STUDENTS' PRONOMINAL SENSE OF REFERENCE IN MATHEMATICS

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> Deductive propositional and predicate logics ultimately are formalizations of the commonsense, correct deductive arguments that people engage in on a day to day basis (Overton, 1990, p. 5).

> Classical propositional calculus is vastly inadequate as a theory about how people reason. Whatever merits one (*e.g.,* a logician) wants to attribute to the propositional calculus, one merit it cannot possess is being a descriptively adequate theory of human reasoning. (Schroyens, 2010, p. 71)

> The agreement, the harmony, of thought and reality consists in this: if I say falsely that something is red, even the red is what it isn't. And when I want to explain the word 'red' to someone, in the sentence 'That is not red', I do it by pointing to something red. (Wittgenstein, 1953, §429)

For those of us continuously engaged in a highly specialized field such as mathematics, it is helpful at times to adopt a foreigner's view of some practices that we take for granted. I thus offer the following fanciful conversation between a mathematics student and her roommate.

1 *Roommate* So, what are you learning in your math class? It seems that none of your assignments even involve numbers.

2 *Student* Hmm. Maybe I could show you. The other day you told me 'My mother is a nurse.' Correct? Well, that sentence is either true or false, so it is a statement.

3 *Roommate* Yes, and I stated it. What of it?

4 *Student* No, it is a statement because it has a truth-value.

5 *Roommate* Well I guess it would not be of much value if it were not true.

6 *Student* No, wait. If I were to say 'My mother is a nurse,' it would be false, so how does this work? Maybe that would be like 'x is even' that depends on the value of x. I guess that means this is an open statement or a predicate.

7 *Roommate* So my mother being a nurse is open to question just because your mother is not?

8 *Student* The idea is that the sentence 'My mother is a nurse' is either true or false for each speaker, so it has a truth set of all the speakers for whom it is true and also a set of speakers who make it false. The word 'my' acts as a variable.

10 *Roommate* So your class taught you that I am in some international fraternity of nurse's children? I have yet to be invited to a meeting.

11 *Student* And since my mother is not a nurse, I can say lots of funny things such as 'If my mother is a nurse, then my father is Darth Vader' and they will be true. And you can say 'My mother is a nurse or my father is Darth Vader' and that would be true.

12 *Roommate* I just thought you might want to call my mom to ask about that cough you had. Forget I even asked about your class.

While this is obviously contrived, I hope it demonstrates how odd and unmotivated some of mathematical logic's framings of language may appear from an outside perspective. The mathematics student made reference to logic's well-known counterintuitive stipulations such as conditionals being true when their antecedent is false, but this essay will explore how other basic and subtle aspects of logic's treatment of language are foreign to students' untrained language use (contrary to the Overton quote above). I shall give account of how undergraduate students in some of my studies treat the reference structures of mathematical terms and how it differs from the common mathematical treatment in terms of truth-values.

In particular, I will demonstrate ways in which students implicitly understand that mathematical terms refer only to objects that satisfy them. I refer to this untrained pattern of interpretation as students' *pronominal sense of reference* (PSR). I call it pronominal because students treat mathematical terms somewhat like the pronouns 'that' or 'something which' to point to objects. I shall throughout compare and contrast this with two of the common logical accounts of mathematical language: propositions yielding truth-values and predicates. The diagrams in Figure 1 portray the basic structure of the three. The key difference between propositions and predicates is that propositions must be true or false (*i.e.,* a 'statement') while a predicate simply must have a truth-value for each value of each variable it contains (conversation turn 6). Thus, '4 is even' is a proposition while 'x is even' is a predicate.

Toggling truth-values

In what follows, I shall depart slightly from discussing students' PSR to demonstrate how the formal accounts of mathematical reference conflict with student thinking, and why student reasoning in such cases is quite reasonable. After that, I shall return to defining the PSR and relating it to normative logical framings.

Instructional units on logic often begin by forming units of language called statements (or propositions) and pointing out that they have truth-values (conversation turn 4). Not all units of language have truth-values, but those are initially set aside (*e.g.*, Chartrand, Polimeni & Zhang, 2013, pp. 37–38; Hammack, 2013, pp. 34–35). Statements with variables such as 'x is even' do not have single truth-values, so they must be considered separately for each x (conversation turn 6). However, the predicate interpretation assumes that x in this case points to all integers. This seems innocuous enough, but my studies with college students have shown that we have already departed from some of our students' intuitions. How can this be? Consider the following true/false tasks that usually served as starting points in my teaching experiments on guided reinvention of logic:

Statement 1 Given any integer x, x is even or x is odd.

Statement 2 15 is even or 15 is odd.

The conventional logical treatment of these two statements is that the first is comprised of all instances of the second where particular values of x have been substituted [1]. The truth-value of Statement 2 directly depends upon the truth-value of the two parts (disjuncts), which is why this meaning of *or* is called 'truth-functional'. Statement 1 is true because all of its instances are true by truth-function. Under this view, the truth of Statement 1 implies the truth of Statement 2. For several of my study participants, the first statement is true and the second statement is false. This clearly diverges from standard mathematical language use, which is why the behavior initially surprised me. Through dialogue with these students and some further reading, I have been able to take an outside view of mathematical language that helps me appreciate some of what makes the formal usage strange.

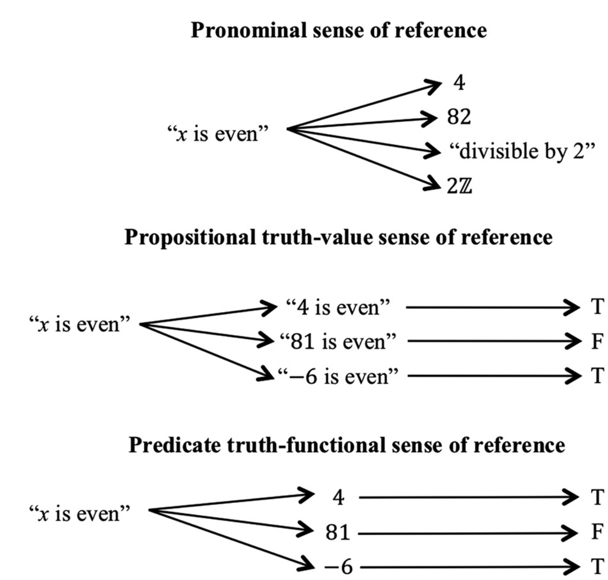

Figure 1: Comparing senses of reference.

Some lessons from pragmatics

Pragmatics helps provide another perspective on mathematical language. Grice (1975) introduced a fruitful set of maxims of conversation that represent assumptions interlocutors use to understand what is meant by another's speech. One maxim states, "Do not say what you believe to be false" (p. 46). Indeed, I find it jarring when I perceive that someone deliberately tells me something they consider false (conversation turn 5). The patently false first part of Statement 2 might naturally leave a hearer curious as to the speaker's intent. As one study participant later explained, "I didn't take into account the *or*. I just, you know, '15 is even', false." Why would anyone include such a claim in Statement 2 instead of simply saying '15 is odd?'. Indeed I doubt that such statements appear in mathematics with any frequency outside of instructional units on logic!

A key point is that Grice's maxim effectively reduces any distance between saying Statement 2 and saying, 'Statement 2 is true' since the very act of speaking the former implicates the latter. I have come to understand that some students assign the answer 'false' to Statement 2 in part to disavow it as an appropriate speech act. This kind of analysis also sheds light on another well-known effect regarding disjunctions. Everyday use of the word *or* varies by context as to whether the statement affords the possibility that both disjuncts are true. The distinction is usually articulated as *inclusive* versus *exclusive* meanings of *or* (Velleman, 2006) which differ with regard to statements such as:

Statement 3 14 is even or 15 is odd.

The classical account of why students' experience discomfort with Statement 3 imputes some misunderstanding of mathematicians' use of *inclusive or*. Grice's maxims point to a different issue that is likely at play.

Some participants in my studies expressed dissatisfaction that Statement 3 did not use the connective *and*. Another of Grice's (1975) maxims states "Make your contribution as informative as is required" (p. 45). I think students are justified in perceiving that Statement 3 violates this. Again, I propose they are not denying the statement on the basis of truth-values, but rather reject the statement as an appropriate speech act (compare conversation turns 10 and 11). Some students reject Statement 2 and Statement 3 explicitly based on their meaning for *or* which is distinct from either the inclusive *or* exclusive truth-functional meaning. Those students note that or should be used in situations where alternatives or possibilities are at play. For instance, two example disjunctions taken as paradigm examples in Velleman's (2006) introduction to proof text are, 'It will either rain or snow tomorrow.' and 'I will go to work either tomorrow or today.'. The former presents possibilities that are predicted but not yet knowable, thus entailing possibilities. The latter presents alternatives available to the speaker that she may choose. It is worth noting that the speaker of the second statement should not be certain that she will go to work today, else she has violated Grice's two maxims. Statements 2 and 3 cite either well-known mathematical facts or blatant falsehoods, which means they lack any sense of possibility or alternative. Thus, my study participants are justified in their assertion that Statements 2 and 3 misuse *or*.

Does quantification add complexity or clarity?

Let us return to our initial mathematical statements. In light of the pragmatic objections that students may raise to Statement 2, we can now see how different Statements 1 and 2 really are. Statement 1 requires both disjuncts in order to be true. Nothing is superfluous and it violates none of Grice's maxims. Further, since the statement is a universally quantified ('given any') disjunction of two predicates ('x is even' and 'x is odd'), the word *or* conveys a sense of possibility or alternatives consistent with the everyday use. I anticipate that the same could be said of any mathematical disjunction that appears outside of a logic unit. Thus, the mathematician's truth-functional framing of these statements closely links Statements 1 and 2 while students' pragmatic understandings of language strongly distinguish the two.

An interesting trend in my teaching experiments is that while many students deny statements like Statement 3, they do not deny the following:

> *Statement 4* Given any even number z, z is divisible by 2 or z is divisible by 3.

Note that for numbers like 6, both parts of this statement are true. Even the study participant who most strongly committed herself to an exclusive *or* reading for non-quantified disjunctions did not initially find any problem with affirming Statement 4. This further suggests that the *or* in Statements 2 and 3 operates quite differently for students than the one in Statements 1 and 4, the key difference being quantification over some set that renders both parts of the statement as 'possible' for different values of the variables. In summary, I posit that though quantified statements are more logically complex (and are formally understood to depend upon the non-quantified, truth-table structure), they are more pragmatically natural and thus may provide more clarity to logic instruction than complexity [2].

To avoid potential misunderstanding: my argument is not that mathematics instruction should avoid unnatural or technical aspects of mathematical language. On the contrary, my goal is that students learn how to use mathematical language in normative ways by first becoming aware of their own language use. I attempted to teach logic through guided reinvention with the explicit goal that students construct mathematical logic as an expression and refinement of their own languaging activity. I concur with previous authors (*e.g.,* Durand-Guerrier, Boero, Douek, Epp & Tanguay, 2012) that mathematics instruction on logic must operate between the extremes of a) expecting students to operate in a fully abstracted formal language and b) throwing out formal logic because it conflicts with everyday language and reasoning. Neither option is tenable for mathematics learning because any account of mathematical logic must simultaneously attend to syntax, semantics and pragmatics. My argument to bypass non-quantified (propositional) logic and use quantified (predicate) logic builds on two claims:

> Non-quantified logic presents many pragmatic barriers to students' learning of mathematical logic, particularly as it relates to reading mathematical language.
>
> Student reasoning about quantified logic allows them to construct all the necessary elements of predicate logic (*i.e.* truth tables) without raising these pragmatic barriers.

Stated another way, beginning logic instruction with non-quantified statements and truth tables is 1) problematic and 2) unnecessary. The earlier sections of this essay argued for the former. What follows is a sketch of an argument for the latter.

Mathematical terms as pointers

We are now ready to turn more directly to the construct introduced in the title of the essay. I find the following principle helps explain much about how students read mathematical language:

> *The Pronominal Sense of Reference (PSR):* Mathematical terms and phrases point only to the objects they describe.

This ultimately describes a conception of mathematical reference that I argue many undergraduate students implicitly hold. In this essay, I justify this claim by using the construct to explain several phenomena that have arisen in my experiments.

First, regarding Statement 1 I have learned that some students do not think the x in 'x is even' and the x in 'x is odd' are ever the same. Each phrase's x only points to the integers that make that predicate true. In other words, students implicitly understand that 'x is even' is not quantified over all integers, but only over the even numbers. Notice that under this reading, the two predicates do not really have truth-values! The statements are always true for the values of x that they point to and thus truth-values are rendered irrelevant. This provides yet another view on why, for these students, Statement 2 is not an instance of Statement 1. Figure 2 contrasts this account of reference with the one proposed by Durand-Guerrier (2003) based on open sentences (non-quantified predicates).

An important extension of this principle arises with regard to conditionals. It is well-known that students find it counterintuitive to claim that a conditional is true when the if condition is false. Introduction to proof textbooks often spend some time developing why this should be the case (*e.g.,* Hammack, 2013, p. 42; Velleman, 2006, p. 45). There are two common approaches. The first explores an everyday situation involving a promise ('If you pass the final exam, you will pass the course.', Hammack, 2013) and notes that the promise is only broken when the *if* condition is satisfied. The second considers a patently true mathematical conditional ('If $x > 2$, then $x^2 > 4$.', Velleman, 2006). The principle in the latter approach (which I initially adopted in my teaching experiments) is that numbers such as $x = 1$ or $x = -5$ should not be counterexamples (because the conditional is clearly true). Thus the statement must be true for those values of x. The PSR helps me understand why my study participants found this second argument utterly unsatisfying [3]. Students balked at my inquiries about whether such a statement were true or false for numbers not greater than 2.

They exhibited a strong sense that x only refers to numbers greater than 2. This PSR effect is only reinforced by the pragmatics of *if*, which students see as an explicit cue to consider only cases satisfying the hypothetical 'if' condition.

A second phenomenon explained by the PSR regards how students dealt with negative categories. Multiple study participants exhibited a strong propensity to replace negative category phrases with positive ones. For instance, 'not even' meant 'odd', 'not acute' meant 'obtuse', and 'not a rectangle' meant 'parallelogram'. Dawkins (2017) explores this pattern in greater depth, but for the present argument I posit that this pattern makes sense. According to the PSR, the word 'even' naturally points students to even numbers and they have to find some way to modify this reference to afford the 'not' modifier. The truth-functional account of 'not' that simply toggles the truth-value of the proposition is of no real value in helping students think about how to modify their sense of reference for particular categories. Under a pronominal reading, negative categories are implicitly viewed as less valuable than positive ones because they do not clearly point to the things being referenced (see the Wittgenstein quote above). This leads some students to seek to clarify the phrase by inserting some positive category term that accomplishes the intended pronominal effect.

I also reported elsewhere (Dawkins & Cook, 2017) that students tend to represent these categories either by examples, properties, or sets. For instance they may think about 'x is even' by imagining 4, 82 and −30 (examples), by thinking about divisibility by 2 (a property) or by imagining the entire set of even numbers (possibly by their spacing on the number line). These are naturally all interrelated and an expert can fluidly alternate between them, but for some students their choices of representation are relatively consistent and at times limiting (Hub & Dawkins, 2018). Once again, the PSR provides some insight. Students do not invoke examples necessarily out of some deep epistemic commitment to their value for proving or otherwise, but rather

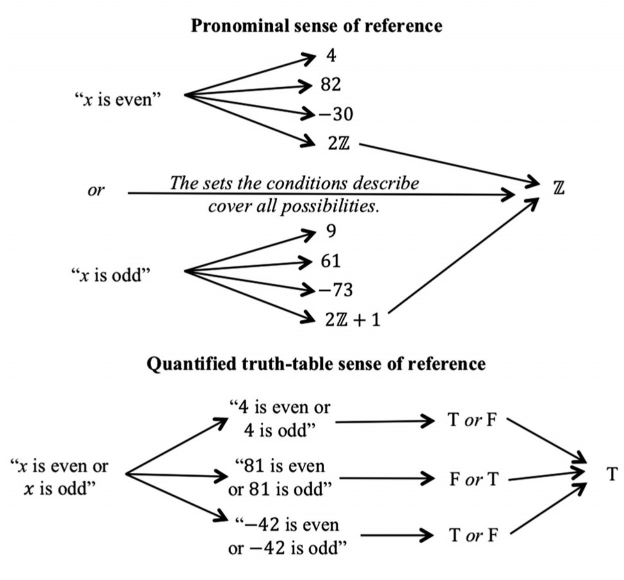

Figure 2. Two senses of reference for quantified disjunctions.

because they understand that is how language works. When I refer to something in speech it seems rather natural for the hearer to mentally invoke the object to which I referred. While mathematics educators have engaged in a fruitful debate about the affordances and limitations of example use in proving and argumentation (*e.g.,* Iannone, Inglis, Mejía-Ramos, Simpson & Weber, 2011; Sandefur, Mason, Stylianides & Watson, 2013), those discussions have focused primarily on epistemic dimensions of the issue. The PSR adds a linguistic and pragmatic explanation to the why students reason with examples.

Once again, I invite the reader to adopt an outside view of how mathematicians use language. In the faux dialogue above, I attempted to contrast the natural and logical framings of the statement 'My mother is a nurse'. For the roommate, this statement clearly referred to a single person and conveyed relevant information about her: she could provide medical advice. The logical interpretation that this statement entails a predicate that is either true or false for each child/mother pair is a rather odd abstraction. I do not mean to deny the utility of this abstraction in mathematics; rather I mean to emphasize its unnatural quality in everyday dialogue.

I further observe that in many mathematical contexts we intend a pronominal reading. When we say '$x = 2$' we generally want students to understand that x refers to 2 rather than to consider this a predicate ranging over all real numbers with the truth set {2}. When we ask, 'What are the solutions to $x^2 = 4$?', we may alternate as to whether we consider this a predicate that is true or false for any value of x or rather an assertion that entails only the possibilities that x is 2 or −2. To consider a more advanced case, when the intermediate value theorem begins with something to the effect of 'f is a continuous function on [a, b]' we do not expect for students to begin by worrying about a number of non-functions or non-continuous functions or to try to conceive of the complexities of the set of all continuous functions of real numbers. Rather, we intend for them to invoke an object for which this condition is true. The pronominal reading is in this case utterly appropriate. Students may later benefit from considering why the intermediate value theorem's conclusions do not always hold when the relation in question is not continuous or not a function, but the most direct reading of the statement itself begins with a pronominal pointing to a continuous function.

A subtlety lurking beneath the surface of this discussion is how the object(s), which mathematical phrases refer to, can be reasoned about with appropriate generality. Common proof techniques hinge upon the use of arbitrary or generic particulars. The cognitive realities of conceiving of such objects is context-dependent (see Dawkins & Karunakaran, 2016) and worthy of more focused study in mathematics education. It is certainly beyond the scope of this essay. My point at present is that mathematics instruction could harness and hone students' PSR for learning rather than implicitly rejecting it through the imposition of propositional treatments of language to examples like Statements 2 and 3 above.

Embracing pronominal reference through quantification

As demonstrated above, quantified mathematical disjunctions and conditionals are much more pragmatically natural (and I believe true to mathematical practice) than are their non-quantified counterparts. My experiments have shown that students do not have much trouble constructing the necessary idea of truth-functions from reading quantified statements, which arise as the conditions under which an example is a counterexample (see Yopp, 2017). However, just because a student understands which examples falsify a disjunction or conditional does not mean that they understand how the categories in the statement partition the set of exemplars in the manner usually represented in a Venn diagram (Dawkins, 2017). In other words, thinking about quantified mathematical disjunctions and conditionals implicitly leads students to formulate truth-functions, but truth-functions alone do not appear sufficient for students to reinvent quantified logical tools such as Venn diagrams. The Venn diagram structure requires students to be able to take any predicate (that may appear in a statement) and imagine partitioning the underlying set into things that make the predicate true and things that make it false. Students are often much more prone to think of sets as structured by familiar categories. For instance, quadrilaterals are organized into squares, rectangles and parallelograms. Students need guidance to see that quadrilaterals can also be partitioned into those with equal length diagonals and those with unequal length diagonals, as well as how such partitions facilitate reasoning about statements.

I posit that if instruction in mathematical logic focused on predicate logic and quantification, it could in many ways make use of students' PSR rather than thrusting it aside. This may be done by inviting students to associate mathematical categories with the sets of objects that satisfy them (attempting to generalize the PSR). Figure 3 portrays the two main lines of reasoning I want my students to develop in thinking about mathematical categories, which I call *reasoning with predicates* (Dawkins, 2017). In this approach, predicates still refer to objects the way that students naively think they do. I simply ask them to consider the more general practice: 'imagine the set of all objects that have this property and the set of those that do not'. This leads to talking about when predicates like 'x is even' or 'x is a multiple of 3' are true and false, but the focus is on finding truth sets rather than just truth-values. Furthermore, because we consistently attend to truth sets and falsehood sets, students develop better ways to think about negative properties, which refer to the complement set of objects.

I argue that this alternative instructional approach will build upon and hone students' PSR. Students often affirm Statement 1 without attending to the reference structure because they know that the statement is true (they understand that 'not even' is equivalent to 'odd'). Students can restructure their understanding of why such statements are true along the following trajectory: <numbered list 1-3>

1. Imagine forming the sets that correspond to the two predicates in the statement.
2. Argue why a counterexample exists or why all of the relevant objects will satisfy one *of* the two predicates (in an or statement).
3. Describe the set of objects that would make the statement false, which allows one to argue why no counterexample to a true statement exists (Yopp, 2017).

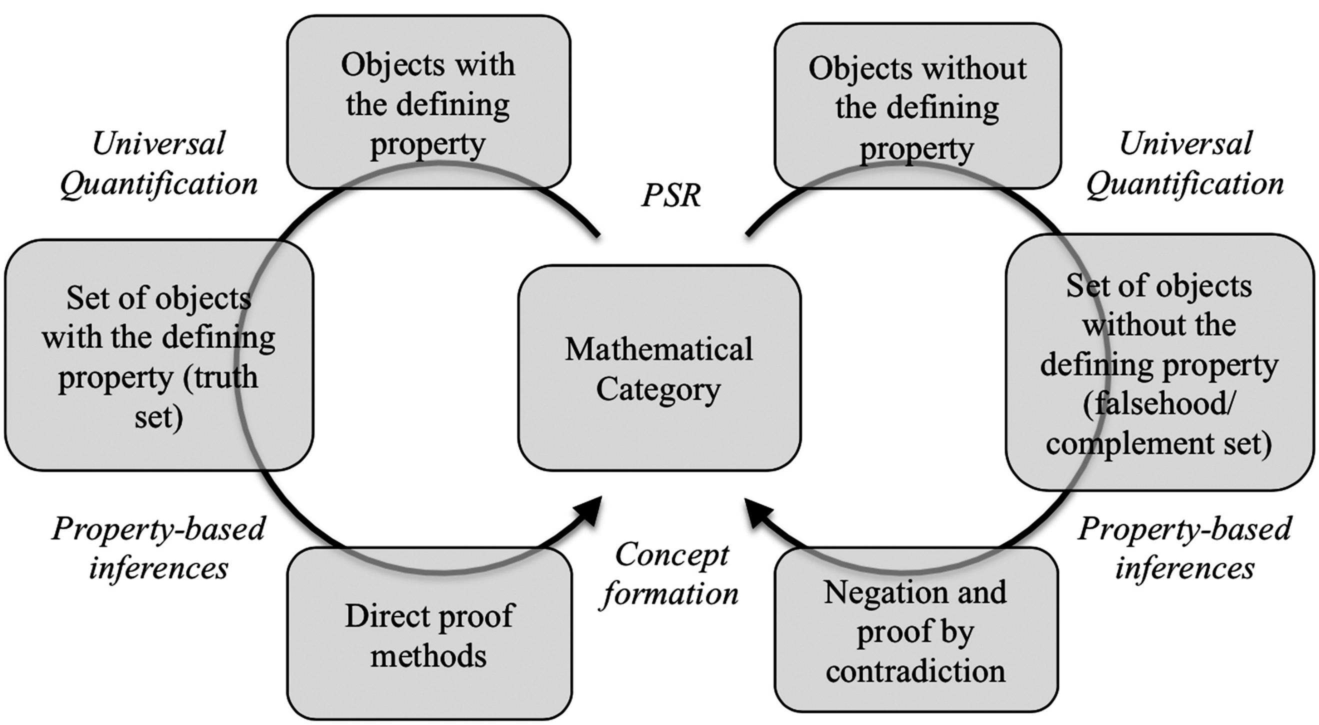

Figure 3. How reasoning with predicates builds upon students' PSR.

Each of these steps involves some important developments as students consider various statements. The first step may involve a development from

- imagining sets corresponding to familiar categories (even, square, multiples of 7),
- to imagining sets corresponding to negative or unfamiliar categories (not a rectangle, quadrilaterals with unequal diagonals, 3 greater than a multiple of 7),
- to imagining the truth set of any predicate ($\{x \in S \mid P(x)\}$ or a compound predicate).

In other words students can progressively expand their PSR to the generalized notion that any predicate has a truth set and a falsehood set (one of which may be empty).

Regarding the second step, Dawkins and Cook (2017) reported some common student strategies that arise in such argumentation about disjunctions. Forming arguments that universal statements are true provides students with the need to partition sets to help them confirm that their argument accounts for all possibilities. This also implicitly leads students to formulate set relations and operations: *or* leads to set union, *and* leads to set intersection and *if, then* statements express a subset relation between truth sets (Hub & Dawkins, 2018).

The third step provides an initial foundation for proofs by contradiction, only stated in the language of sets rather than the language of propositional calculus. Consistent with the PSR, students often do not find this last move very natural when the statement in question is true (because the falsehood set does not refer to anything). Rather, this method should be developed in two sub-steps. First, consider the set of counterexamples to a false statement such as:

Statement 5 Given any even integer z, z is a multiple of 3 or z is a multiple of 4.

Students know that 2 is a counterexample. They may need prompting to explain that fact in terms of the necessary compound property: 'z is not a multiple of 3 and z is not a multiple of 4'. Second, students can adapt this strategy to true claims like Statement 4. A counterexample would need to be an even integer such that 'z is not a multiple of 2 and z is not a multiple of 3'. Since no numbers satisfy this condition, there are no counterexamples and Statement 4 must be true. This also supports student reinvention of the negation of a predicate and moves students toward negating statements, though the role of quantifiers requires more time and attention.

I describe this instructional sequence to demonstrate how key logical principles can still be constructed in a manner that embraces the PSR. Indeed, I propose that such instructional tools may be valuable in exploring how students may construct mental models (Johnson-Laird, 1983) that afford reasoning with generic particulars and coherent understanding of key proof techniques. Much future work needs be done to play out these possibilities in various mathematical contexts.

Notes

[1] As Durand-Guerrier (2003) noted, this relies on the common assumption that mathematical statements with arbitrary objects are universally quantified over all such objects. Phrasings like 'given any' are linguistically ambiguous and are often interpreted to begin predicates whose truth-value depends upon the x chosen (meaning 'Given any integer x, x is divisible by 3 or x is divisible by 4' is neither true nor false). The analysis I provide here abides by the assumption of universal quantification. The participants in the studies I will refer to in this paper have all adopted this assumption without guidance.

[2] It is beyond the scope of this essay to address multiple quantification which is naturally the much more challenging aspect of quantification.

[3] I have found an alternative explanation more palatable to students. They initially claim Velleman's statement is only about number greater than 2. I acknowledge this and defer a bit. I raise the issue again after we demonstrate contrapositive equivalence. How can 'If $x > 2$, then $x^2 > 4$' be equivalent to 'If $x^2 \leq 4$, then $x \leq 2$' if they are about completely different sets of numbers? Since the first statement provides information about the numbers for which $x^2 \leq 4$, it is reasonable to say that it is really about all real numbers (not just those greater than 2).

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