Communications

Order of Operations: On Conventions, Mnemonics and Knowledge-in-Use

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In many Canadian schools the acronym BEDMAS is used as a mnemonic to assist students in remembering the order of operations: Brackets, Exponents, Division, Multiplication, Addition, and Subtraction. In the USA the mnemonic is PEMDAS, where 'P' denotes parentheses, along with the phrase "Please Excuse My Dear Aunt Sally". In the UK the mnemonic BIDMAS is used, where I stands for indices. Other variations include BODMAS, POMDAS or PODMAS, where O stands for Order.

While researchers argue that mnemonics do not support conceptual understanding and may lead to mistakes (*e.g.,* Ameis, 2011, Hewitt, 2012), still they are used by many teachers. Note that while 'parentheses' (P) and 'brackets' (B) are synonyms, the order of division and multiplication (D and M) is reversed in PEMDAS vs. BEDMAS. This order is of interest in this paper, following an explicit assertion of a student, that "division should be performed before multiplication".

A problem with mnemonic

Order of operation errors are common. Researchers noted students' incorrect answers when the mnemonic PEMDAS is interpreted as the appropriate rigid order (*e.g.,* Glidden, 2008). That is, students' mistakes resulted from giving priority to multiplication over division or from giving priority to addition over subtraction.

Various attempts have been made to emphasize that multiplication and division have the same priority. This includes explicit statements in textbooks, presenting the order of operation as hierarchical levels (*e.g.,* Ameis, 2011), or creative ways to record the mnemonic, (*e.g.,* PE(M/D)(A/S)). Regardless of the instructional attempts and written materials, what often remains in memory, and is used, is the mnemonic.

Mathematical Landscape and Knowledge-in-Use: Theoretical Underpinning

Wasserman (2016a) describes how knowledge of "advanced mathematics may positively impact instruction" (p. 29), where 'advanced mathematics' refers to topics typically studied in university, such as abstract algebra. (It is notable that there are significant similarities in tertiary mathematics curricula at different institutions.) Focusing on teaching, Wasserman (2016b) introduces the topological metaphor of a *mathematical landscape.* He considers the *local mathematical landscape* to be the mathematics being taught and the *nonlocal mathematical landscape* consisting of "ideas that are farther away" (p. 380). He suggests that this division "tackles the notion of mathematical knowledge beyond what one teaches" (p. 380).

In relation to teachers' mathematical knowledge Wasserman (2016b) uses the terms 'nonlocal' and 'advanced' as synonyms, as in most cases advanced mathematics is not taught in school. Wasserman distinguished three perceptions on the significance of exposure to nonlocal/advanced mathematics: advanced mathematics for its own sake, advanced mathematics as connected to school mathematics, and advanced mathematics as connected to the *teaching* of school mathematics. He asserted that "teachers' development of and understandings about advanced mathematics must not only relate to the content of school mathematics, but to the teaching of school mathematics content" (p. 386). This is because exposure to nonlocal mathematics helps teachers in developing Key Developmental Understandings (KDUs) (Simon, 2006), which change perceptions about content and influence mathematical connections, so in turn, have an impact on teaching.

In what follows I refine Wasserman's construct of *nonlocal mathematical landscape,* presenting a story of nonlocal, but not necessarily advanced, knowledge, which shapes instructional interaction.

A Story in Two Accounts

In presenting the story I rely on Mason (2002) in distinguishing between *account-of* and *accounting-for.* The term 'account-of' provides a brief description of the key elements of the story, suspending as much as possible emotion, evaluation, judgment or explanations. This serves as data for 'accounting-for', which provides explanation, interpretation, value judgment or theory-based analysis.

The story is situated in a course *Foundations of Mathematics* for secondary mathematics teachers, which is a part of the Master's program in mathematics education. Building and strengthening connections between advanced mathematics and school mathematics was one of the explicit goals of the course.

Account-of: part 1, conventions task

One of the tasks for secondary mathematics teachers was to consider mathematical conventions. This task followed discussion on the choice of a particular mathematical convention, the use of superscript (-1) in different contexts.

The focus on superscript (-1) provoked interest in the choice of other mathematical conventions, conventions that are often introduced and perceived as arbitrary, rather than necessary (Hewitt, 1999), without any particular explanation (see Kontorovich and Zazkis, 2017, for extended discussion on conventions). The *Mathematical conventions* task was designed to address this interest further. In this task the teachers were asked to write a script for a dialogue between a teacher and students, or between students, where interlocutors explore a particular mathematical convention of their choice and a reason behind it. The idea behind this task was to extend a conversation on the choice of conventions, and acknowledge either the arbitrary nature or the reasoning underlying some of these choices.

Account-of: part 2, order of operations convention

One of the repeated choices for a mathematical convention was order of operations when performing arithmetic calculations. Below is an excerpt from the script written by Andy, who described a conversation occurring in Grade 8 class. In this script student-characters consider a 'long' arithmetic expression, and argue about the order in which the operations are to be carried out. After they consider left-to-right and right-to-left orders, Jane makes a suggestion to which the teacher-character Mr. X seems to agree.

Jane Well I did the division first, followed by multiplication, addition and subtraction.

$25 + 5 \times 7 - 2 \times 10 \div 5$
$25 + 5 \times 7 - 2 \times 2$
$25 + 35 - 4$
$60 - 4$
56

Tom I don't understand why you started with division. Why would you start there?

Jane Everybody knows that's the proper order to do operations.

Mr. X Jane is correct. That is the correct order to do those operations.

Sam But why?

Mr. X A long time ago a group of people had a very similar situation that we have now. They were confused and couldn't figure out who had the correct solution to a problem [...]. It happened [in] the Kingdom of Math. The King of Math, as it were, was a very intelligent leader [...]

Mary Very funny, Mr. X.

Tom So what did the King do?

Mr. X The King commanded his most trusted advisors, members of the Order of Knowledge, to look into the problem. [...] They proposed that the only way to solve this problem was for the King to proclaim an order to the operations so that everyone would know the correct way to solve the mathematical problem.

Sam That makes sense. Then everyone would follow the same order and no one would be confused about what steps to do first.

Andy offered the following comment at the end of the assignment:

> I felt that there was only one reason that I could [tell] students: "We need to have an order that everyone follows so we can be consistent". "This is the way we all do it". 'We' being us in the math community. [...] This is because we've all agreed to use the same order so as to have the same understanding of the operations. I tried to find an actual history of the order of operations, but couldn't find anything concrete. So I decided to make up a story that would hopefully give them some connection to the problem and some entertainment along the way.

Accounting-for: mathematics of 'division first'

It is clear from Andy's commentary that accompanied the script that he perceives the convention of order of operations as an arbitrary decision. The reasoning behind this choice, other than the need for consistency, was unclear to Andy and was not found when sought. Other secondary school teachers agreed with this view.

The teacher-character's agreement with Jane's statement in the script, "division first followed by multiplication, addition and subtraction" could have been overlooked, as the result was correct. Nevertheless, both the claim of 'division first', and the order in which the operations were performed in Jane's example, attracted my attention, as it was in discord with my knowledge of order of operations.

For simplicity, consider only Jane's last short computation that involves multiplication and division. Performing 'division first' in $2 \times 10 \div 5$ means interpreting this calculation 'as if' there are parentheses around the operation of division $2 \times (10 \div 5)$. And this is exactly what Jane did, given that this part calculation was simplified as 2×2.

But actually, 'division first' and 'in order of appearance' yield the same result $2 \times 10 \div 5 = (2 \times 10) \div 5 = 20 \div 5 = 4$ and $2 \times (10 \div 5) = 2 \times 2 = 4$, exemplifying the 'general case' that $(a \times b) \div c = a \times (b \div c)$. The equality easily confirmed attending to (a) division is an inverse operation of multiplication and (b) multiplication is associative. Therefore, division can be performed 'out of order', as

$$(a \times b) \div c = (a \times b) \times \frac{1}{c} = a \times (b \times \frac{1}{c}) = a \times (b \div c)$$

Account-of: part 3, discussion on 'division first' assertion

As a consequence of 'division first' suggestion in Andy's script, the following assertion was presented to a class discussion: *According to the established order of operations, division should be performed before multiplication.* It was presented a student's claim, for which a teacher's response was sought.

Four teachers (out a class of 16, including Andy) agreed with the claim, while others insisted on the 'left-to-right' order, when only division and multiplication appear in a computation. BEDMAS was the presented argument that supported the assertion. However, majority of teachers claimed that 'division first' was wrong and attempted to find a counterexample, where giving priority of division over multiplication vs. performing these operations in order they appear will lead to different results. (An analogous idea of 'multiplication first' or 'order does not matter' was suggested, but immediately rejected by a counterexample.)

When 'simple' computations did not lead to a counterexample, teachers turned to more complicated examples. These examples included a longer chain of computations, fractions, and negative numbers. In a class session, a search for a counterexample lasted for about 25 minutes. There were occasional exclamations of "Eureka!" that eventually resulted in double checking that uncovered computational errors. A failure to come up with a counterexample resulted in a conjecture that prioritizing division over multiplication will always work.

Considerable scaffolding was needed to prove this conjecture. When someone suggested that "it works" because "division is just an inverse of multiplication", I countered the claim with "multiplication is just an inverse of division" and "it doesn't work". The suggestion of associativity was voiced only after teachers were asked explicitly to consider in what ways division and multiplication were different. As a result of focusing on this difference, the assertion was rephrased: *Division can be (rather than should be) performed before multiplication.*

Accounting for: analysis of discussion

The assertion presented for discussion for secondary school teachers caused a cognitive conflict to both supporters and objectors. Those who supported the conjecture based on BEDMAS were surprised to find out that performing addition before subtraction (A before S) does not lead to an expected result. Those who believed that the assertion was false, and claimed that division and multiplication have the same priority and should be performed in left-to-right order, were surprised to find out that giving priority to division indeed 'works'. Each group exhibited a robust "strength of belief" (Ginsburg, 1997), based on knowledge that was entrenched and never questioned, as evident in a lengthy search for a counterexample.

The justification and reformulation of the assertion, based on associativity of multiplication, was readily accepted, and even came with an AHA! experience for some teachers. As such, it is curious why such an argument was hard for teachers to find on their own. The theoretical constructs of local and nonlocal mathematical landscape presented above provide a possible explanation.

I suggest that the notion of associativity does not belong to secondary school teachers' local mathematical landscape. That is to say, they do not teach associativity, and even when this property is acknowledged together with other properties of arithmetic operations, it is mentioned together with commutativity. To elaborate, operations discussed in school mathematics, are either both commutative and associative, or neither commutative nor associative, which results in frequent confusions between the two (Hadar & Hadass, 1981, Zaslavsky & Peled, 1996). Associativity appears as a property 'on its own' in relation to groups and their structure when learning advanced mathematics, and being exposed to examples of operations that are associative but not commutative. As such, while the notion itself does not require advanced background, knowledge of advanced mathematics reshapes how associativity is perceived.

For secondary school teachers, associativity appeared to be found in the nonlocal environment, and the connection between local and nonlocal mathematics was not immediately articulated. Addressing the evoked by the assertion cognitive conflict helped teachers in connecting their nonlocal mathematics (associativity) to local mathematics (order of operations) they are teaching.

Conclusion

Wasserman suggests that "knowledge of nonlocal mathematics becomes potentially productive for teaching at the moment that such knowledge alters teachers' perceptions of or understandings about the local content they teach" (2016b, p. 382). The example of order of operations demonstrates that 'nonlocal' and 'advanced' should not be used interchangeably. Nonlocal can refer to content situated beyond teachers' 'active repertoire' of knowledge used in teaching, but which is not necessarily 'advanced' mathematics.

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Building Inclusive Mathematics Classrooms for Students with Disabilities

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In the United States, inequities in the mathematics education, in terms of access and outcomes, are widely discussed for marginalized groups such as Native Americans, African Americans, and Hispanics. Yet, inequities involving students with disabilities are often excluded from these conversations. This article suggests three supportive principles in designing inclusive curricula and environments toward advancing mathematical equity, access, and empowerment for students with disabilities.

By inclusive and disabilities, I encompass all categories listed in the Individuals with Disability Improvement Act (IDEA, 2004), from those who may be 'mildly' affected by the disability (*e.g.,* language impairment) to those more 'severely' affected (*e.g.,* autism). In the U.S., many students across these disabilities categories spend a majority of their school day in segregated special education learning environments (U.S. Department of Education, 2016), 'mainstreamed' with general education students mostly during non-core academic subjects such as physical education and art. Equity, access, and empowerment are important concepts in mathematics education to counter the exclusion of individuals with disabilities.

In this article, I suggest three supportive principles aimed at advancing inclusive mathematics classrooms and practices, specifically focusing on students with disabilities who are most likely to experience mathematics in segregated and self-contained learning environments, including those with more 'severe' disabilities. These three principles support inclusive practices by incorporating conceptual tools and intentionally structuring mathematics classrooms for and of all.

Reframing and relocating disability

The first principle in moving toward inclusive mathematics classrooms is to reframe and constantly reflect upon our perceived concept of disability and its location. In schools and society, disability is typically conceived as a fixed deficit or problem located within an individual. Moving away from such disability concepts may not only be productive (Lambert & Tan, 2017), but it is also socially just. In particular, while we have a strong understanding of effective teaching and learning of mathematics, such understanding is focused on limited sensory capacities such as those related to written and spoken forms of communication. There remains much that we still do not fully understand about the variety of ways individuals represent, communicate, and engage mathematically (de Freitas, 2015). Typically, educators ask students to verbally or in written form engage in and express mathematics under the assumption that such sensory forms (*i.e.,* talking and writing) are the only modalities worth exploring, perhaps for the sake of efficiency. However, students with disabilities might not be as fluent attending to those specific sensory modes, requiring teachers to better understand and attune to other sensory capacities. Indeed, researchers have long known that by exploring other sensory capacities, students with disabilities often excel in creative productions (Carter, Richmond, & Bundschuh, 1973). Students may be making sense of and engaging in mathematics using sensory capacities (*e.g.,* rhythmic, sound, feel) that are not well understood in schools and classrooms. Lack of attention to and understanding of multiple sensory capacities can lead to students being identified with a learning disability and dismissing some as not doers and thinkers of mathematics.

Consequently, framing disability as a difference (Lewis, 2014) or uniqueness more adequately captures the social construction of disability—calling attention to and locating disability within inaccessible and inequitable mathematics curricula. If we only allow students to participate verbally and with handwritten work, then this decision contributes to our perceptions of a disability as a deficit. Indeed, to improve curricula accessibility, assistive technologies such as text-to-speech mobile applications and computer software may be useful for students with limited oral or fine motor skills to engage in the process of representing or communicating their thinking. Critical reflective questions may also help guide the shift of relocating disability. For example: Which students typically benefit from current mathematics educational standards and practices? What are some of the ways that standards, curriculum, and practices enable/disable certain students? Is this the way we want and intend things to be? What needs to be transformed to adjust the imbalances such that all students benefit and are empowered? (Tan & Patrick, in press).

While reframing and relocating disabilities in mathematics education is perhaps the most important of the three principles, it is also the most difficult to espouse as it requires substantially challenging normatively held assumptions about disability and navigating related social forces (Tan & Kastberg, in press). Indeed, access to meaningful mathematics education for students with disabilities (as opposed to rote and low-cognitive mathematics that students with disabilities are typically subjected to) hinges on the progress of this difficult work. As such, the extent to which teachers reframe and relocate disabilities may directly influence subsequent principles of advancing inclusive mathematics education.

Surface and assign mathematical competence

Van den Heuvel-Panhuizen's (1996) work with middle school students identified as having intellectual disabilities illustrates the transition from reframing and relocating disabilities to surfacing mathematical competence. In particular, van den Heuvel-Panhuizen viewed widely used assessments of ratio knowledge to be problematic for these students, and instead utilized strategies to construct more equitable and accessible instruments to gain a more indepth view of students' ratio reasoning. The results of van den Heuvel-Panhuizen's work were astonishing. Despite not having had any formal ratio instruction, the students displayed sophisticated ratio problem-solving strategies such as multiplicative reasoning and use of abstract and concrete models. Thus, the role of educators is to not only diligently look for evidence of competence within students but also to presume that it exists. As evident from van den Heuvel-Panhuizen's work, educators can enact practices of reframing and relocating disabilities toward surfacing competency.

As educators begin to routinize such practices, they may assign competence through peer-mediated learning supportive structures, which is a powerful way to make visible the message of presuming and assigning mathematical competence. Ideas from Complex Instruction (Cohen & Lotan, 2014) are useful in structuring peer-mediated learning to facilitate more balanced social and mathematics status relationships between group members. This is especially crucial as students are inclined to establish individual dominance, and group hierarchies wherein students with disabilities fall on the lower end (Sale & Carey, 1995). As such, supporting group work is thereby grounded in interdependency.

Mathematics for and of all

As educators begin to design accessible and empowering mathematics education through the first two principles, not only do students with disabilities benefit but so do all mathematics learners. Indeed, mathematics becomes not only *for all* but also *of all*—the third principle. An effective framework to advance this principle is Universal Design for Learning ([UDL], CAST, 2016). Central to the UDL framework is that individuals learn, engage in learning, and express learning differently. The UDL framework is also based on recent advances in learning sciences and technological developments such that educators consider multiple means of representations, expressions, and engagement in designing curricula. Importantly, these considerations are made before and while designing curriculum rather than as an afterthought. That is, rather than making modifications and adaptations to designs for specific students after curricula with the majority of students in mind, the UDL framework requires teachers to consider the preferences and needs of all of their students upfront in the planning and designing curricula process.

In mathematics classrooms, UDL can be a very powerful framework to guide learning experiences. UDL principles, for example, can be applied to a unit on solving ratio problems (*e.g.,* finding the ratio, comparing ratios, producing equivalent ratios, and finding proportions). As educators design this unit, they examine the needs, preferences, and strengths of all of their students. From better understanding their students, educators may incorporate accessible features to elicit problem-solving strategies such as providing multiple options for representing materials (*e.g.,* providing oral and written instruction; numerical and non-numerical ratio problems), engaging in the unit (*e.g.,* creating meaningful problems; allow work with peers or individually), and expressing and communicating (*e.g.,* allow use of computers and assistive technology, and options for use of kinesthetic movements). Educators may already be incorporating some of these options for learning in their instructions and classrooms, but other options may be unfamiliar. Such dichotomy reflects the nature of UDL in requiring educators to engage in the process of revisiting and refining their methods based on the uniqueness and preferences of their students. Hence, UDL supports students in making sense of and engaging in mathematics using a variety of sensory capacities—underscoring the notion of mathematics as a living subject for all and of all.

The three principles described herein are simultaneously dispositions and actionable steps to frame the construction of inclusive mathematics classrooms. That is, educators can use the three elements as a guide to design inclusive mathematics curriculum for all students, and to fulfill the field's longstanding equity commitment. Also, a useful process in addressing the three principles is the UDL Discussion Facilitation tool which we next describe.

A tool for advancing the three principles

The UDL Discussion tool utilizes questions to guide teams of two or more educators to examine issues related to accessibility of mathematics curricula and to determine possible solutions. Importantly, educators should have opportunities to grapple with the three principles before engaging with the tool as such opportunities may create more productive conversations. Engagement with the tool involves at least one general educator and one special educator. However, conversations around the tool can be just as effective with different permutations consisting of various educators' roles (*e.g.,* discussion between two middle school mathematics teachers or between a mathematics teacher and the school principal). A major attribute of the tool is that while it aims to design accessible mathematics curricula with one or a few students in mind, all students have the option of utilizing those accessible features.

This conversation addresses the first principle by reframing disability from student deficits (*e.g.,* 'he doesn't have number sense') toward preferences for learning (*e.g.,* 'he prefers to use his body to make sense of numbers'). At the same time, the concept of 'fixing' shifts from 'fixing' the student toward making the curriculum more accessible to this student and for all students (principle #3). Lastly, the questions are intentionally framed to assume students' mathematical competency (principle #2). Thus, while not necessarily ignoring challenges, the responses do and should gravitate towards students' strengths, preferences for learning, and uniqueness. In turn, the tool allows for teachers to be in a better position to surface these mathematical competencies in action and be able to point those out for others to see and understand (principles #2 and #3). Moreover, while the questions in the facilitation tool focus on one particular student, the options and strategies discussed are consistent with UDL ideologies and should be available for all students in the class.

Conclusion

The shift toward inclusive mathematics education involves students across the range of abilities working together to make meaning of mathematics and one another as a community of learners. Creating equitable, accessible, and empowering mathematics curricula is inherently complex requiring substantial investment in time, collaboration and cultivating partnerships with students, educators, and parents. In turn, designing inclusive mathematics classrooms supports a work-in-progress persistently challenging normative notions of disability; and of our role as humans as we work with others to continually learn more about the variety of ways that this beautiful subject of mathematics is manifested and expressed both in all and as all of us.

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Through the Sociological Lens: Learning Mathematics in a Mumbai Classroom

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I entered a seventh grade mathematics classroom at a senior secondary school located in Santa Cruz, Mumbai. The students were sitting at desks made for two, arranged in three rows. Of the 13 girls, 11 were sitting in the right-hand row, nearest the door. The other two girls were sitting at a separate desk to the teacher's left, and one of them was wearing the 'Class prefect' badge. The 19 boys were in the centre and left-hand rows. The lesson was taught in English, on 'An introduction to linear equations in one variable', and the teacher, a native of Mumbai, had a B.Ed. and M.Sc. in Mathematics.

The school is an unaided private institution affiliated to the Central Board of Secondary Education. There is a policy of providing financial aid to needy students, particularly to those who had only one working parent. The problems that plague public and low-fee private schools in India are well known, but this is a well-established school with trained teachers and resource-rich classrooms. What is mathematics learning like for students here?

Gender differences

The teacher explained the concept of linear equations very briefly, and then got back to writing questions on the board, waiting for the students to solve and respond. She often called out names to invite answers; the names followed a pattern rotating through four of five students. The teacher's gaze after writing down a question was almost always directed towards the students who had answered the previous questions correctly. The classroom dynamic was very competitive. The four boys sitting in the front desks were often the first to respond, but they were closely challenged several times by the class prefect and her partner. The other boys further back were not responding. Some of them seemed engaged, but the boys at the very back, as well as the girls in the right-hand row made no apparent effort to respond or take notes.

What does this stark categorization based on seating mean for mathematics learning in this classroom? It informally legitimizes stratifications, 'labeling' the students, and likely pushing them towards acting in accordance with what was expected of them (Rist, 1977, p. 296). Notably, one stratification in this classroom is based on gender. I remember from my experience of teaching students in lower primary grades that differences between the sexes are not much pronounced then, a fact that is corroborated by Forsgasz and Leder (2001). But I could see these differences widening in the secondary grade classroom I was observing. Boys clearly outpaced girls in classroom participation. The focus on speed and competition in the classroom tasks could be favouring the male students, as is evident from Fennema's research as well (Fennema & Peterson, 1986). Sporadic efforts to work collaboratively made by the female students were dismissed twice by the teacher, with firm remarks to "focus on their own work".

Another visible aspect is that the classroom stratification comprises sub-layers within layers. For instance, one must realize that the universe of female students engaging with mathematics itself (which one often considers to be facing injustice as a whole) is further segregated into multiple levels, as is evident from the visible gap in engagement between the class prefect and her partner, and the other girls.

Socioeconomic status differences

A majority of the students came from middle to high income families, hence class inequities were not very conspicuous. But I had an interesting conversation with the teacher regarding students receiving financial aid grants from the school:

SK The department head told me that the school supports needy students.

Teacher Yes. And I have to pay extra attention to such students.

SK Why is that?

Teacher Sometimes students are not motivated in class. At least their parents push them since they are paying the fee. Taking out that economic incentive is not good for the classroom sometimes.

This excerpt can be looked at from two angles. First, it is interesting to see how a lack of academic motivation is automatically being attributed to students from financially weaker backgrounds. This is much in line with what Skovsmose has highlighted regarding students' dispositions (Skovsmose, 2007, p. 87). An automatic assumption of mediocrity from the students' financial *background*, would also go on to further harm their *foregrounds* (their perception of opportunities available, and hence their aspirations). This ruined foreground does little to motivate the students, and they are left with diminished intentions to learn, hence initiating a downward spiral. Secondly, the teacher also assumes here that economic incentive is the primary motivation for parents to push their children. Such assumptions about their parents will almost always get communicated to the students, restricting their foregrounds even further. Students with such skewed dispositions will construct restricted meanings, which will push them further down the spiral.

Discussion

The school has good infrastructure, the teachers are qualified and well trained and the majority of students come from relatively privileged socio-economic backgrounds (as compared to the city's demographics). The school also organizes quarterly workshops to update teachers on latest developments in their respective fields. Despite ticking all the right boxes (aspects found missing in many public and low-fee private schools in the region), the classroom proceedings were found lacking from a sociological perspective.

A continuous theme running throughout my observations, as well as during my conversations with the staff, was a strong spirit of competition and rivalry. This competitive spirit possibly emanated from competition among elite schools in the region, as parents from relatively privileged backgrounds prefer to get their children admitted to schools showcasing the best outcomes. This was also conspicuous among teachers competing to be the best performing section within a grade. The notion percolates down to some students as well, as is evident from the eagerness of the boys at the front to contest and answer questions in a flash. Such vibrant energy and quick responses pouring in might even provide the superficial impression of a well-engaged classroom. But a closer look reveals that cracks do persist, as we have seen. This becomes even more problematic when a perpetual emphasis on achievement of a minority overshadows the learning needs of a majority of students.

SK From your experience, which approach do you think makes the students learn better?

Teacher I think that we can give students independence and it is good. S1, S2, S3 *[three boys at the front desks]* do well regardless of the approach.

SK How well do the girls engage in learning mathematics?

Teacher Girls are good as well! S4 *[the class prefect]* is very good in mathematics. S5 *[her partner]* has also done well by collaborating with her often. They both work very hard.

The teacher tends to stay restricted to a minority of well-performing boys and girls even when asked about the classroom as a whole. This fits with Keddie's analysis (1971, p. 66), where equal rights are not granted to each pupil based on his or her normal status (which here could be based on the perceived ability of particular groups of students). Further, the fact that academically weaker students did not appear to know formal mathematical terms went against them. Their attempts to understand concepts in an everyday context and language were not well appreciated. Moreover, rather than attempting to inform the weaker student's understanding (S6 as seen below), the teacher ignores him and moves ahead to ask other (and possibly well informed) students.

Teacher Very good *[to the boys at the front desks]*. What do we mean by linear equations in one variable?

S6 *[sitting at a desk third from the back]* Adding and subtracting

Teacher *[interrupting]* Not just that, how do we solve linear equations? *[looking back to the front desks]*

Other aspects that came out during our conversations pertained to the teacher's assertion of her belief in 'inclusive schooling', as well as the school's promotion of a model of 'self-paced learning'. If we couple these notions with an emphasis on competition and achievement, the situation gets oddly skewed in favour of academically brighter students. In other words, an inclusive model of self-paced learning would theoretically embrace the pace of the slowest student. But that would be highly inefficient given this context's competitive demands. Hence, in reality, the classroom effectively embraces the pace of the fastest students answering questions in quick succession. The ones left behind must fend for themselves.

I observed that problems related to social factors can persist in a resource-rich and well-regarded mathematics classroom. Although this is only one example, it raises important questions. What could cause a mathematics class in a well-established school to underperform on sociological dimensions, despite having all the recommended ingredients for excellence and inclusion? Does a fierce sense of competition among (and within) such elite schools really shift the focus towards a well performing minority, and consequently obscure issues faced by the majority? Are similar sociological issues overlooked in public and low-fee private schools, possibly due to larger problems at hand? Further research is needed to address these questions. From a sociological perspective, a truly inclusive mathematics classroom must allow leeway for multiple definitions and multiple pathways of learning (and not just multiple paces along a single path, such as in self-paced learning). This becomes imperative, for instance, in light of research findings pointing out that thinking of mathematics as a domain of 'reason and rationality' could be detrimental to female students (Paechter, 2001). Similar arguments could be drawn for students belonging to any other layer of social stratification—be it based on class, caste, ethnicity, or religion. Given the plethora of resources at hand, well-resourced 'elite' schools are best placed to address these issues by innovating and experimenting with more flexible teaching-learning approaches. The successful prototypes can then be developed for cost-effective promulgation to other schools as well.

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From the Archives

The following is an edited excerpt from *Excellence and equity in mathematics classrooms* by George M.A. Stanic, Laurie Hart Reyes, in issue 7(2), pp. 27-31.

Were John Dewey still alive, he would probably look at the present arguments about excellence and equity in education as reflections of a false dichotomy. Just as he reconstructed the relationships between interest and effort and between the child and the curriculum, he would help us see the relationship between excellence and equity in a new light.

In this article, we will not do what Dewey would have done because we are still struggling with ideas of excellence and equity. Instead, we would like to express our concerns about achieving equity, especially at a time when excellence has become a rallying cry for people inside and outside the field of education. The main point of this article is that even if the goal of equity is seen as crucial and threatened by the "excellence movement" in education, the most difficult problem is to determine exactly what constitutes equitable treatment of students in schools and classrooms. Given that most, if not all, teachers would agree that their interactions with students should reflect fairness and justice (*i.e.,* equity), what can they do to ensure that equity exists in their classrooms? […]

The excellence movement in education has come to focus not just on students' level of achievement but also on what subject matter they are studying. One way to critique this movement and to argue for the importance of equity would be to focus on what subject matter is considered valuable by those who are calling for excellence in education. Although a discussion of high status and low status knowledge is necessary, our discussion of the relationship between excellence and equity begins with the assumption that knowledge of mathematics is important for everyone. In order to participate fully in our democratic processes and to be unrestricted in career choice and advancement, individuals must be able to understand and apply mathematical ideas Therefore, although mathematics educators do need to examine their assumptions about the importance of mathematics, we begin this paper with the belief that all people should know about and be able to do mathematics.

Originally we wanted to argue not just that excellence and equity are compatible, but that true excellence cannot be achieved without equity. To make this argument, however, the basic meaning of excellence has to be changed. As long as excel means to surpass, to be superior to, or to outdo others, it would be difficult to argue that fairness, or justice, or equity is a necessary condition for excellence. One could, of course, try to argue for basing excellence on outdoing oneself, or surpassing some criterion, so that all of us can be excellent. But outdoing oneself is a problematic concept, and recent calls for excellence in education do not seem to be based on the goal of all people surpassing some criterion. Instead, competition with others is central to the excellence movement, whether the "others" are the Soviets, the Japanese, students in another state, or even one's own classmates.

Although not all forms of competition in our society are necessarily bad, there is reason to worry about the role of competition in our system of education in general and in classrooms in particular. When we look into classrooms, it becomes clear that striving for excellence operationalized as competing against and attempting to be superior to others can seriously threaten our quest for equity. Research on sex-related differences in mathematics performance provides at least some evidence that we have reason to be concerned. The work of Elizabeth Fennema and Penelope Peterson (1985) indicates that certain students, particularly girls, do better in cooperative learning environments than in competitive ones. The problem, of course, is that most mathematics classrooms are already competitive in nature, thus inhibiting the performance of those students who would benefit from a more cooperative environment. And the danger associated with the excellence movement lies in the fact that it will reinforce rather than change many of the taken-for-granted realities of classroom life, whereas a focus on equity would lead us to ask how we might change classrooms to make them more fair and more just for all students. The *Nation at Risk* report (USDE, 1983), if not the beginning at least the creed of the excellence movement, tells us that Americans all have the same goals for our education system, that we need simply to carry out what we know is the right thing to do. A concern for equity on the other hand, would make us constantly ask: What is the right thing to do in our schools?

Striving for excellence is, then, empty at best and harmful at worst without a concomitant concern for equity. Equity, however, is a much more difficult goal to achieve than is excellence. It is easier to show that certain students have done better than others than it is to show that all students have been treated fairly. That is why lip-service to equity as we strive for excellence is simply not enough. And that is why teachers have such a profoundly difficult task in trying to address the goals of both excellence and equity.

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