfeld (1991) put forward the notion that one of the functions of symbols is "pointing" (p. 56) to specific re-presentations, "without, however, obliging the proficient symbol user to produce the re-presentations there and then" (p. 56). For example, in the context of learning number, a proficient symbol user knows that a certain whole number implies or *points* to a place in the sequence of the natural numbers, without needing to carry out the count. And, at the same time, the proficient user is *able* to carry out the count if needed.

In the first volume of this journal, Mason (1980) suggested the following characterisation of three modes of representation, re-interpreting Bruner (1974): "ENACTIVE—confidently manipulable; ICONIC—having a sense or image of; SYMBOLIC—having an articulation of" (p. 11). Mason suggests that for young children encountering "1" and "2" in written form, these numbers are "truly symbolic", having "little or no meaning" (p. 11). It is only later, after many and varied experiences with number, that children gain a sense of one-ness or two-ness (which is iconic). And, Mason suggests, "[t]o proceed with arithmetic it is essential that 1, 2, 3 become enactive elements, become friends. If they remain as unfriendly symbols then arithmetic must be a source of great mystery" (p. 11). Mason writes of a spiral in which enactive elements become iconic and then symbolic and back to enactive, although intriguingly his example demonstrates a movement in the opposite direction, from symbolic to iconic to enactive. In either case, what seems significant in learning mathematics is that a link to the enactive or iconic meaning is maintained, or that the "pointing" function of symbols is preserved.

Mason, like Hughes, suggests paying attention to the transitions between forms of representation. More recent characterisations of the growth of student understanding in mathematics (Hershkowitz, Schwarz & Dreyfus, 2001; Pirie & Kieran, 1994) emphasise the dynamic and recursive cycling between different forms of representation whilst, like Mason, questioning the tacit assumption of the necessity of moving from the concrete to the abstract in learning (mathematics).

Davydov (1990), in prioritising labour and action, suggested the movement is in the reverse direction and the only scientifically correct method of understanding how we, as humans, are able to contemplate unities out of diversities is the "method of ascent from the abstract to the concrete" (p. 128). One of Davydov's insights here was that any description of, say, "concrete" objects, requires words that *already* delineate and classify those objects in an abstract manner. Our first experiences of objects are through a "reduction" of their concrete "fullness", into "a relationship that can be contemplated" (p. 132). It is only through further development that we are able to "ascend" from these abbreviated abstractions in a process of synthesis, to the "concrete" which for Davydov implies "a phenomenon or object is taken in combination with a whole, is considered in connection with other manifestations of it and in connection with its essence, with a universal source (law)" (p. 137).

Symbolic fluency

Dick Tahta suggested:

> We do not pay enough attention to the actual techniques involved in helping people gain facility in the handling of mathematical symbols. (Tahta, 1985, p. 49)

Any mathematical symbol exists in a rich web of connections to other symbols and to a myriad of representations, not to mention possible "real world" contexts. Any learning of mathematical symbols must of necessity begin with a restricted sense of what the symbols can "point" to. Tahta challenges us to investigate and question the contexts in which we introduce students to symbolism. To take seriously the notion of the "ascent to the concrete" implies that a students' first encounter with a symbol need not be in the context of a "pointing" to something concrete. We can begin, for example, with pointing to other symbols or to relationships between objects.

In the pages of this journal, attention has been paid to working with symbols by, among others, Hewitt (1996) and Arcavi (1994, 2005). Hewitt (1996) suggested symbolic fluency can arise through activities that subordinate the learner's attention to some other purpose, in pursuit of which symbolic manipulations are necessary. Arcavi (1994) introduced the notion of "symbol sense", to parallel the more commonly used "number sense". Arcavi distilled eight aspects of what he considered to be important in developing symbol sense, incorporating the gaining of fluency with symbolic manipulations. One important element, for Arcavi, in developing symbol sense is the capacity to tolerate *not* knowing and being able to live with "partial understandings and with the knowledge that sometimes meaning may emerge from meaninglessness" (2005, p. 45).

Arcavi, Hewitt and Dayvdov speak, in different ways, to the paradox of learning. If learning is seen as the recognition of something new, how is this ever possible, since to recognise something, we need to have a pre-existing idea of what it is? Arcavi suggests we have to get comfortable with partial understanding and that practice and drill (within a culture that supports reflection) is one mechanism to allow for the emergence of meaning. Hewitt (1994) is emphatic that the use of drill is inefficient (in terms of student time) as a mechanism to escape from the learning paradox. His notion of subordinating the use of symbols links to the ideas of Gattegno that inform the rest of this article.

Gattegno's pedagogy

Gattegno drew inspiration for his mathematics curriculum from observations of how children learn their first language (1971). Hughes's notion of "translations" (1986, p. 169) between representations is also suggestive of a linguistic approach to learning mathematics. It is a common observation that young children babble before they speak. Children pick up the rhythms and cadences of the language they are immersed in, before any words. Later in their development, children can recite the number name sequence before being able to show recognition of the link between number names and collections of objects. In other words, it is possible to develop a kind of linguistic (symbolic) fluency in a playful and metonymic manner. In learning language, it seems we are (or at least, were) happy to engage in playing with sounds without worrying about what they mean. Another skill we exhibit early in our lives is the capacity to make linguistic transformations, "my brother" may also be "your son"; "this pen" might also be "that pen" to you; "A is on top of B" means "B is under A". Gattegno employed the powers we possess of linguistic transformation to aid the learning of mathematics.

Gattegno (1974) offers powerful examples of how learners may be offered a space to work on symbolism, drawing

Figure 1. Representations of the number 2.

on relationships within tangible/visible resources. In his own hands, these spaces provoked rapid learning; Gattegno claimed he could teach, to mastery, the entire 5-year secondary curriculum (ages 11-16) in 18 months or less. In Gattegno's curriculum, algebra is introduced before arithmetic, in the sense that children are first invited to work on the relationships between numbers (bigger than, smaller than, ordering, then "double" and "half") before any number names are introduced. The first number symbol to be used is "2" (1970, p. 25), to stand for the relationship between two lengths of rod (see Figure 1). The approach, in the early years, is strikingly similar to Davydov's curriculum (see Venenciano & Dougherty, 2014, p. 20).

Soon after introducing "2", the symbol "½" is introduced, to represent the complementary view of the relationship in Figure 1. Symbols are introduced consistently to name a relationship that children experience enactively, for example, through play with the rods. As these symbols are used, Gattegno suggests the rods are initially always available to the children in order for them to check out statements made or questions asked. But the symbol does not represent the rod—it represents a *relationship* between the rods. So, although the rods themselves can be seen as concrete, the symbolism is not tied to objects in a direct manner. The symbol, from the very start, points to something abstract, yet with a visible and tangible representation. From the start it is clear that the visual representation of number (Figure 1) is just one among many alternative representations. The symbol "2" is not tied to one particular image or object.

The image I discuss in this article is one devised by Gattegno for later in the curriculum, variously called the "Gattegno chart", "Gattegno tens chart", or "tens chart" (see Figure 2). There are several different versions of this chart, displaying different rows (and for working with bases, cutting off columns from the right). The units row is important to always have visible, but, for example, decimals need not be shown. The alignment of numbers in any one column is

0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900

Figure 2. An example of a Gattegno tens chart.

significant. A question I was once asked about this chart is why Gattegno did not rotate it by 90 degrees, so that all the units would be in the same column all the "tens" in the same column and so on. Such an arrangement would perhaps emphasise the place value meaning of the numbers. Gattegno's arrangement (Figure 2) instead emphasises how number names are written and said. I can literally shift the "2" over the "20" and would see the number "22".

In the next section, I hope to show how the Gattegno chart can support students in symbolising relationships, as they gain a playful and creative fluency with numbers and operations.

Using the Gattegno chart

I illustrate some possibilities for work with the Gattegno chart, drawing on evidence from a funded project that took place between 2011 and 2013 in primary schools in the southwest of the UK. The aim of the project was to support students who are underachieving in mathematics, through a focus on the creative processes of the subject. What the project team took creativity to mean, in this context, was students asking their own questions, choosing their own methods of representation, noticing patterns and making predictions or conjectures.

The project was a collaboration between myself and the charity "5 × 5 × 5 = creativity" (5 × 5 × 5). The name 5 × 5 × 5 refers to the origin of the charity, which was placing 5 artists in 5 schools in 5 local authorities (a local authority in the UK is a regional area of governance, with responsibility for education among other things). The artists linked to 5 × 5 × 5 work with children in schools to develop projects that arise out of the children's own interests. The 5 × 5 × 5 approach draws inspiration from the Reggio Emilio pre-schools in Italy (Rinaldi, 2006) and the idea of following children's fascinations. Projects run by 5 × 5 × 5 are generally arts based. One prompt for the project reported in this article was the idea of trying out a mathematics based 5 × 5 × 5 project, where instead of an artist going into schools, I would act as a mathematician and aim to provoke mathematical work in schools.

My own teaching background was in secondary education (ages 11-18) and I had developed some conviction about productive ways of working in the secondary school. These ways of working, influenced by Gattegno, centred around the idea of students "becoming a mathematician" and supporting them in working on noticing pattern, making conjectures and finding counter-examples (see Coles, 2013). At the start of the 5 × 5 × 5 project, we had no idea how relevant or useful these ideas would be in a primary school context (children aged 4 to 10).

In 2011-12, I worked with one primary (ages 4-11) and two infant schools (ages 4-7), spending half a day a week visiting and running sessions during the autumn and spring terms. It was usually possible to visit two schools each week and over the year an average of 12 visits were made to each school. Field notes were taken during these visits by whoever was not teaching (*i.e.*, by me if I was observing, or by the usual teacher if I was teaching). Selective photocopies of students' work were made by teachers, as part of their own documentation of their learning on the project. The teachers involved in the project met with me, as a group, after school five times during the academic year. During these meetings, the teachers reflected on their work with their students and we discussed and planned ideas for the future. These meetings were audio recorded, although the data is not analysed in this article, where the focus is on the classroom work.

The Gattegno tens chart (Figure 2) was the tool that teachers on the project spoke about as providing the most innovation in their teaching. I used the chart at some point in each of the schools to set up a range of activities and teachers continued to use the chart and develop ways of working on their own.

I exemplify the use made of the Gattegno chart through a narrative account of one project in one school, which lasted for four 1-hour sessions, each of them one week apart. I chose this project because it was the most fully documented of those that made use of the Gattegno chart. I had been asked by the teacher to support the students in working on multiplication and division. The school is a rural state primary school, with levels of attainment in line with UK national averages. Narrative accounts, taken from field notes and my emailed reflections (written soon after the lesson and sent to the group of teachers), are indented and analysed immediately afterwards, in relation to evidence of symbol use. The class was a mixed year 3-4 (ages 7-9) with students from across the range of attainment within the school.

Beginnings

> I stuck the Gattegno chart to the wall at the front of the classroom (without the decimal rows, but going up to hundred thousands) and students gathered on the carpet. I began the class by getting the students chanting, to get familiar with how it works. I tapped on a number in the units row and got the class to chant back in unison the number name, then continued for other numbers of the units row and extended to numbers in the tens row. I was conscious of wanting to generate responses from every student, not shouted, but said confidently.
>
> I then wanted to link unit names with tens names. I tap on "4" (class chant FOUR) and then "40" (class chant FOUR-TY); tap on "6" and then "60"; tap on "8" and then "80". I focus attention on how the number name changes. All we do is add "-ty" to the end of the digit number. We practice this. Having established these names, I tap on "40" followed by "2"—students need to chant back "FOUR-TY-TWO", again, more practice at saying these two digit numbers.
>
> I tap on a number, and invite the class to chant back the number that is 10 times bigger. The first response needed confirmation and repetition to ensure the whole class called back the answer. I initially chose single digit numbers, and the class chanted back the number 10 times bigger, then progressed to other numbers, returning to single digit ones if the class lost confidence. After a few minutes, I invited someone to say how, on the chart, they were getting their answers. I focused attention on how you can get the answer simply by moving down one row on the chart. Returning to chanting with the awareness of movement, I then pointed on a number and invited the class to chant back that number divided by 10 (which is a movement up one row). I repeated the process for multiplication and division by 100.

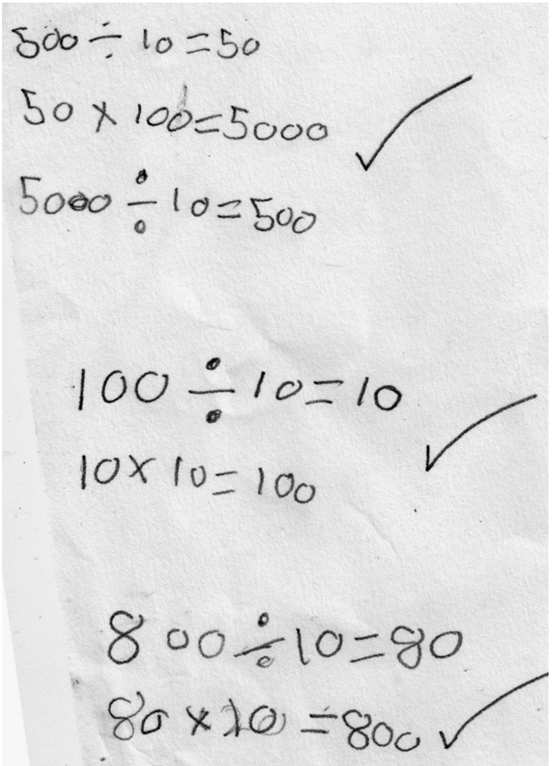

Figure 3. Three journeys.

The Gattegno chart offers a perceptual structuring of the number system. A linguistic connection between units and tens rows (adding "-ty") is established. The use of chanting and the association of multiplication and division by powers of 10, with visual movements up and down a row aim to establish an immediate (habitual) response to relationships within the chart. The students have to respond to the spoken stimulus "multiply by 10" and a tap on the chart by calling out the number in the row below, or "divide by 10" and the number in the row above. The (formal) symbols in the chart are not linked to anything concrete, in the sense of collections of objects—the chart and the work in chanting offer connections between the symbols themselves. Students are developing habitual responses in relation to the structure and relationships embedded in the chart. At the point when I stopped the class and asked what you have to do in order to multiply by 10 (getting answers such as "move up one row") there was an articulation in words of one aspect of this visual structure.

First journeys

I offered a challenge, to choose a number on the chart, go on a journey multiplying or dividing by 10, 100 and to get back to where they started. We did two journeys all together so I could demonstrate how I wanted them written out. Students suggested the movements and I supported the writing. In keeping with the principles of the project, I then asked students what questions they might want to explore linked to their journeys. Students made these suggestions: how big can you make the number? How long can you make the journey? How many ways can you do the same journey? Before students returned to their desks, I checked that everyone knew the number they were going to start with.

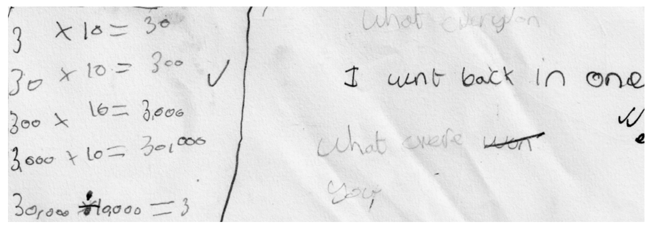

Figure 4. A student extends to division by 10 000: "I went back in one".

The student whose work is shown in Figure 3 completed three journeys and ticked their own work since they recognized that they got back to where they started (we suggested they tick their work in this way). The use of ÷, × suggests a symbolic representation of the movements up and down the chart. There is evidence of students' confident use of inverse operations and the distinction between directions of movement on the chart, linked to the symbols ÷ and ×. All students were able to produce these kinds of journey; some stayed with the same type of journey done from different starting points, others branched out to try more and different combinations of operations.

In the first lesson, I was struck by one student (see Figure 4) who had chosen a number then done × 10, × 10, ×10, × 10 and had worked out that to get back in one go to where she started, she needed to divide by 10 000. I asked her how she knew this and she said to undo × 10 you divide by 10, then × 10 × 10 needs divide by 100, × 10 × 10 × 10 needs you to divide by 1000 and so on. It was interesting that several students wanted to undo × 10 ×10 by dividing by 20 (rather than 100). This seems like a common misconception and a good one to be working on.

There is evidence here of the student extending the pattern of how to divide by 10 and 100, to work out what she must do as the inverse of × 10, × 10, × 10, × 10. Here perhaps we also see evidence of a playful exploration of symbols, extending a relationship observed between successive multiplications by 10. As a class we had not worked on division by numbers greater than 100. I interpret this student as having noticed something about the symbols themselves. In Mason's terms, this student appears to be getting a "sense" of how these multiplications and divisions operate and working on relationships between the relationships visible in the chart.

Student noticing

Towards the end of the first lesson, I asked all the students to stop their work on the journeys and then spend a couple of minutes writing down anything they had noticed about what they had done, or anything they had felt or any questions they still wanted to ask. We then came to the carpet at the front of the room and students shared some of their ideas with each other.

I interpret the student text in Figure 5a as: "I've noticed it is clever because if ÷ by 10 you × 10 then get back" and

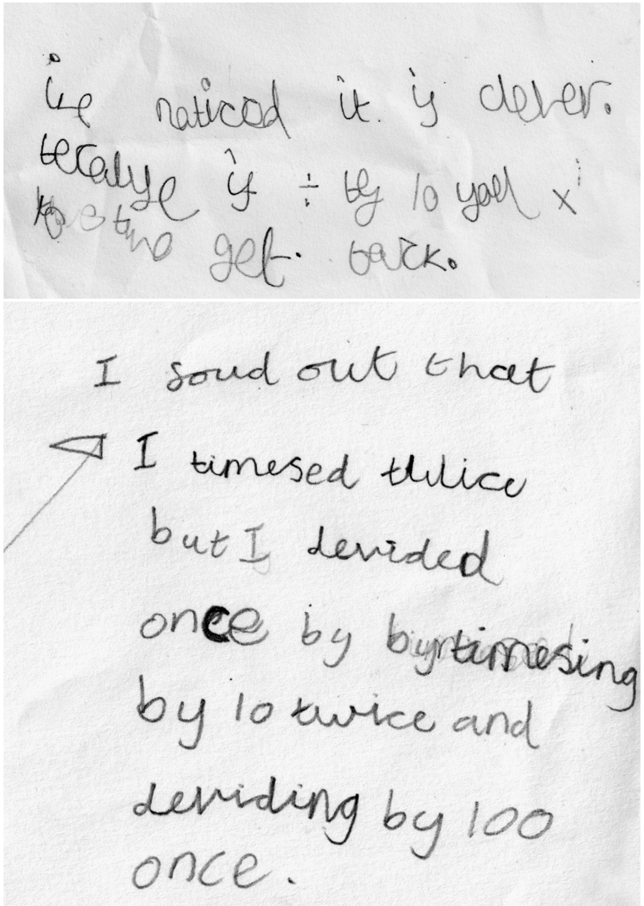

Figure 5a (top) and 5b (bottom). *Connections made.*

that in Figure 5b as: "I found out that I timesed twice but I divided once by timesing by 10 twice and dividing by 100 once". The student who wrote Figure 5a is articulating a key mathematical awareness and offers an example of the connection between multiplication and division. The student offers a representation, expressed in language, of an inverse relationship. A further awareness is offered by a different student in this first session (Figure 5b). This student has noticed the inverse connection between multiplication by 10 twice and division by 100. Students are articulating the awarenesses they have developed about the relationships between the symbols they have been using. There is no evidence of these students having made, or needing to make, links between their symbol use and an awareness of the relative sizes of the numbers they are operating with. The symbols for the operators stand for movements or relationships within the chart and, starting from these symbols, students were able to uncover further connections.

Moving to decimals

A few students had seen decimal rows (printed on the back of the chart I was using) in the first lesson and others had talked about wanting to use them at the end of that class. At the start of the second session (one week later), I worked again with students, chanting multiplication and division by powers of ten, this time moving into decimals as well. Students were invited to continue work on their journeys and again I asked students what questions they wanted to work on, which were written on a flipchart. The idea of "getting back in one" was shared with the whole class. The structure of the tens chart meant that there was little difficulty for students in extending what they had done with journeys in whole numbers, to journeys into decimals (see Figure 6).

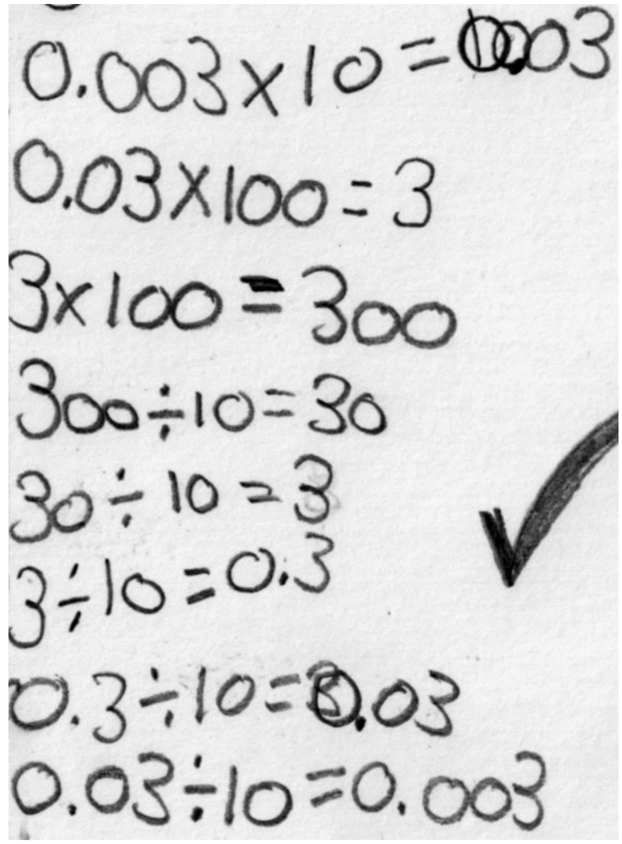

Figure 6. A journey into decimals.

The student who wrote Figure 6 has used the operations × 10, × 100 and ÷ 10. The combination of these different operations suggests the student has some control of these symbols and is gaining fluency in their manipulation. There is no apparent discontinuity or difficulty in incorporating symbols for decimal numbers into the movements on the chart and the operations with powers of 10. The focus on multiplication and division as a relationship that is expressed visually on the chart means there is no difficulty in extending the operations into decimals.

Moving away from powers of 10

At the end of the second lesson, one student asked if they could make a journey from one column to a different column (all journeys to this point had been in a single column of the chart, representing multiplication and division by powers of 10). As teachers and researchers we had not expected this question and I had never used the chart to do this. However, we took on the challenge and in the third lesson, I offered the idea that to go from, say,

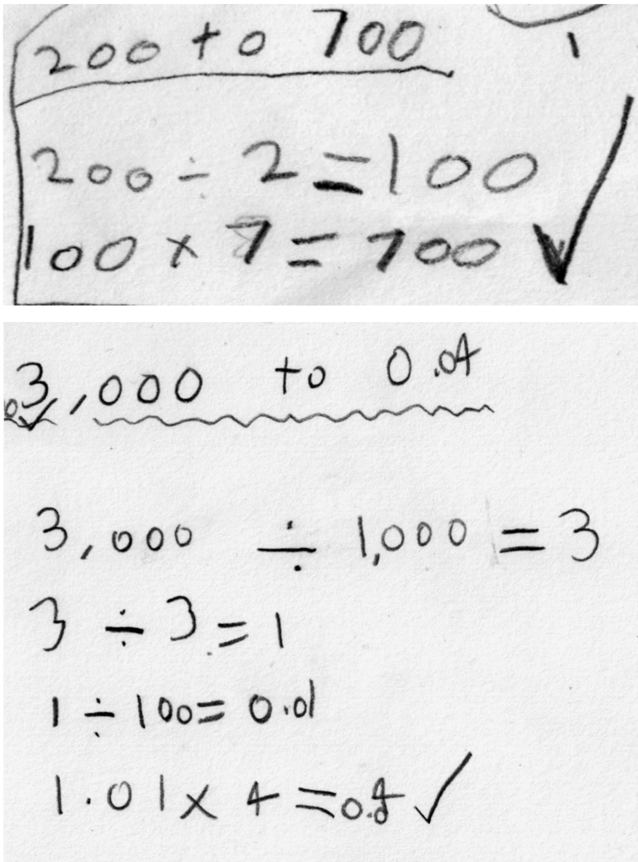

Figure 7a (top) and 7b (bottom). Journeys from one column to a different column.

> 7 to 4 on this chart, since we only use multiplication and division, we would have to divide by 7 to get to 1 and then multiply by 4 to get to 4. With this new possibility, students were then challenged to choose a starting point and a different end point for their journeys.

An example of the simplest of these new kinds of journey is shown in Figure 7a. A more complex challenge one student set themselves can be seen in Figure 7b. Although there is a small error in the last line (1.01 was written instead of 0.01), this student demonstrates a striking fluency in the use of these symbols. The student who wrote Figure 7b was seen by the school as having low levels of prior attainment in mathematics. Part of the evidence for the fluency or control over these symbols is the efficiency of the method employed. The student has, in effect, discovered or at least used a unitary method. In the most efficient manner possible (in the context of the work on this problem) the student has reduced 3 000 to 1 and then gone from 1 to 0.04, again in the most efficient manner possible (given that movement left and right on the chart has only been offered as a possibility along one row).

Discussion

One of the teachers, in reflecting on this particular sequence of lessons, commented on how she had always, previously, approached multiplication through a concrete manipulation and grouping of objects. In this treatment, we have observed students being able to begin with a visual tool that is already removed from concrete objects (Figure 2) and build further abstractions, through the focus on multiplication and division as a relationship that can be observed within the chart. At some point, no doubt, students will need to re-connect their use of multiplication and division with physical objects, but in this treatment it appears students are able to work with numbers (*e.g.*, decimals) and operations (*e.g.*, division by 10 000) that would not normally be introduced until later in the curriculum in the UK. There is the potential to extend such work into a consideration of exponents (see Davis, 2014). The work on journeys supported students in gaining awareness of the inverse relation between multiplication and division, awareness of place value (without this being explicitly mentioned) and, in the work moving between columns (Figure 7b), the beginnings of the "unitary method" for solving problems. I make no claims about what students do or do not understand about multiplication and division. What I observe is that they have become energised by gaining fluency in symbolic manipulations and have developed awarenesses linked to these symbols. The "partial understandings" these students are working with did not appear to provoke anxiety and some of the joy and creativity of the exploration with these symbols can perhaps be guessed from their written work.

Conclusion

It seems clear from the story of the "journeys" project that all students were able to make the transition between a visual or enactive representation of a movement/relationship and its symbolic or formal description and go on to use this symbolism with confidence. The Gattegno chart was displayed throughout the project and some students asked for smaller, personal copies in order to support their work. The students were drawing on the image to aid their (symbolic) writing of their journeys. It took less than one hour for students to begin making mathematical connections between numbers and operations. However, if the Gattegno chart is a "transitional device", it is transitional in allowing movement between the formal and the concrete only if the concrete is interpreted as Davydov suggests, *i.e.*, a movement towards seeing objects in relation to a "whole" and in their variety of manifestations. The Gattegno chart offers a structuring of the number system that allows symbols (for multiplication and division) to be linked to relationships between rows and columns. Students gained symbolic fluency with these operators, unencumbered by the need for a laboured one-to-one linking to concrete objects. Students made connections in their written work between and within the formal representations themselves; this work, for some, was their first introduction to using multiplication and division symbols in school.

One possible challenge arising from this project, is to find other transitional devices that can allow mathematical symbolism to be linked to relationships that can be perceived. Much has been written about the ambiguity of mathematical objects and the need for students to appreciate, for example, how symbols can stand for both processes and objects. There was seemingly no difficulty for students in the number "10", for example, representing a position on the chart and being linked to an operation between positions on the chart. The presentation of symbols as linked to relationships perhaps offers students powerful access to their ambiguity. Figure 1, in representing the symbols 2 and ½, can be seen as a representation of the process of halving the longer rod, or as an object, filling up half of the longer rod. The symbol "× 10" acts as a process in most students' work and yet in statements about the relationship between multiplication and division, students demonstrate they can consider these operations as objects that have relationships to each other. Transitional devices, conceived as linking relationships to symbols, may offer an opportunity to work explicitly, from the beginning, on connections and translations that are used by successful mathematicians, but that all too often are opaque to learners.

And what of the connection to a transitional object? For Winnicott, the transitional object supports the development of the child on the journey to independence from their parents. The object symbolises the relationship with the parent so that, for a while, being with the object provides the child with comfort. After some time the object is abandoned, but never rejected, it simply loses significance. In the same way, perhaps, the Gattegno chart allows students, in the early stages, to perceive the relationships that symbols stand for and provides them a reference (a comfort?) to which they can return if necessary, to ground symbolic transformations. After some time the chart becomes unnecessary, it is abandoned, but can be invoked, for example as a mind image, if needed.

Acknowledgements

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Many working mathematicians are puzzled about what proofs are for if they do not prove. On the one hand they know from experience that proofs are fallible but on the other hand they know from their dogmatist indoctrination that *genuine* proofs must be infallible. *Applied mathematicians* usually solve this dilemma by a shamefaced but firm belief that the proofs of the *pure mathematicians* are "complete", and so *really* prove. Pure mathematicians, however, know better—they have such respect only for the "complete proofs" of *logicians*. If asked what is then the use, the function, of their "incomplete proofs", most of them are at a loss. For instance, G. H. hardy had a great respect for the logicians' demand for formal proofs, but when he wanted to characterise mathematical proof "as we working mathematicians are familiar with it", he did it in the following way: "There is strictly speaking no such thing as mathematical proof; we can, in the last analysis, do nothing but point; … proofs are what Littlewood and I call gas, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils".

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